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NEW GENERALIZATION OF GAUSS–PÓLYA'S INEQUALITY

S. ABRAMOVICH, J. PEČARIĆ AND S. VAROŠANEC

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Abstract. We consider inequality of Hölder's type

$$\frac{\int_a^b w_3(x)g(x)dx}{\int_a^b w_3(x)dx} \leqslant \prod_{i=1}^{2}\left(\frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx}\right)^{1/p_i}, \quad \frac{1}{p_1}+\frac{1}{p_2}=1,$$

and give a number of results about functions w_1, w_2, w_3 which satisfy the above-mentioned inequality. Also, in a similar way, we consider an inequality of Minkowski's type.

In this paper we show a new way of generalization of Gauss' result about moments. Namely, in [8] the following theorem was proved.

THEOREM 1. *Let $g : [a,b] \to \mathbf{R}$ be a nonnegative nonincreasing function, $x_i : [a,b] \to \mathbf{R}$, $i = 1,2,\dots,n$, be nonnegative nondecreasing functions with a continuous first derivative and $x_i(a) = 0$ for all $i = 1,2,\dots,n$. If p_i $i = 1,2,\dots,n$, are positive real numbers such that $\sum_{i=1}^{n}\frac{1}{p_i} = 1$ then*

$$\prod_{i=1}^{n}\left(\int_a^b x_i'(t)g(t)dt\right)^{1/p_i} \geqslant \int_a^b \left(\prod_{i=1}^{n} x_i(t)^{1/p_i}\right)' g(t)dt. \tag{1}$$

This Theorem is a generalization of so–called Gauss–Pólya's inequality, [10]. Namely, for $n = 2$, $a = 0$, $p_1 = p_2 = 2$, $x_1(t) = t^{2u+1}$, $x_2(t) = t^{2v+1}$, $u, v > -\frac{1}{2}$ we have

$$\left(\int_0^b x^{u+v}g(x)dx\right)^2 \leqslant \left(1-\left(\frac{u-v}{u+v+1}\right)^2\right)\int_0^b x^{2u}g(x)dx \int_0^b x^{2v}g(x)dx, \tag{2}$$

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where g is nonincreasing function. Setting in inequality (2) $b \to \infty$, $u = 0$, $v = 2$ we have inequality between the second and the fourth order moments which is due to C.F. Gauss, [3].

This Gauss–Pólya's inequality (2) can be generalized in various ways, some of them are given in the papers [2], [6], [7], [9], [11], [12], [13], [14].

In the first part of this paper we present a number of helpful lemmas one of which is due to Hardy [4] and the others can be proved using integration by parts, [1]. In the second part of paper we the improve above mentioned Theorem 1 from [8]. And finally, in the third part of text we will give an analogue result for Minkowski's type inequality.

In this paper if the inequality has a number (n) then its reverse version (the reversed inequality) is denoted by (Rn).

1. Preliminary results

Let us suppose that H is an integrable function on $[a,b]$.

LEMMA 1. *a) If S is a nonnegative and nondecreasing function on $[a,b]$ and*

$$\int_x^b H(t)dt \leqslant 0 \textit{ for all } \quad x \in [a,b],$$

then

$$\int_a^b H(t)S(t)dt \leqslant 0.$$

b) If S is a nonnegative and nonincreasing function on $[a,b]$ and

$$\int_a^x H(t)dt \leqslant 0 \textit{ for all } \quad x \in [a,b],$$

then

$$\int_a^b H(t)S(t)dt \leqslant 0.$$

The statement in case a) was proved in [4, p.298] for interval $[0,1]$.

LEMMA 2. *If S is a nonnegative and symmetrical function on $[a,b]$, (i.e. $S(a+x) = S(b-x)$ for all $x \in [0, \frac{b-a}{2}]$), nonincreasing on $[\frac{a+b}{2}, b]$ and if*

$$\int_{a+x}^{b-x} H(t)dt \leqslant 0 \quad \textit{for} \quad x \in [0, \tfrac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \leqslant 0.$$

Proof. Using integration by parts and the symmetry of the function S we have

$$\int_a^{\frac{a+b}{2}} H(t)S(t)dt = S(a)\int_a^{\frac{a+b}{2}} H(t)dt + \int_a^{\frac{a+b}{2}} \left(\int_x^{\frac{a+b}{2}} H(t)dt\right) dS(x).$$

$$\begin{aligned}\int_{\frac{a+b}{2}}^b H(t)S(t)dt &= S(b)\int_{\frac{a+b}{2}}^b H(t)dt - \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^x H(t)dt\right) dS(x)\\ &= S(a)\int_{\frac{a+b}{2}}^b H(t)dt - \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^x H(t)dt\right) dS(a+b-x)\end{aligned}$$

$$\begin{aligned}&= S(a)\int_{\frac{a+b}{2}}^b H(t)dt - \int_{\frac{a+b}{2}}^a \left(\int_{\frac{a+b}{2}}^{a+b-\tau} H(t)dt\right) dS(\tau)\\ &= S(a)\int_{\frac{a+b}{2}}^b H(t)dt + \int_a^{\frac{a+b}{2}} \left(\int_{\frac{a+b}{2}}^{a+b-\tau} H(t)dt\right) dS(\tau).\end{aligned}$$

$$\begin{aligned}\int_a^b H(t)S(t)dt &= \int_a^{\frac{a+b}{2}} H(t)S(t)dt + \int_{\frac{a+b}{2}}^b H(t)S(t)dt\\ &= S(a)\int_a^{\frac{a+b}{2}} H(t)dt + \int_a^{\frac{a+b}{2}} \left(\int_x^{\frac{a+b}{2}} H(t)dt\right) dS(x)\\ &\quad + S(a)\int_{\frac{a+b}{2}}^b H(t)dt + \int_a^{\frac{a+b}{2}} \left(\int_{\frac{a+b}{2}}^{a+b-\tau} H(t)dt\right) dS(\tau)\\ &= S(a)\int_a^b H(t)dt + \int_a^{\frac{a+b}{2}} \left(\int_x^{a+b-x} H(t)dt\right) dS(x). \qquad (3)\end{aligned}$$

The terms $\int_a^b H(t)dt$ and $\int_x^{a+b-x} H(t)dt$ are nonpositive, and the term $S(a)$ is nonnegative and S is nondecreasing on $[a, \frac{a+b}{2}]$, so, the whole term (3) is nonpositive and the lemma is proved. □

Similarly, the following four lemmas could be deduced.

LEMMA 3. *Let S be a nonnegative right balanced function on $[a,b]$ (right balanced means that $S(a+x) \leqslant S(b-x)$ for all $x \in [0, \frac{b-a}{2}]$), and let S be nondecreasing on $[a, \frac{a+b}{2}]$. If*

$$H(x) \leqslant 0 \quad \textit{for all} \quad x \in [\tfrac{a+b}{2}, b]$$

and

$$\int_{a+x}^{b-x} H(t)dt \leqslant 0 \quad \textit{for} \quad x \in [0, \tfrac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \leqslant 0.$$

LEMMA 4. *Let S be a nonnegative left balanced function on $[a,b]$ (left balanced means that $S(a+x) \geqslant S(b-x)$ for all $x \in [0, \frac{b-a}{2}]$), and let S be nondecreasing on $[a, \frac{a+b}{2}]$. If*

$$H(x) \leqslant 0 \quad \textit{for all} \quad x \in [\tfrac{a+b}{2}, b]$$

and

$$\int_{a+x}^{b-x} H(t)dt \geqslant 0 \quad \textit{for} \quad x \in [0, \tfrac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \geqslant 0.$$

LEMMA 5. *Let S be a nonnegative left balanced function on $[a,b]$ and let S be a nonincreasing on $[\frac{a+b}{2}, b]$. If*

$$H(x) \leqslant 0 \quad \textit{for all} \quad x \in [a, \tfrac{a+b}{2}]$$

and

$$\int_{a+x}^{b-x} H(t)dt \leqslant 0 \quad \textit{for} \quad x \in [0, \tfrac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \leqslant 0.$$

LEMMA 6. *Let S be a nonnegative right balanced function on $[a,b]$ and let S be nonincreasing on $[\frac{a+b}{2}, b]$. If*

$$H(x) \leqslant 0 \quad \textit{for all} \quad x \in [a, \tfrac{a+b}{2}]$$

and

$$\int_{a+x}^{b-x} H(t)dt \geqslant 0 \quad \textit{for} \quad x \in [0, \tfrac{b-a}{2}],$$

then

$$\int_a^b H(t)S(t)dt \geqslant 0.$$

2. Main results

THEOREM 2. *Let $w_i, i = 1,2,3$ be nonnegative and integrable functions on $[a,b]$ and let W_i be defined by*

$$W_i(x) = \frac{\int_a^x w_i(t)dt}{\int_a^b w_i(t)dt} \quad i = 1,2,3.$$

Let p_1, p_2 be positive real numbers such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

a) If g is a nonnegative nonincreasing function on $[a,b]$ and

$$W_1^{1/p_1}(x)W_2^{1/p_2}(x) \geqslant W_3(x) \quad \text{for all} \quad x \in [a,b], \tag{4}$$

then

$$\frac{\int_a^b w_3(x)g(x)dx}{\int_a^b w_3(x)dx} \leqslant \prod_{i=1}^{2} \left(\frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx} \right)^{1/p_i}. \tag{5}$$

b) If g is a nonnegative nondecreasing function on $[a,b]$ and (R4) *holds then inequality* (5) *is reversed.*

Proof. a) Using integration by parts and Hölder's inequality [5], we obtain

$$\begin{aligned}
\prod_{i=1}^{2} \left(\frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx} \right)^{1/p_i} &= \prod_{i=1}^{2} \left(g(b) + \int_a^b W_i(x)d\overline{g}(x) \right)^{1/p_i} \\
&\geqslant g(b) + \prod_{i=1}^{2} \left(\int_a^b W_i d\overline{g}(x) \right)^{1/p_i} \\
&\geqslant g(b) + \int_a^b W_1^{1/p_1}(x)W_2^{1/p_2}(x)d\overline{g}(x) \\
&\geqslant g(b) + \int_a^b W_3(x)d\overline{g}(x) \\
&= \int_a^b W_3'(x)g(x)dx = \frac{\int_a^b w_3(x)g(x)dx}{\int_a^b w_3(x)dx},
\end{aligned}$$

where $\overline{g} = -g$.

b) The proof is similar, only instead of discrete Hölder's inequality we use Popoviciu's inequality, [5, p118]. □

Let us note that assumption (4) are used in the inequality

$$g(b) + \int_a^b W_1^{1/p_1}(x)W_2^{1/p_2}(x)d\overline{g}(x) \geqslant g(b) + \int_a^b W_3(x)d\overline{g}(x).$$

Assumption (4) could be substituted by weaker assumptions by using Lemma 1 – Lemma 6. Namely, the following theorem is valid.

THEOREM 3. *Let g be a nonnegative function on $[a,b]$. In cases a)–g) we will suppose that g is a differentiable function.*

a) If $g'(x) \leqslant 0$ for all $x \in [a,b]$, g is convex on $[a,b]$ and

$$\int_a^x W_3(t)dt \leqslant \int_a^x \prod_{i=1}^2 W_i(t)^{1/p_i} dt \text{ for all } x \in [a,b] \tag{6}$$

then (5) holds.

If $g'(x) \geqslant 0$ for all $x \in [a,b]$ and g is concave on $[a,b]$ and if (6) holds then (R5) is valid.

b) If $g'(x) \leqslant 0$ for all $x \in [a,b]$, g is concave on $[a,b]$ and

$$\int_x^b W_3(t)dt \leqslant \int_x^b \prod_{i=1}^2 W_i(t)^{1/p_i} dt \text{ for all } x \in [a,b] \tag{7}$$

then (5) holds.

If $g'(x) \geqslant 0$ for $x \in [a,b]$, g is convex on $[a,b]$ and (7) is valid then (R5) holds.

c) If g' is a nonpositive symmetrical function on $[a,b]$ and is nondecreasing on $[\frac{a+b}{2}, b]$ and if

$$\int_{a+x}^{b-x} W_3(t)dt \leqslant \int_{a+x}^{b-x} \prod_{i=1}^2 W_i(t)^{1/p_i} dt \text{ for all } x \in [0, \tfrac{b-a}{2}] \tag{8}$$

then (5) holds.

If g' is a nonnegative symmetrical function on $[a,b]$ and is nonincreasing on $[\frac{a+b}{2}, b]$ and if (8) holds then (R5) holds.

d) If g is concave on $[a, \frac{a+b}{2}]$, g' is nonpositive left balanced on $[a,b]$ and if

$$W_3(x) \leqslant \prod_{i=1}^2 W_i(x)^{1/p_i} \text{ for all } x \in [\tfrac{a+b}{2}, b] \tag{9}$$

and if (8) holds, then (5) holds.

If g' is a nonnegative right balanced function on $[a,b]$, g is convex on $[a, \frac{a+b}{2}]$ and if (9) and (8) hold, then (R5) holds.

e) If g' is a nonpositive right balanced function on $[a,b]$, g is concave on $[a, \frac{a+b}{2}]$, (R9) and (8) hold, then (5) holds.

If g' is a nonnegative left balanced function on $[a,b]$, g is convex on $[a, \frac{a+b}{2}]$, (R9) and (8) hold, then (R5) holds.

f) If g is convex on $[\frac{a+b}{2}, b]$, g' is nonpositive right balanced on $[a,b]$ and if

$$W_3(x) \leqslant \prod_{i=1}^2 W_i(x)^{1/p_i} \text{ for all } x \in [a, \tfrac{a+b}{2}] \tag{10}$$

and if (8) *holds, then* (5) *holds.*

If g' *is a nonnegative left balanced function on* $[a,b]$, g *is concave on* $[\frac{a+b}{2},b]$ *and if* (10) *and* (8) *hold, then* (R5) *holds.*

g) If g' *is a nonpositive left balanced function on* $[a,b]$, g *is convex on* $[\frac{a+b}{2},b]$, (R10) *and* (8) *are valid, then* (5) *holds.*

If g' *is a nonnegative right balanced function on* $[a,b]$, g *is concave on* $[\frac{a+b}{2},b]$, (R10) *and* (8) *are valid, then* (R5) *holds.*

h) If g *is a nonnegative nonincreasing function on* $[a,b]$.

$$W_3'(x) \leqslant \left(\prod_{i=1}^{2} W_i^{1/p_i}\right)'(x), \ \ for \ \ x \in [a, \tfrac{a+b}{2}],$$

$$W_3(b-x) - W_3(a+x) \leqslant \prod_{i=1}^{2} W_i^{1/p_i}(b-x) - \prod_{i=1}^{2} W_i^{1/p_i}(a+x) \tag{11}$$

for $x \in [0, \frac{b-a}{2}]$, *then* (5) *holds.*

i) If g *is a nonnegative nondecreasing function on* $[a,b]$.

$$W_3'(x) \geqslant \left(\prod_{i=1}^{2} W_i^{1/p_i}\right)'(x), \ \ for \ \ x \in [\tfrac{a+b}{2}, b]$$

and if (R11) *holds then* (R5) *holds.*

Proof. First, we will prove cases a)–g). When $g' \leqslant 0$ then in cases a), b), c), d) and f) putting $S = -g'$ and $H = W_3 - W_1^{1/p_1} W_2^{1/p_2}$ and applying Lemmas 1b), 1a), 2, 3 and 5 respectively we obtain the statements there. In cases e) and g), if $g' \leqslant 0$ we put $S = -g'$ and $H = W_1^{1/p_1} W_2^{1/p_2} - W_3$ and apply Lemmas 4 and 6 respectively.

When $g' \geqslant 0$ we replace $S = g'$ and the same method is used.

Let us prove the case h). Setting $S = g$ and $H = W_3' - (W_1^{1/p_1} W_2^{1/p_2})'$ and applying Lemma 5 we get

$$\int_a^b (W_1^{1/p_1} W_2^{1/p_2})'(x) g(x) dx \geqslant \int_a^b W_3'(x) g(x) dx.$$

Now, using the proof of Theorem 2 we have

$$\begin{aligned}\prod_{i=1}^{2}\left(\frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx}\right)^{1/p_i} &\geqslant g(b)+\int_a^b W_1^{1/p_1}(x)W_2^{1/p_2}(x)d\overline{g}(x)\\ &= \int_a^b (W_1^{1/p_1}W_2^{1/p_2})'(x)g(x)dx\\ &\geqslant \int_a^b W_3'(x)g(x)dx\\ &= \frac{\int_a^b w_3(x)g(x)dx}{\int_a^b w_3(x)dx}.\end{aligned}$$

Case i) is proven similarly, applying Lemma 3 on $S=g$ and $H=(W_1^{1/p_1}W_2^{1/p_2})'-W_3'$. □

REMARK 1. Obviously Theorem 2 can be generalized in this way: Let the functions w_i, $i=1,2,\dots,n+1$, be nonnegative and integrable functions and W_i defined as in Theorem 2.

a) If g is a nonnegative nonincreasing function on $[a,b]$ and

$$\prod_{i=1}^{n} W_i^{1/p_i}(x)\geqslant W_{n+1}(x)\ \text{ for all }\ x\in[a,b], \tag{12}$$

then

$$\frac{\int_a^b w_{n+1}(x)g(x)dx}{\int_a^b w_{n+1}(x)dx}\leqslant\prod_{i=1}^{n}\left(\frac{\int_a^b w_i(x)g(x)dx}{\int_a^b w_i(x)dx}\right)^{1/p_i}. \tag{13}$$

where p_i are positive real numbers such that $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i}=1$.

Also, we get this generalization in all other cases of the Theorems 2 and 3.

REMARK 2. Now, we will show that Theorem 1 is a consequence of Theorem 2a) when in (4) equality holds, (in fact, we will deal with $n+1$ function– see Remark 1).

Let x_i be a function as defined in Theorem 1. Setting: $w_i=x_i'$ for $i=1,2,\dots,n$ and $w_{n+1}=(\prod_{i=1}^{n}x_i^{1/p_i})'$ we get:

$$W_i(x)=\frac{x_i(x)}{x_i(b)},\quad i=1,2,\dots,n,\quad W_{n+1}(x)=\frac{\prod_{i=1}^{n}x_i(x)^{1/p_i}}{\prod_{i=1}^{n}x_i(b)^{1/p_i}}$$

and

$$\prod_{i=1}^{n}W_i^{1/p_i}=W_{n+1}.$$

Then by using Theorem 2 we get inequality (1). Therefore, Theorems 2 and 3 give us a new way of generalization of Gauss–Pólya's inequality.

3. Inequalities of Minkowski's type

The following theorems give us inequalities of Minkowski's type.

THEOREM 4. *Let* w_i, *and* W_i $i = 1,2,3$, *be functions on* $[a,b]$ *defined as in Theorem* 2.

a) Let p *be a real number greater than 1 or less than 0. If* g *is a nonnegative nonincreasing function on* $[a,b]$ *and if*

$$\left(p_1 W_1(x)^{1/p} + p_2 W_2(x)^{1/p}\right)^p \geqslant W_3(x) \text{ for all } x \in [a,b], \tag{14}$$

where p_1, p_2 *are positive real numbers such that* $p_1 + p_2 = 1$, *then*

$$\left(p_1\left(\frac{\int_a^b w_1(t)g(t)dt}{\int_a^b w_1(t)dt}\right)^{1/p} + p_2\left(\frac{\int_a^b w_2(t)g(t)dt}{\int_a^b w_2(t)dt}\right)^{1/p}\right)^p \geqslant \frac{\int_a^b w_3(t)g(t)dt}{\int_a^b w_3(t)dt}. \tag{15}$$

If g *is a nonnegative nondecreasing function and if* (14) *holds then* (R15) *holds.*

b) If $0 < p < 1$, g *is a nonnegative nonincreasing function and if* (R14) *holds then* (R15) *holds.*

If g *is a nonnegative nondecreasing function and if* (R14) *holds then* (15) *holds.*

Proof. Let us suppose that $p > 1$, g is a nonnegative nonincreasing function and (14) holds. Using integration by parts and discrete and integral versions of the Minkowski inequality we have

$$\begin{aligned}
& p_1\left(\frac{\int_a^b w_1(t)g(t)dt}{\int_a^b w_1(t)dt}\right)^{1/p} + p_2\left(\frac{\int_a^b w_2(t)g(t)dt}{\int_a^b w_2(t)dt}\right)^{1/p} \\
= \;& p_1\left(g(b) + \int_a^b W_1(t)d\overline{g}(t)\right)^{1/p} + p_2\left(g(b) + \int_a^b W_2(t)d\overline{g}(t)\right)^{1/p} \\
\geqslant \;& \left(\left(p_1 g(b)^{1/p} + p_2 g(b)^{1/p}\right)^p \right.\\
& \left. + \left(p_1\left(\int_a^b W_1(t)d\overline{g}(t)\right)^{1/p} + p_2\left(\int_a^b W_2(t)d\overline{g}(t)\right)^{1/p}\right)^p\right)^{1/p} \\
\geqslant \;& \left(g(b) + \int_a^b \left(p_1 W_1^{1/p} + p_2 W_2^{1/p}\right)^p (t)d\overline{g}(t)\right)^{1/p} \\
\geqslant \;& \left(g(b) + \int_a^b W_3(t)d\overline{g}(t)\right)^{1/p} \\
= \;& \left(\int_a^b W_3'(t)g(t)dt\right)^{1/p} = \left(\frac{\int_a^b w_3(t)g(t)dt}{\int_a^b w_3(t)dt}\right)^{1/p}
\end{aligned} \tag{16}$$

where in inequality (16) condition (14) is used.

In the case that $p < 0$ the Bellman inequality is used instead of the discrete Minkowski inequality, [5, p.118]. □

As in the previous theorems requirement (14) could be given in the weaker form. Here we will give only the case when $p > 1$ or $p < 0$. The similar results hold for $0 < p < 1$.

THEOREM 5. *Let g be a nonnegative function on $[a,b]$. In cases a)–g) we will suppose that g is differentiable function.*

a) If $g'(x) \leqslant 0$ for all $x \in [a,b]$, g is convex on $[a,b]$ and

$$\int_a^x W_3(t)dt \leqslant \int_a^x \left(p_1 W_1(t)^{1/p} + p_2 W_2(t)^{1/p}\right)^p dt \ \text{for all } x \in [a,b] \tag{17}$$

then (15) *holds.*

If $g'(x) \geqslant 0$ for all $x \in [a,b]$ and g is concave on $[a,b]$ and if (17) *holds then* (R15) *is valid.*

b) If $g'(x) \leqslant 0$ for all $x \in [a,b]$, g is concave on $[a,b]$ and

$$\int_x^b W_3(t)dt \leqslant \int_x^b \left(p_1 W_1(t)^{1/p} + p_2 W_2(t)^{1/p}\right)^p dt \ \text{for all } x \in [a,b] \tag{18}$$

then (15) *holds.*

If $g'(x) \geqslant 0$ for $x \in [a,b]$, g is convex on $[a,b]$ and (18) *is valid then* (R15) *holds.*

c) If g' is a nonpositive symmetrical function on $[a,b]$ and is nondecreasing on $[\frac{a+b}{2}, b]$ and if

$$\int_{a+x}^{b-x} W_3(t)dt \leqslant \int_{a+x}^{b-x} \left(p_1 W_1(t)^{1/p} + p_2 W_2(t)^{1/p}\right)^p dt \ \text{for all } x \in [0, \tfrac{b-a}{2}] \tag{19}$$

then (15) *holds.*

If g' is a nonnegative symmetrical function on $[a,b]$ and is nonincreasing on $[\frac{a+b}{2}, b]$ and if (19) *holds then* (R15) *holds.*

d) If g is concave on $[a, \frac{a+b}{2}]$, g' is nonpositive left balanced on $[a,b]$ and if

$$W_3(x) \leqslant \left(p_1 W_1(x)^{1/p} + p_2 W_2(x)^{1/p}\right)^p \ \text{for all } x \in [\tfrac{a+b}{2}, b] \tag{20}$$

and if (19) *holds, then* (15) *holds.*

If g' is a nonnegative right balanced function on $[a,b]$, g is convex on $[a, \frac{a+b}{2}]$ and if (20) *and* (19) *hold, then* (R15) *holds.*

e) If g' is a nonpositive right balanced function on $[a,b]$, g is concave on $[a, \frac{a+b}{2}]$, (R20) *and* (19) *hold, then* (15) *holds.*

If g' is a nonnegative left balanced function on $[a,b]$, g is convex on $[a, \frac{a+b}{2}]$, (R20) *and* (19) *hold, then* (R15) *holds.*

f) If g is convex on $[\frac{a+b}{2}, b]$, g' is nonpositive right balanced on $[a,b]$ and if

$$W_3(x) \leqslant \left(p_1 W_1(x)^{1/p} + p_2 W_2(x)^{1/p}\right)^p \quad \text{for all } x \in [a, \tfrac{a+b}{2}] \tag{21}$$

and if (19) *holds, then* (15) *holds.*

If g' is a nonnegative left balanced function on $[a,b]$, g is concave on $[\frac{a+b}{2}, b]$ and if (21) *and* (19) *hold, then* (R15) *holds.*

g) If g' is a nonpositive left balanced function on $[a,b]$, g is convex on $[\frac{a+b}{2}, b]$, (R21) *and* (19) *are valid, then* (15) *holds.*

If g' is a nonnegative right balanced function on $[a,b]$, g is concave on $[\frac{a+b}{2}, b]$, (R21) *and* (19) *are valid, then* (R15) *holds.*

h) If g is a nonnegative nonincreasing function on $[a,b]$.

$$W_3'(x) \leqslant \left(\left(p_1 W_1(x)^{1/p} + p_2 W_2(x)^{1/p}\right)^p\right)', \quad \text{for} \quad x \in [a, \tfrac{a+b}{2}],$$

$$W_3(b-x) - W_3(a+x) \leqslant$$
$$\leqslant \left(p_1 W_1^{1/p} + p_2 W_2^{1/p}\right)^p (b-x) - \left(p_1 W_1^{1/p} + p_2 W_2^{1/p}\right)^p (a+x) \tag{22}$$

for $x \in [0, \frac{b-a}{2}]$, then (15) *holds.*

i) If g is a nonnegative nondecreasing function on $[a,b]$.

$$W_3'(x) \geqslant \left(\left(p_1 W_1^{1/p} + p_2 W_2^{1/p}\right)^p\right)'(x), \quad \text{for} \quad x \in [\tfrac{a+b}{2}, b]$$

and if (R22) *holds then* (R15) *holds.*

The proof is similar to the proof of Theorem 3.

REMARK 3. Let us denote that in these two theorems the term $\left(p_1 W_1^{1/p} + p_2 W_2^{1/p}\right)^p$ is, in fact, a weighted means of the order $1/p$ for the pair (W_1, W_2).

REMARK 4. A natural generalization of these theorems are to consider not only a pair of functions (W_1, W_2), but n–tuple $(W_1, W_2, \ldots, W_n)$.

REMARK 5. In [11] the following theorem is given.

THEOREM 6. *Let $g : [a,b] \to \mathbf{R}$ be a nonnegative and nondecreasing function, $x_i : [a,b] \to \mathbf{R}$ $(i = 1, \ldots, n)$ be nonnegative and nondecreasing functions with continuous first derivative. If $p > 1$, then*

$$\left(\int_b^a \left((\sum_{i=1}^n x_i(t))^p\right)' g(t)dt\right)^{1/p} \geqslant \sum_{i=1}^n \left(\int_a^b (x_i^p(t))' g(t)dt\right)^{1/p}. \tag{23}$$

If g is a nonincreasing function and $x_i(a) = 0$ for all $i = 1, \ldots, n$, then the reverse inequality of (23) *is valid.*

When g is nonincreasing, inequality (R23) is a simple consequence of Theorem 4 when equality holds in (14). Namely, putting:

$$p_1 = \frac{x_1(b)}{x_1(b)+x_2(b)}, \quad p_2 = \frac{x_2(b)}{x_1(b)+x_2(b)},$$

$$w_i = (x_i^p)', \quad i = 1,2, \quad w_3(x) = \left(\left(\frac{x_1(x)+x_2(x)}{x_1(b)+x_2(b)}\right)^p\right)',$$

we have (R23) for $n = 2$.

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Shoshana Abramovich
Department of Mathematics
University of Haifa
Haifa 31905, Israel
e-mail: rsma757@uvm.haifa.ac.il

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10 000 Zagreb, Croatia
e-mail: pecaric@hazu.hr

Sanja Varošanec
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
e-mail: varosans@math.hr