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OSCILLATION OF EVEN ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. Oscillation criteria for even order differential equations of the following form

$$z^{(n)}(t)+p(t)\phi(z^{(n-1)}(t))+q(t)|x(\sigma(t))|^{\alpha}\operatorname{sgn}[x(\sigma(t))]=0,$$

where

$$z(t)=x(t)+a(t)x(\tau(t)),\quad \alpha>0,\quad \text{and } n \text{ is even}$$

are obtained via comparison with second order differential inequalities. It is shown that existence of no eventually positive solution of a certain second order delay differential inequality is sufficient for every solution $x(t)$ of the above equation to be oscillatory.

1. Introduction

In this paper, we study the oscillatory behavior of solutions of the even order neutral differential equations with damping

$$z^{(n)}(t)+p(t)\phi(z^{(n-1)}(t))+q(t)|x(\sigma(t))|^{\alpha}\operatorname{sgn}[x(\sigma(t))]=0, \qquad (E_\alpha)$$

where n is even, $\alpha>0$, and $z(t)=x(t)+a(t)x(\tau(t))$. The following conditions will be assumed without further mention:

(a) $p(t)$ and $q(t)$ are continuous and nonnegative on $R_+=(0,\infty)$, and $q(t)$ is not identically zero on any half line of the form $[T,\infty)$, $T\geqslant 0$.

(b) $a(t)$ is continuous on R_+ and $0\leqslant a(t)<1$.

(c) $\phi(u)$ is continuous on R and $0<u\phi(u)\leqslant M|u|^{\gamma+2}$, where $M>0$ and $\gamma\geqslant 0$ are real numbers.

(d) $\tau(t)$ and $\sigma(t)$ are continuous on R_+, $\tau(t)<t$, $\sigma(t)\leqslant t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$

Throughout this paper, we restrict our attention only to solutions of (E_α) which exist on some half-line $[t_0,\infty)$, where $t_0\geqslant 0$ may depend on the particular solution. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (E_α) is called oscillatory if all its solutions are oscillatory.

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In the absence of damping the corresponding equation

$$[x(t)+a(t)x(\tau(t))]^{(n)}+q(t)|x(\sigma(t))|^{\alpha}\operatorname{sgn}[x(\sigma(t))]=0, \tag{E_0}$$

has been extensively studied by many authors [1,7,13-15]. However, to the best of our knowledge there seems to be nothing known regarding the oscillatory behavior of solutions of (E_α). For some results concerning the oscillation of (E_α) in the special case $a(t)\equiv 0$, see [3-6,10] and references cited therein.

Therefore the purpose of this work is to study the effect of the middle term on the solutions of (E_α), and establish some new sufficient conditions which ensure that every solution $x(t)$ of (E_α) is oscillatory when n is even.

Our method is based on the assumption that the second order differential inequality

$$w''(t)+Q(t)[w(\sigma(t))]^{\alpha}\leqslant 0 \quad \alpha>0, \tag{I}$$

has no eventually positive solution. It should be noted that there are several explicit conditions which are sufficient for inequality (I) to have no eventually positive solution. We choose to mention the following results.

Sublinear case ($\alpha<1$): If

$$\int^{\infty}[\sigma(t)]^{\alpha}Q(t)dt=\infty, \tag{C_1}$$

then inequality (I) cannot have an eventually positive solution, see see [2].

Superlinear case ($\alpha>1$): Suppose that $\sigma'(t)\geqslant C$ for some positive constant C, and that

$$\int^{\infty} t\,Q(t)dt=\infty. \tag{C_2}$$

Then, inequality (I) cannot have an eventually positive solution, see [12].

2. Main results

In what follows, $A(t,T)$ denotes

$$A(t,T)=\begin{cases}\displaystyle\int_T^t\left(1+c\int_T^s p(r)dr\right)^{-1/\gamma}ds & \text{if } \gamma>0\\[2ex] \displaystyle\int_T^t \exp\left(-\int_T^s p(r)dr\right)ds & \text{if } \gamma=0.\end{cases}$$

We start with a lemma which we will rely on later.

LEMMA 1. *Let $u(t)$ be a nonoscillatory solution of*

$$u^{(n)}(t)+p(t)\phi(u^{(n-1)}(t))\leqslant 0. \tag{I_ϕ}$$

If for every $c>0$ and $T\geqslant 0$,

$$\lim_{t\to\infty} A(t,T) = \infty, \tag{1}$$

then eventually $u(t)u^{(n-1)}(t) > 0$.

Proof. Without loss of generality we may assume that $u(t)$ is eventually positive solution of (I_ϕ), since otherwise the proof can be accomplished by replacing $u(t)$ by $-u(t)$.

We first claim that $u^{(n-1)}(t)$ is nonoscillatory. For if it is oscillatory, then $u^{(n-1)}(t_1) = 0$ for some $t_1 > t_0$. From (I_ϕ), we see that $u^{(n)}(t_1) < 0$, showing that $u^{(n-1)}(t)$ cannot have another zero after it vanishes at $t = t_1$. Thus $u^{(n-1)}(t)$ is of fixed sign for all large t.

Suppose that there exists a $t_2 \geqslant t_0$ such that $u^{(n-1)}(t) < 0$ for $t \geqslant t_2$. Setting $v(t) = -u^{(n-1)}(t)$ we have

$$v'(t) + Mp(t)v^{\gamma+1}(t) \geqslant 0 \quad \text{for} \quad t \geqslant t_2. \tag{2}$$

It follows from (2) that

$$v(t) \geqslant v(t_2)\left(1 + \gamma v^\gamma(t_2)M\int_{t_2}^{t} p(s)ds\right)^{-1/\gamma} \quad \text{if} \quad \gamma > 0$$

and

$$v(t) \geqslant v(t_2)\exp\left(-\int_{t_2}^{t} p(s)ds\right) \qquad \text{if} \quad \gamma = 0.$$

Therefore,

$$u^{(n-2)}(t) \leqslant u^{(n-2)}(t_2) + u^{(n-1)}(t_2)A(t,t_2). \tag{3}$$

In view of (1) and the fact that $u^{(n-1)}(t_2) < 0$, we may conclude from (3) that $\lim_{t\to\infty} u^{(n-2)}(t) = -\infty$. This, however, is a contradiction with $u(t)$ being eventually positive. The proof is complete.

We are now ready to state and prove the main result of this paper, (cf. [5,11,14]).

THEOREM 1. *Suppose that condition (1) holds. If the second order delay differential inequality*

$$y''(t) + \frac{1}{(n-1)!}(\sigma(t)-T)^\beta(1-a(\sigma(t)))^\alpha q(t)[y(\sigma(t))]^\alpha \leqslant 0 \tag{I_1}$$

where $\beta = n + (\alpha-1)l - (\alpha+1)$, *has no eventually positive solution for every* $T > 0$ *and every* $1 \leqslant l \leqslant n-1$, *then equation* (E_α) *is oscillatory when* n *is even*

Proof. Assume that (E_α) has a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is eventually positive. Since $\tau(t)$, $\sigma(t) \to \infty$ as $t \to \infty$, $x(\sigma(t))$ and $z(t)$ are also eventually positive. It is clear from (E_α) that for t sufficiently large,

$$z^{(n)}(t) + p(t)\phi(z^{(n-1)}(t)) \leqslant 0.$$

Employing Lemma 1, we see that $z^{(n-1)}(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$, and therefore

$$z^{(n)}(t) + q(t)[x(\sigma(t))]^{\alpha} \leqslant 0, \quad t \geqslant t_1. \qquad (I_\alpha)$$

Now since $z(t)z^{(n)}(t)$ is eventually positive, it follows from a lemma of Kiguradze [9] that there are a $T > t_1$ and an integer $l \in \{0, 1, ..., n-1\}$ with $n-l$ odd such that

$$\begin{aligned} z^{(i)}(t) &> 0 \text{ on } [T, \infty) \text{ for } 0 \leqslant i \leqslant l \\ (-1)^{i-l} z^{(i)}(t) &> 0 \text{ on } [T, \infty) \text{ for } l \leqslant i \leqslant n \end{aligned} \qquad (4)$$

By Taylor's formula with remainder, we may write

$$z^{(l)}(t) = \sum_{j=0}^{n-l-1} (-1)^j \frac{z^{(l+j)}(\tau)}{j!}(\tau - t)^j + \frac{1}{(n-l-1)!}\int_t^{\tau} (s-t)^{n-l-1}(-z^{(n)}(s))ds.$$

Using (4), we get

$$z^{(l)}(t) \geqslant \frac{1}{(n-l-1)!}\int_t^{\tau} (s-t)^{n-l-1} q(s)[x(\sigma(s))]^{\alpha} ds, \quad T \leqslant t \leqslant \tau.$$

As $\tau \to \infty$, we have

$$z^{(l)}(t) \geqslant \frac{1}{(n-l-1)!}\int_t^{\infty} (s-t)^{n-l-1} q(s)[x(\sigma(s))]^{\alpha} ds, \quad t \geqslant T.$$

Integrating the above inequality from T to t, it follows that

$$\begin{aligned} z^{(l-1)}(t) &\geqslant z^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_T^t \Big[\int_s^{\infty} (r-s)^{n-l-1} q(r)[x(\sigma(r))]^{\alpha} dr\Big] ds \\ &= z^{(l-1)}(T) + \frac{1}{(n-l-1)!}\int_T^t \Big[\int_T^r (r-s)^{n-l-1} ds\Big] q(r)[x(\sigma(r))]^{\alpha} dr \\ &\quad + \frac{1}{(n-l-1)!}\int_t^{\infty} \Big[\int_T^t (r-s)^{n-l-1} ds\Big] q(r)[x(\sigma(r))]^{\alpha} dr \end{aligned}$$

for $t \geqslant T$. Hence, by virtue of the inequality

$$\int_T^t (r-s)^{n-l-1} ds \geqslant \frac{1}{(n-l)}(t-T)(r-T)^{n-l-1}, \quad T \leqslant t \leqslant r,$$

we obtain

$$\begin{aligned} z^{(l-1)}(t) &\geqslant z^{(l-1)}(T) + \frac{1}{(n-l)!}\int_T^t (r-T)^{n-l} q(r)[x(\sigma(r))]^{\alpha} dr \\ &\quad + \frac{(t-T)}{(n-l)!}\int_t^{\infty} (r-T)^{n-l-1} q(r)[x(\sigma(r))]^{\alpha} dr. \end{aligned} \qquad (5)$$

Let us denote the right-hand side of (5) by $y(t)$. It is easy to verify that $y(t)$ is positive and satisfies

$$y''(t) + \frac{1}{(n-l)!}(t-T)^{n-l-1}q(t)[x(\sigma(t))]^{\alpha} = 0, \quad t \geqslant T. \qquad (E_1)$$

Since n is even, we have $l \geqslant 1$, and so $z(t)$ is increasing. It is clear that

$$z(t) \leqslant x(t) + a(t)z(\tau(t)) \leqslant x(t) + a(t)z(t)$$

or

$$x(t) \geqslant (1-a(t))z(t). \qquad (6)$$

On the other hand, it can be show that (see [8])

$$z(t) \geqslant \frac{1}{l!}(t-T)^{l-1}z^{(l-1)}(t), \quad t \geqslant T. \qquad (7)$$

Now, from (5), (6) and (7) we have

$$\begin{aligned} x(\sigma(t)) &\geqslant (1-a(\sigma(t)))z(\sigma(t)) \geqslant \frac{1}{l!}(1-a(\sigma(t)))(\sigma(t)-T)^{l-1}z^{(l-1)}(\sigma(t)) \\ &\geqslant \frac{1}{l!}(1-a(\sigma(t)))(\sigma(t)-T)^{l-1}y(\sigma(t)). \end{aligned}$$

In view of this last inequality and equation (E_1), we obtain the following inequality

$$y''(t) + \frac{1}{(n-1)!}(\sigma(t)-T)^{\beta}(1-a(\sigma(t)))^{\alpha}q(t)[y(\sigma(t))]^{\alpha} \leqslant 0, \quad t \geqslant T \qquad (I_1)$$

where $\beta = n + (\alpha-1)l - (\alpha+1)$ and $1 \leqslant l \leqslant n-1$.

Thus, we have shown that (I_1) has an eventually positive solution. This, however, contradicts the hypothesis of the theorem.

Several oscillation criteria for (E_α) can now be obtained from known oscillation criteria already exist for (I_1) by means of Theorem 1. To illustrate a possible usage of Theorem 1, we give the following results, which are connected to conditions (C_1) and (C_2) stated earlier in this paper.

COROLLARY 1. *Suppose that condition (1) holds and* $0 < \alpha < 1$. *if*

$$\int^{\infty} (\sigma(t))^{\alpha(n-1)}(1-a(\sigma(t)))^{\alpha}q(t)dt = \infty \qquad (8)$$

then every solution of $(E_{\alpha<1})$ *is oscillatory.*

Proof. Condition (8) is sufficient for (C_1) to hold with $Q(t) = (1/(n-1)!)(\sigma(t)-T)^{\beta}(1-a(\sigma(t)))^{\alpha}q(t)$. Note that if the condition is satisfied for $l=1$, it holds for all l, $1 \leqslant l \leqslant n-1$. So (I_1) connot have eventually positive solution.

COROLLARY 2. *Suppose that condition (1) holds and* $\alpha > 1$. *Suppose also that there exist* $C > 0$ *and* $t_1 \geqslant t_0$ *such that* $\sigma(t)' \geqslant C$ *for all* $t \geqslant t_1$. *If*

$$\int^{\infty} t\,(\sigma(t))^{n-2}(1 - a(\sigma(t)))^{\alpha} q(t)dt = \infty \tag{9}$$

then every solution of $(E_{\alpha>1})$ *is oscillatory.*

Proof. If we take $Q(t) = (1/(n-1)!)(\sigma(t) - T)^{\beta}(1 - a(\sigma(t)))^{\alpha} q(t)$, then (9) gives (C_2), where l can be fixed as $l = n - 1$.

REMARK. It is clear that if $p(t) \equiv 0$ then condition (1) is satisfied. Therefore, the oscillation of (E_α) with no damping is obtained as a special case. It is also easy see that if equation (E_0) is oscillatory and (1) holds then (E_α) is also oscillatory. However, if (1) is violated this conclusion may not be true as illustrated in the following example.

EXAMPLE. Consider

$$z^{(n)}(t) + e^{2t}[z^{(n-1)}(t)]^3 + 6\,[x(t/5)]^5 = 0, \quad z(t) = x(t) + e^{-1}x(t-1), \quad n \text{ even.}$$

It is easy to check that all conditions except condition (1) of Corollary 2 are satisfied, and $x(t) = e^{-t}$ is a nonoscillatory solution of the equation.

It would be interesting to find oscillation criteria for (E_α) in the case when condition (1) fails to hold.

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