Mathematical
Inequalities
& Applications
Volume 1, Number 2 (1998), 239–246

DISTORTION INEQUALITIES FOR RUSCHEWEYH DERIVATIVES

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(*communicated by J. Pečarić*)

Abstract. Let $\mathscr{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathscr{U}$ with $f(0) = 0$ and $f'(0) = 1$. For $f(z) \in \mathscr{A}$, the Ruscheweyh derivative of order λ is denoted by $\mathscr{D}^\lambda f(z)$. The object of the present paper is to derive several distortion inequalities involving $\mathscr{D}^\lambda f(z)$ for certain classes of univalent functions $f(z)$ by applying known properties of generalized hypergeometric functions.

1. Introduction

Let $\mathscr{A}$ be the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathscr{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of all *univalent* functions in $\mathscr{U}$. Further, let $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ be the subclasses of $\mathscr{A}$ consisting, respectively, of functions which are *starlike of order* α $(0 \leqslant \alpha < 1)$ and *convex of order* α $(0 \leqslant \alpha < 1)$ in $\mathscr{U}$.

It is well-known (*cf.* Robertson [1]) that

$$f \in \mathscr{S}^*(\alpha) \Rightarrow |a_k| \leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} \qquad (k \in \mathbb{N} \setminus \{1\}; \quad \mathbb{N} := \{1, 2, 3, \cdots\}) \tag{1.2}$$

and

$$f \in \mathscr{K}(\alpha) \Rightarrow |a_k| \leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} \qquad (k \in \mathbb{N} \setminus \{1\}). \tag{1.3}$$

Mathematics subject classification (1991): Primary 30C45; Secondary 33C05, 33C20.

Key words and phrases: Analytic functions, univalent functions, Ruscheweyh derivatives, starlike functions, convex functions, generalized hypergeometric functions, Pochhammer symbol, distortion inequalities, Hadamard product (or convolution).

Paper MIA-01-22

For $f_j(z) \in \mathscr{A}$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}\, z^k \qquad (j = 1, 2), \tag{1.4}$$

the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{1.5}$$

Using the convolution (1.5), Ruscheweyh [2] introduced what is now referred to as the Ruscheweyh derivative $\mathscr{D}^{\lambda} f(z)$ of order λ of $f(z) \in \mathscr{A}$ by

$$\mathscr{D}^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1). \tag{1.6}$$

It follows that

$$\mathscr{D}^0 f(z) = f(z), \quad \mathscr{D}^1 f(z) = zf'(z),$$

and, in general,

$$\mathscr{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \qquad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right).$$

Furthermore, we have

$$\mathscr{D}^{\lambda} f(z) = z + \sum_{k=2}^{\infty} C(\lambda, k)\, a_k\, z^k, \tag{1.7}$$

where

$$C(\lambda, k) = \frac{\prod_{j=1}^{k-1} (j + \lambda)}{(k-1)!} \qquad \left(k \in \mathbb{N} \setminus \{1\}\right). \tag{1.8}$$

The generalized hypergeometric function ${}_pF_q(z)$ is given by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; z\right) \\ &= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \, \frac{z^k}{k!} \qquad (p \leqslant q+1), \end{aligned} \tag{1.9}$$

where p and q are non-negative integers, a_j $(j = 1, \cdots, p)$ and b_j $(j = 1, \cdots, q)$ are complex numbers with $b_j \neq 0, -1, -2, \cdots$. Here $(\lambda)_k$ denotes the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N}). \end{cases} \tag{1.10}$$

If we set

$$\omega = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j \ , \tag{1.11}$$

we see that the series ${}_pF_q(z)$, with $p = q+1$, is

(i) absolutely convergent for $|z| = 1$, if $\mathrm{Re}\,(\omega) > 0$,

(ii) conditionally convergent for $|z| = 1$ $(z \neq 1)$ if $-1 < \mathrm{Re}\,(\omega) \leqslant 0$, and

(iii) divergent for $|z| = 1$ if $\mathrm{Re}\,(\omega) \leqslant -1$.

If $p < q+1$ and $\mathrm{Re}\,(\omega) > 0$, then the ${}_pF_q(z)$ series (1.9) is absolutely convergent for $|z| < \infty$.

2. Distortion Inequalities for Starlike Functions

Our first distortion inequality involving Ruscheweyh derivatives is contained in

THEOREM 1. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{S}^*(\alpha)$, *then*

$$|\mathscr{D}^{\lambda} f(z)| \leqslant M(n, \lambda, \alpha;\ |z|) + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\,|z|^n\ {}_3F_2\left(\begin{matrix} n+\lambda, n+1-2\alpha, 1; \\ n,\ n; \end{matrix}\ |z|\right), \tag{2.1}$$

where $n \in \mathbb{N} \setminus \{1, 2\}$ *and*

$$M(n, \lambda, \alpha;\ |z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\,|z|^k. \tag{2.2}$$

Proof. We begin by noting that

$$C(\lambda, k) = \frac{\prod_{j=1}^{k-1}(j+\lambda)}{(k-1)!} = \frac{(1+\lambda)_{k-1}}{(k-1)!}\,, \tag{2.3}$$

and

$$|a_k| \leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{(k-1)!} = \frac{(2-2\alpha)_{k-1}}{(k-1)!} \qquad (f \in \mathscr{S}^*(\alpha))\,. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}|\mathscr{D}^{\lambda} f(z)| &\leqslant |z| + \sum_{k=2}^{\infty} C(\lambda,k)\,|a_k|\,|z|^k \\ &\leqslant |z| + \sum_{k=2}^{\infty} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\,|z|^k \\ &= \left\{ |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\,|z|^k \right\} \\ &\qquad + \sum_{k=n}^{\infty} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{\{(k-1)!\}^2}\,|z|^k \\ &= M(n,\lambda,\alpha;\,|z|) + \sum_{k=0}^{\infty} \frac{(1+\lambda)_{k+n-1}(2-2\alpha)_{k+n-1}}{\{(k+n-1)!\}^2}\,|z|^{k+n}.\end{aligned} \tag{2.5}$$

Since

$$(1+\lambda)_{k+n-1} = (1+\lambda)_{n-1}(n+\lambda)_k, \tag{2.6}$$

$$(2-2\alpha)_{k+n-1} = (2-2\alpha)_{n-1}(n+1-2\alpha)_k\,, \tag{2.7}$$

and

$$(k+n-1)! = (n-1)!\,(n)_k\;, \tag{2.8}$$

we see that

$$\begin{aligned}|\mathscr{D}^{\lambda} f(z)| &\leqslant M(n,\lambda,\alpha;\,|z|) \\ &\quad + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\,|z|^n \sum_{k=0}^{\infty} \frac{(n+\lambda)_k(n+1-2\alpha)_k}{\{(n)_k\}^2}\,|z|^k \\ &= M(n,\lambda,\alpha;\,|z|) \\ &\quad + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\,|z|^n\;{}_3F_2\left(\begin{matrix} n+\lambda,\ n+1-2\alpha,\ 1; \\ n,\ n; \end{matrix}\ |z|\right).\end{aligned} \tag{2.9}$$

COROLLARY 1. *If a function $f(z)$ given by (1.1) belongs to the class $\mathscr{S}^*(\alpha)$, then*

$$\begin{aligned}|\mathscr{D}^{m} f(z)| &\leqslant M(n,m,\alpha;\,|z|) + \frac{(1+m)_{n-1}(2-2\alpha)_{n-1}}{\{(n-1)!\}^2}\,|z|^n \\ &\cdot \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(n+1-2\alpha)_k(1)_k}{\{(n)_k\}^2}\;\frac{|z|^k}{(1-|z|)^{k+2-2\alpha}}\;{}_2F_1\left(\begin{matrix} 2\alpha-1,\ n-1; \\ n+k; \end{matrix}\ |z|\right)\right\},\end{aligned} \tag{2.10}$$

where $m \in \mathbb{N}$.

Proof. Note that (*cf.* Srivastava [3])

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix} b_1+m, & a_2,\cdots,a_p; \\ b_1, & b_2,\cdots,b_q; \end{matrix}\; z\right) \\
&\quad = \sum_{k=0}^{m}\binom{m}{k}\frac{\prod_{j=2}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k} z^k \; {}_{p-1}F_{q-1}\left(\begin{matrix} a_2+k,\cdots,a_p+k; \\ b_2+k,\cdots,b_q+k; \end{matrix}\; z\right)
\end{aligned}
\tag{2.11}
$$

for $m \in \mathbb{N}$, and

$$
{}_2F_1\left(\begin{matrix} a_1,a_2; \\ b_1; \end{matrix}\; z\right) = (1-z)^{b_1-a_1-a_2}\; {}_2F_1\left(\begin{matrix} b_1-a_1,\, b_1-a_2; \\ b_1; \end{matrix}\; z\right). \tag{2.12}
$$

Therefore, we have

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} n+m, & n+1-2\alpha, & 1; \\ & n, & n; \end{matrix}\; |z|\right) \\
&= \sum_{k=0}^{m}\binom{m}{k}\frac{(n+1-2\alpha)_k(1)_k}{\{(n)_k\}^2}|z|^k\; {}_2F_1\left(\begin{matrix} n+1+k-2\alpha, & 1+k; \\ & n+k; \end{matrix}\; |z|\right) \\
&= \sum_{k=0}^{m}\binom{m}{k}\frac{(n+1-2\alpha)_k(1)_k}{\{(n)_k\}^2}\;\frac{|z|^k}{(1-|z|)^{k+2-2\alpha}}\; {}_2F_1\left(\begin{matrix} 2\alpha-1, & n-1; \\ & n+k; \end{matrix}\; |z|\right).
\end{aligned}
\tag{2.13}
$$

The assertion of Corollary 1 follows from (2.13).

COROLLARY 2. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{S}^*\left(\frac{1}{2}\right)$, *then*

$$
\begin{aligned}
|\mathscr{D}^\lambda f(z)| &\leqslant M\left(n,\lambda,\frac{1}{2};\,|z|\right) \\
&\quad + \frac{(1+\lambda)_{n-1}}{(n-1)!}\;\frac{|z|^n}{(1-|z|)^{1+\lambda}}\; {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\; |z|\right),
\end{aligned}
\tag{2.14}
$$

where

$$
M\left(n,\lambda,\tfrac{1}{2};\,|z|\right) = |z| + \sum_{k=2}^{n-1}\frac{(1+\lambda)_{k-1}}{(k-1)!}|z|^k. \tag{2.15}
$$

Further, taking $\alpha = 0$ in Theorem 1, we have

COROLLARY 3. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{S}^*$, *then*

$$\left|\mathscr{D}^{\lambda} f(z)\right| \leqslant M(n,\lambda,0;\,|z|) + \frac{n(1+\lambda)_{n-1}}{(n-1)!}\,|z|^n \left\{ \frac{1}{(1-|z|)^{1+\lambda}}\ {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\ |z|\right) + \frac{n+\lambda}{n^2}\ \frac{|z|}{(1-|z|)^{2+\lambda}}\ {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n+1; \end{matrix}\ |z|\right)\right\}, \tag{2.16}$$

where

$$M(n,\lambda,0;\,|z|) = |z| + \sum_{k=1}^{n-1} \frac{k(1+\lambda)_{k-1}}{(k-1)!}\,|z|^k. \tag{2.17}$$

3. Distortion Inequalities for Convex Functions

For the Ruscheweyh derivatives of convex functions belonging to the class $\mathscr{K}(\alpha)$, we have

THEOREM 2. *If a function* $f(z)$ *given by* (1.1) *belongs to the class* $\mathscr{K}(\alpha)$, *then*

$$\left|\mathscr{D}^{\lambda} f(z)\right| \leqslant N(n,\lambda,\alpha;\,|z|) + \frac{(1+\lambda)_{n-1}(2-2\alpha)_{n-1}}{n!\,(n-1)!}\,|z|^n\ {}_3F_2\left(\begin{matrix} \lambda+n, & n+1-2\alpha, & 1; \\ & n, & n+1; \end{matrix}\ |z|\right), \tag{3.1}$$

where $n \in \mathbb{N}\setminus\{1,2\}$ *and*

$$N(n,\lambda,\alpha;\,|z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}(2-2\alpha)_{k-1}}{k!\ (k-1)!}\,|z|^k. \tag{3.2}$$

Proof. Using the fact that

$$|a_k| \leqslant \frac{\prod_{j=2}^{k}(j-2\alpha)}{k!} = \frac{(2-2\alpha)_{k-1}}{k!} \qquad (k\in\mathbb{N}\setminus\{1\};\quad f\in\mathscr{K}(\alpha)), \tag{3.3}$$

we readily arrive at the inequality (3.1) by applying the proof of Theorem 1 *mutatis mutandis*.

COROLLARY 4. *If a function $f(z)$ given by (1.1) belongs to the class $\mathscr{K}(\alpha)$, then*

$$|\mathscr{D}^m f(z)| \leqslant N(n,m,\alpha;\ |z|) + \frac{(1+m)_{n-1}(2-2\alpha)_{n-1}}{n!\ (n-1)!} \cdot \left\{ \sum_{k=0}^{m} \binom{m}{k} \frac{(n+1-2\alpha)_k (1)_k}{(n)_k\,(n+1)_k}\ \frac{|z|^k}{(1-|z|)^{k+1-2\alpha}}\ {}_2F_1\left(\begin{matrix} 2\alpha,\ n; \\ n+k+1; \end{matrix}\ |z| \right) \right\}, \tag{3.4}$$

where $m \in \mathbb{N}$.

Setting $\alpha = \frac{1}{2}$ in Theorem 2, we have

COROLLARY 5. *If a function $f(z)$ given by (1.1) belongs to the class $\mathscr{K}\left(\frac{1}{2}\right)$, then*

$$\left|\mathscr{D}^\lambda f(z)\right| \leqslant N\left(n,\ \lambda,\ \frac{1}{2};\ |z|\right) + \frac{(1+\lambda)_{n-1}}{n!}\ \frac{|z|^n}{(1-|z|)^\lambda}\ {}_2F_1\left(\begin{matrix} 1-\lambda,\ n; \\ n+1; \end{matrix}\ |z| \right), \tag{3.5}$$

where

$$N\left(n,\lambda,\tfrac{1}{2};\ |z|\right) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}}{k!}\ |z|^k. \tag{3.6}$$

Finally, letting $\alpha = 0$ in Theorem 2, we have

COROLLARY 6. *If a function $f(z)$ given by (1.1) belongs to the class $\mathscr{K}$, then*

$$\left|\mathscr{D}^\lambda f(z)\right| \leqslant N(n,\lambda,0;\ |z|) + \frac{(1+\lambda)_{n-1}}{(n-1)!}\ \frac{|z|^n}{(1-|z|)^{1+\lambda}}\ {}_2F_1\left(\begin{matrix} -\lambda, & n-1; \\ & n; \end{matrix}\ |z| \right), \tag{3.7}$$

where

$$N(n,\lambda,0;\ |z|) = |z| + \sum_{k=2}^{n-1} \frac{(1+\lambda)_{k-1}}{(k-1)!}\ |z|^k. \tag{3.8}$$

Acknowledgments. The present investigation was supported, in part, by the Japanese Ministry of Education, Science and Culture under a Grant-in-Aid for General Scientific Research and, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353. Some of the results of this paper were included in a lecture delivered by the second-named author at the Annual Meeting of the Mathematical Society of Japan held at the University of Tokyo from September 30 to October 3, 1997.

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(Received October 28, 1997)

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