Mathematical
Inequalities
& Applications
Volume 1, Number 4 (1998), 491–506

MIXED MEANS AND HARDY'S INEQUALITY

A. ČIŽMEŠIJA AND J. PEČARIĆ

(*communicated by B. Mond*)

Abstract. Integral means of arbitrary order, with power weights, and their companion means are introduced and related mixed-means inequalities are derived. These results are then used in proving inequalities of Hardy and Levin-Cochran-Lee type. Also, new proofs of Hardy and Carleman inequality for finite and infinite series are given by using discrete mixed-means.

1. Introduction

We start with the mixed arithmetic–geometric mean inequality:

THEOREM 1. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The arithmetic mean of the numbers*

$$a_1, \sqrt{a_1a_2}, \sqrt[3]{a_1a_2a_3}, \ldots, \sqrt[n]{a_1a_2\ldots a_n}$$

does not exceed the geometric mean of the numbers

$$a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}, \ldots, \frac{a_1+a_2+a_3+\cdots+a_n}{n} .$$

There is equality if and only if $a_1 = a_2 = \cdots = a_n$.

This result was first given by F. Holland in [7] in the form of a conjecture and then independently proved by K. Kedlaya in [8] and T. Matsuda in [12]. Kedlaya's proof was strictly combinatorial, while Matsuda proposed an inductive proof that uses a little analysis.

In their paper [14] B. Mond and J. Pečarić gave a generalization of Theorem 1, involving two arbitrary power means. Let $a = (a_1, a_2, \ldots, a_n)$ be a positive real n–tuple and $r \in \mathbf{R}$. We denote the non-weighted mean of order r by

$$M_n^{[r]}(a) = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} & , \quad r \neq 0 \\ \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} & , \quad r = 0 . \end{cases}$$

Mathematics subject classification (1991): 26D10, 26D15.

Key words and phrases: Mixed-means, Hardy inequality, Levin-Cochran-Lee inequality, Carleman inequality.

In fact, the mean of order 1 is the arithmetic mean of numbers $a_1, \ldots, a_n$, i.e. $A_n(a) = M_n^{[1]}(a)$, and the mean of order 0 is the geometric mean of these numbers, i.e. $G_n(a) = M_n^{[0]}(a)$.

So, in [14] Mond and Pečarić proved the following

THEOREM 2. *Let $a = (a_1, a_2, \ldots, a_n)$ be a positive real n–tuple and let $r < s$. The mean of order s of*

$$a_1, M_2^{[r]}(a), \ldots, M_n^{[r]}(a)$$

does not exceed the mean of order r of

$$a_1, M_2^{[s]}(a), \ldots, M_n^{[s]}(a) \quad .$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

The remarkable idea of generalizing Theorem 1 as in Theorem 2 was due to T. S. Nanjundiah, [17], but unfortunately he never published his proof, as it was noticed in the paper [3] of P. S. Bullen. In that paper the statement of Theorem 2 was called Nanjundiah's inequality and it was proven for the case $n = 2$ and $0 < r < s$. Moreover, Nanjundiah's proof of Theorem 1 will be published in [4].

Theorem 1 has some further generalizations on matrices and positive operators. In [15] Mond and Pečarić gave an analogous mixed arithmetic–mean, harmonic–mean inequality for Hermitian matrices and the mixed arithmetic–geometric mean inequality for noncommutative positive definite matrices. In [16] they extended these results to positive linear operators.

The aim of this paper is to generalize these results to integrals. We introduce integral power means and give relations between two means of different order, i.e. mixed (r, s)–means inequality. Also, we give companion results of these inequalities and demonstrate how powerful mixed means are by showing how the well-known Hardy and Cochran–Lee inequalities in their discrete and integral versions can be obtained as limit cases of given results.

The analysis used in proofs is based on Minkowski's integral inequality (see [6], Theorem 202), properties of the means and on some basic theorems of real analysis.

In what follows, without further mention, we assume that all integrals exist on the respective domains of their definitions.

2. Mixed (r, s)–means inequalities

All results are based on

LEMMA 1. *Let $a, b \in \mathbf{R}$, $a < b$, and let f be non-negative function integrable* *on* $[a,b]$. *Suppose* $r,s\in\mathbf{R}$, $r<s$, $r,s\neq 0$, *and* $\alpha,\gamma\in\mathbf{R}$. *Then*

$$\left\{\frac{1}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}\left[\frac{1}{(x-a)^{\gamma}}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}}dx\right\}^{\frac{1}{r}}$$
$$\geqslant\left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}\left[\frac{1}{(x-a)^{\alpha}}\int_a^x (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}}dx\right\}^{\frac{1}{s}}. \qquad (1)$$

Proof. Using the change $t=a+u(x-a)$ of the independent variable in the inner integral, the left side of (1) becomes

$$\left\{\frac{1}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}\left[\int_0^1 u^{\gamma-1}f^s(a+u(x-a))du\right]^{\frac{r}{s}}dx\right\}^{\frac{1}{r}}$$
$$\geqslant\left\{\int_0^1 u^{\gamma-1}\left[\frac{1}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}f^r(a+u(x-a))dx\right]^{\frac{s}{r}}du\right\}^{\frac{1}{s}}. \qquad (2)$$

The second row in (2) is obtained by an application of the integral version of Minkowski's inequality. Substituting back $a+u(x-a)=t$ and considering the notation $\tilde{u}=a+u(b-a)$, the right side of (2) is equal to

$$\left\{\int_0^1 u^{\gamma-1}\left[\frac{1}{u^{\alpha}(b-a)^{\alpha}}\int_a^{\tilde{u}} (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}}du\right\}^{\frac{1}{s}}$$
$$=\left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}\left[\frac{1}{(x-a)^{\alpha}}\int_a^x (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}}dx\right\}^{\frac{1}{s}}.$$

The last equality is due to the substitution $x=a+u(b-a)$, that completes the proof. □

Using appropriate changes of independent variables, we can easily obtain the companion result of Lemma 1:

LEMMA 2. *Let* $b\in\mathbf{R}$, $b>0$, *and let* f *be non-negative function integrable on* $[b,\infty\rangle$. *If* $r,s\in\mathbf{R}$ *are such that* $r<s$, $r,s\neq 0$, *and* $\alpha,\gamma\in\mathbf{R}$, *then inequality*

$$\left\{\frac{1}{b^{\alpha}}\int_b^{\infty} x^{\alpha-1}\left[\frac{1}{x^{\gamma}}\int_x^{\infty} t^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}}dx\right\}^{\frac{1}{r}}$$
$$\geqslant\left\{\frac{1}{b^{\gamma}}\int_b^{\infty} x^{\gamma-1}\left[\frac{1}{x^{\alpha}}\int_x^{\infty} t^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}}dx\right\}^{\frac{1}{s}} \qquad (3)$$

holds.

Proof. Define function g by $g(x) = f\left(\frac{1}{x}\right)$, for $x \in \langle 0, \frac{1}{b}]$. Then, for function g, using scalars $0, \frac{1}{b}, -\alpha, -\gamma$ instead of a, b, α, γ respectively, inequality (1) can be written in the form

$$\left\{ \frac{1}{b^\alpha} \int_0^{\frac{1}{b}} \left(\frac{1}{u}\right)^{\alpha+1} \left[u^\gamma \int_0^u \left(\frac{1}{v}\right)^{\gamma+1} g^s(v)dv \right]^{\frac{r}{s}} du \right\}^{\frac{1}{r}}$$

$$\geqslant \left\{ \frac{1}{b^\gamma} \int_0^{\frac{1}{b}} \left(\frac{1}{u}\right)^{\gamma+1} \left[u^\alpha \int_0^u \left(\frac{1}{v}\right)^{\alpha+1} g^r(v)dv \right]^{\frac{s}{r}} du \right\}^{\frac{1}{s}}. \tag{4}$$

Let us transform correspondingly the inner integrals at both sides of (4) by the transformation $t = \frac{1}{v}$, hence

$$\left\{ \frac{1}{b^\alpha} \int_0^{\frac{1}{b}} \left(\frac{1}{u}\right)^{\alpha+1} \left[u^\gamma \int_{\frac{1}{u}}^{\infty} t^{\gamma-1} f^s(t)dt \right]^{\frac{r}{s}} du \right\}^{\frac{1}{r}}$$

$$\geqslant \left\{ \frac{1}{b^\gamma} \int_0^{\frac{1}{b}} \left(\frac{1}{u}\right)^{\gamma+1} \left[u^\alpha \int_{\frac{1}{u}}^{\infty} t^{\alpha-1} f^r(t)dt \right]^{\frac{s}{r}} du \right\}^{\frac{1}{s}}. \tag{5}$$

Inequality (3) is obtained by putting $x = \frac{1}{u}$ in first integrals at both sides of (5). □

Now we introduce integral power means. Let $c, d \in \mathbf{R}$, $c < d$, and let f be a non-negative function integrable on $[c, d]$. For $\alpha > 0$ and $r \in \mathbf{R}$, $r \neq 0$, as in [6], we define the mean of order r, $M^{[r]}(f; c, d, \alpha)$, of f by

$$M^{[r]}(f; c, d, \alpha) = \left[\frac{\alpha}{(d-c)^\alpha} \int_c^d (x-c)^{\alpha-1} f^r(x)dx \right]^{\frac{1}{r}}. \tag{6}$$

If f is positive, we can also define the geometric mean of f as

$$\begin{aligned} G(f; c, d, \alpha) &= M^{[0]}(f; c, d, \alpha) \\ &= \exp\left(\frac{\alpha}{(d-c)^\alpha} \int_c^d (x-c)^{\alpha-1} \ln f(x)dx \right). \end{aligned} \tag{7}$$

On the other hand, for $c, \alpha, r \in \mathbf{R}$, $c > 0, \alpha < 0, r \neq 0$, and non-negative function f integrable on $[c, \infty\rangle$ let $M_*^{[r]}(f; c, \alpha)$ denote the companion power mean of order r of f,

$$M_*^{[r]}(f; c, \alpha) = \left[\frac{-\alpha}{c^\alpha} \int_c^\infty x^{\alpha-1} f^r(x)dx \right]^{\frac{1}{r}}. \tag{8}$$

If $f > 0$, let

$$G_*(f; c, \alpha) = M_*^{[0]}(f; c, \alpha) = \exp\left(\frac{-\alpha}{c^\alpha} \int_c^\infty x^{\alpha-1} \ln f(x)dx \right) \tag{9}$$

be the companion geometric mean of f.

Since

$$\int_c^d (x-c)^{\alpha-1}dx = \frac{(d-c)^\alpha}{\alpha}, \quad \text{if } \alpha > 0,$$

and

$$\int_c^\infty x^{\alpha-1}dx = \frac{c^\alpha}{-\alpha}, \quad \text{if } \alpha < 0,$$

one can see that relations (6)–(9) really represent the means. Those means have further properties:

$$M^{[r]}(f;c,d,\alpha) \leqslant M^{[s]}(f;c,d,\alpha), \tag{10}$$

$$M_*^{[r]}(f;c,\alpha) \leqslant M_*^{[s]}(f;c,\alpha), \tag{11}$$

for $r < s$, and

$$\lim_{r\to 0} M^{[r]}(f;c,d,\alpha) = G(f;c,d,\alpha), \tag{12}$$

$$\lim_{r\to 0} M_*^{[r]}(f;c,\alpha) = G(f;c,\alpha). \tag{13}$$

A very important consequence of Lemma 1 is

THEOREM 3. *If $a,b \in \mathbf{R}$, $a < b$, f is a non-negative function integrable on $[a,b]$, and $r,s,\alpha,\gamma \in \mathbf{R}$ such that $r < s$, $r,s \neq 0$, and $\alpha,\gamma > 0$, then*

$$\left\{\frac{\alpha}{(b-a)^\alpha}\int_a^b (x-a)^{\alpha-1}\left[\frac{\gamma}{(x-a)^\gamma}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}}$$

$$\geqslant \left\{\frac{\gamma}{(b-a)^\gamma}\int_a^b (x-a)^{\gamma-1}\left[\frac{\alpha}{(x-a)^\alpha}\int_a^x (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}} dx\right\}^{\frac{1}{s}},$$

i.e. $M^{[r]}(M^{[s]}(f;a,x,\gamma);a,b,\alpha) \geqslant M^{[s]}(M^{[r]}(f;a,x,\alpha);a,b,\gamma)$.

Proof. Directly from Lemma 1, replacing f by $\alpha^{\frac{1}{r}}\gamma^{\frac{1}{s}}f$. □

This is the integral mixed (r,s)–means inequality. The companion result is given by

THEOREM 4. *If the conditions of Lemma 2 are satisfied with $\alpha,\gamma < 0$, then*

$$\left\{\frac{-\alpha}{b^\alpha}\int_b^\infty x^{\alpha-1}\left[\frac{-\gamma}{x^\gamma}\int_x^\infty t^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}}$$

$$\geqslant \left\{\frac{-\gamma}{b^\gamma}\int_b^\infty x^{\gamma-1}\left[\frac{-\alpha}{x^\alpha}\int_x^\infty t^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}} dx\right\}^{\frac{1}{s}},$$

i.e. $M_*^{[r]}(M_*^{[s]}(f;x,\gamma);b,\alpha) \geqslant M_*^{[s]}(M_*^{[r]}(f;x,\alpha);b,\gamma)$.

Proof. Applying Lemma 2 to the function $(-\alpha)^{\frac{1}{r}}(-\gamma)^{\frac{1}{s}}f$ instead of f. □

The next two lemmas describe the case $r = 0$, i.e. the case involving the geometrical mean.

LEMMA 3. *Suppose $a, b \in \mathbf{R}$, $a < b$. If f is a positive function integrable on $[a, b]$, $\alpha, \gamma \in \mathbf{R}$ such that $\alpha > 0$, and $s \in \mathbf{R}$, $s > 0$, then*

$$\left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}\left[\exp\left(\frac{\alpha}{(x-a)^{\alpha}}\int_a^x (t-a)^{\alpha-1}\ln f(t)dt\right)\right]^s dx\right\}^{\frac{1}{s}}$$
$$\leqslant \exp\left\{\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}\ln\left[\frac{1}{(x-a)^{\gamma}}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{1}{s}} dx\right\}. \tag{14}$$

Proof. Writing the result of Lemma 1 for the function $\alpha^{\frac{1}{r}}f$, instead of (1) we have

$$\left\{\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (x-a)^{\alpha-1}\left[\frac{1}{(x-a)^{\gamma}}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}}$$
$$\geqslant \left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}\left[\frac{\alpha}{(x-a)^{\alpha}}\int_a^x (t-a)^{\alpha-1}f^r(t)dt\right]^{\frac{s}{r}} dx\right\}^{\frac{1}{s}}. \tag{15}$$

For $x \in [a, b]$ define

$$h(x) = \left[\frac{1}{(x-a)^{\gamma}}\int_a^x (t-a)^{\gamma-1}f^s(t)dt\right]^{\frac{1}{s}}.$$

The left side of (15) is then equal to $M^{[r]}(h; a, b, \alpha)$ and by (12) we obtain that $\lim_{r\to 0} M^{[r]}(h; a, b, \alpha) = G(h; a, b, \alpha)$, i.e. the right side of (14). The right side of (15) can be written as

$$R(r) = \left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}\left[M^{[r]}(f; a, x, \alpha)\right]^s dx\right\}^{\frac{1}{s}}. \tag{16}$$

Since $M^{[r]}(f; a, x, \alpha) \geqslant 0$, combining (10) and (12), the monotone convergence of $M^{[r]}(f; a, x, \alpha)$ to $G(f; a, x, \alpha)$ as r decreases to 0 is obvious. Using Lebesgue's monotone convergence theorem, equality

$$\lim_{r\searrow 0} R(r) = \left\{\frac{1}{(b-a)^{\gamma}}\int_a^b (x-a)^{\gamma-1}[G(f; a, x, \alpha)]^s dx\right\}^{\frac{1}{s}}$$

holds, i.e. the left side of (14). Consequently, (14) follows from (15) by taking $\lim_{r\searrow 0}$. □

LEMMA 4. *Let $b \in \mathbf{R}$, $b > 0$, and f be a positive function integrable on $[b,\infty\rangle$. If $s, \alpha, \gamma \in \mathbf{R}$ are such that $s > 0$, $\alpha < 0$, then*

$$\left\{\frac{1}{b^{\gamma}}\int_b^{\infty} x^{\gamma-1}\left[\exp\left(\frac{-\alpha}{x^{\alpha}}\int_x^{\infty} t^{\alpha-1}\ln f(t)dt\right)\right]^s dx\right\}^{\frac{1}{s}} \leqslant \exp\left\{\frac{-\alpha}{b^{\alpha}}\int_b^{\infty} x^{\alpha-1}\ln\left[\frac{1}{x^{\gamma}}\int_x^{\infty} t^{\gamma-1}f^s(t)dt\right]^{\frac{1}{s}} dx\right\}. \tag{17}$$

Proof. From Lemma 2, applied to the function $(-\alpha)^{\frac{1}{r}}f$, by using (11), (13) and the monotone convergence theorem. □

Lemma 1 and Lemma 2 also imply

THEOREM 5. *If conditions of Lemma 3 hold true with $\gamma > 0$, then*
(i) $M^{[s]}(G(f;a,x,\alpha);a,b,\gamma) \leqslant G(M^{[s]}(f;a,x,\gamma);a,b,\alpha)$, *for* $s > 0$,
(ii) $M^{[s]}(G(f;a,x,\alpha);a,b,\gamma) \geqslant G(M^{[s]}(f;a,x,\gamma);a,b,\alpha)$, *for* $s < 0$.

Proof. From Lemma 1, by the same tools as in the proof of Lemma 3. □

THEOREM 6. *Under conditions of Lemma 4 with $\gamma < 0$, inequalities*
(i) $M_*^{[s]}(G(f;x,\alpha);b,\gamma) \leqslant G(M_*^{[s]}(f;x,\gamma);b,\alpha)$, *for* $s > 0$,
(ii) $M_*^{[s]}(G(f;x,\alpha);b,\gamma) \geqslant G(M_*^{[s]}(f;x,\gamma);b,\alpha)$, *for* $s < 0$,
hold.

Proof. Directly from Lemma 2, as in the proofs of Lemma 3 and Lemma 4. □

3. Hardy type inequalities

The results given in the previous section can be used in proving some well-known inequalities. Here we give new proofs of these inequalities and improve one result related to finite sums.

The discrete version of the famous Hardy inequality (see [6], Theorem 327 and [13], Chapter IV, Theorem 1) is contained in

THEOREM 7. *If $p > 1$, $a_n \geqslant 0$, and $A_n = a_1 + a_2 + \cdots + a_n$, then*

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{18}$$

unless all the a_n are zero. The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

Proof. By Theorem 2, for $r=1$, $s=p$ and $k\in\mathbf{N}$

$$\left[\frac{1}{k}\sum_{n=1}^{k}\left(\frac{A_n}{n}\right)^p\right]^{\frac{1}{p}} \leqslant \frac{1}{k}\sum_{n=1}^{k}\left(\frac{1}{n}\sum_{l=1}^{n}a_l^p\right)^{\frac{1}{p}}$$

and then

$$\sum_{n=1}^{k}\left(\frac{A_n}{n}\right)^p \leqslant k^{1-p}\left[\sum_{n=1}^{k}\left(\frac{1}{n}\sum_{l=1}^{n}a_l^p\right)^{\frac{1}{p}}\right]^p. \tag{19}$$

Since $\sum_{l=1}^{n}a_l^p \leqslant \sum_{l=1}^{k}a_l^p = S_k$, for all $n\in\mathbf{N}$, $n\leqslant k$, the right side of (19) is less than or equal to

$$k^{1-p}S_k\left[\sum_{n=1}^{k}\left(\frac{1}{n}\right)^{\frac{1}{p}}\right]^p \leqslant k^{1-p}S_k\left(\frac{k^{1-\frac{1}{p}}}{1-\frac{1}{p}}\right)^p = \left(\frac{p}{p-1}\right)^p S_k. \tag{20}$$

The inequality in (20) is a consequence of the integrability of the function $f(x)=x^{-\frac{1}{p}}$ in $\langle 0,k\rangle$, since the sum $\sum_{n=1}^{k}n^{-\frac{1}{p}}$ is the lower Darboux sum of f. So,

$$\sum_{n=1}^{k}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{k}a_n^p \tag{21}$$

and inequality (18) holds by taking limit as $k\to\infty$. □

Inequality (21) describes what happens if both series in (18) are restricted to a finite number of terms. The constant that appeared in this situation is the same as in (18). But, if we look at the proof of Theorem 7 carefully, it is obvious that relations (19) and (20) give us an inequality with a smaller constant, since

$$k^{1-p}\left(\sum_{n=1}^{k}n^{-\frac{1}{p}}\right)^p < \left(\frac{p}{p-1}\right)^p.$$

So, the best possible constant in the infinite case is not the best possible for the finite series. We just proved

THEOREM 8. *If $p>1$, $a_n\geqslant 0$, and $A_n=a_1+a_2+\cdots+a_n$, then*

$$\sum_{n=1}^{k}\left(\frac{A_n}{n}\right)^p < k^{1-p}\left(\sum_{n=1}^{k}n^{-\frac{1}{p}}\right)^p\sum_{n=1}^{k}a_n^p,$$

for all $k\in\mathbf{N}$, unless all the a_n are zero.

REMARK 1. Our result gives a smaller explicit bound for the best possible constant λ_k for the relation

$$\sum_{n=1}^{k}\left(\frac{A_n}{n}\right)^p < \lambda_k \sum_{n=1}^{k} a_n^p$$

and our constant depends on the number of terms in the sums. The best possible constant for the finite section of the discrete Hardy inequality was investigated by H. S. Wilf in [20], but only for the case $p = 2$. He established only the asymptotic behavior of λ_k as $k \to \infty$ and showed that

$$\lambda_k = 4 - \frac{16\pi^2}{(\ln k)^2} + O\left(\frac{\ln\ln k}{(\ln k)^3}\right).$$

For further details, see [13], Chapter IV.

The corresponding theorem of Theorem 7 for integrals is (see [6], Theorem 326 or [13], Chapter IV, Theorem 2)

THEOREM 9. *If* $p > 1$, $f(x) \geqslant 0$, *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p dx,$$

unless $f \equiv 0$. *The constant is the best possible.*

In this paper we shall prove Hardy's generalization of Theorem 9, stated in [6], Theorem 330, and [13], Chapter IV, p. 145:

THEOREM 10. *If* f *is a non-negative function integrable on* $[0,\infty\rangle$ *and real numbers* p, r, α *are such that* $p, r > 1$, *then*

$$\int_0^\infty x^{-r}\left[\int_0^x t^{\alpha-1} f(t)dt\right]^p dx \leqslant \left(\frac{p}{r-1}\right)^p \int_0^\infty t^{-r}\left[t^\alpha f(t)\right]^p dt. \tag{22}$$

Proof. Relation (22) is obtained as the limit of (1) as $b \to \infty$. Let $0 < r < s$, $a = 0$ and $\gamma = \alpha p - r + 1$. In this case we have $\frac{s}{r} > 1$ and $\alpha p - \gamma > 0$, so (1) can be written as

$$\begin{aligned}
&\left\{\int_0^b x^{\gamma-1}\left[\frac{1}{x^\alpha}\int_0^x t^{\alpha-1} f^r(t)dt\right]^{\frac{s}{r}} dx\right\}^{\frac{1}{s}} \\
&\leqslant b^{\frac{\gamma}{s}-\frac{\alpha}{r}}\left\{\int_0^b x^{\alpha-1}\left[\frac{1}{x^\gamma}\int_0^x t^{\gamma-1} f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}} \\
&= \left\{b^{\gamma\frac{r}{s}-\alpha}\int_0^b x^{\alpha-1-\gamma\frac{r}{s}}\left[\int_0^x t^{\gamma-1} f^s(t)dt\right]^{\frac{r}{s}} dx\right\}^{\frac{1}{r}}.
\end{aligned} \tag{23}$$

Denote

$$I_b = \int_0^b t^{\gamma-1} f^s(t)dt.$$

Since $\int_0^x t^{\gamma-1} f^s(t)dt \leqslant I_b$, $0 \leqslant x \leqslant b$, the last row of (23) is less than or equal to

$$\left\{ b^{\gamma\frac{r}{s}-\alpha} \cdot I_b^{\frac{r}{s}} \cdot \int_0^b x^{\alpha-1-\gamma\frac{r}{s}} dx \right\}^{\frac{1}{r}} = \left\{ b^{\gamma\frac{r}{s}-\alpha} \cdot \frac{b^{\alpha-\gamma\frac{r}{s}}}{\alpha-\gamma\frac{r}{s}} \cdot I_b^{\frac{r}{s}} \right\}^{\frac{1}{r}}$$

$$= \left(\frac{I_b^{\frac{r}{s}}}{\alpha-\gamma\frac{r}{s}} \right)^{\frac{1}{r}} = \left(\frac{\frac{s}{r}}{\alpha\frac{s}{r}-\gamma} \right)^{\frac{1}{r}} I_b^{\frac{1}{s}}.$$

By raising to the s-th power we have

$$\int_0^b x^{\gamma-1} \left[\frac{1}{x^\alpha} \int_0^x t^{\alpha-1} f^r(t)dt \right]^{\frac{s}{r}} dx \leqslant \left(\frac{\frac{s}{r}}{\alpha\frac{s}{r}-\gamma} \right)^{\frac{s}{r}} I_b. \tag{24}$$

Putting $p = \frac{s}{r}$ and f^r instead of f in (24) one obtains

$$\int_0^b x^{\gamma-\alpha p-1} \left[\int_0^x t^{\alpha-1} f(t)dt \right]^p dx \leqslant \left(\frac{p}{\alpha p-\gamma} \right)^p \int_0^b t^{\gamma-1} f^p(t)dt,$$

or, in the original notation,

$$\int_0^b x^{-r} \left[\int_0^x t^{\alpha-1} f(t)dt \right]^p dx \leqslant \left(\frac{p}{r-1} \right)^p \int_0^b t^{-r} \left[t^\alpha f(t) \right]^p dt.$$

Inequality (22) holds by taking $\lim_{b\to\infty}$. □

The companion inequality of (22) (see [6], Theorem 330, and [13], Chapter IV, p. 145) is given by

THEOREM 11. *Let f be non-negative function integrable on $[0,\infty\rangle$ and real numbers p, r, α such that $p > 1$ and $r < 1$. Then*

$$\int_0^\infty x^{-r} \left[\int_x^\infty t^{\alpha-1} f(t)dt \right]^p dx \leqslant \left(\frac{p}{1-r} \right)^p \int_0^\infty t^{-r} \left[t^\alpha f(t) \right]^p dt. \tag{25}$$

Proof. Recall Lemma 2 for $r = 1$, $s = p > 1$ and parameters α and $\gamma = \alpha p - r + 1$. By raising to the p–th power inequality (3) becomes

$$\frac{1}{b^\gamma} \int_b^\infty x^{\gamma-1} \left[\frac{1}{x^\alpha} \int_x^\infty t^{\alpha-1} f(t)dt \right]^p dx$$
$$\leqslant \left\{ \frac{1}{b^\alpha} \int_b^\infty x^{\alpha-1} \left[\frac{1}{x^\gamma} \int_x^\infty t^{\gamma-1} f^p(t)dt \right]^{\frac{1}{p}} dx \right\}^p,$$

or

$$\int_b^\infty x^{\gamma-\alpha p-1}\left[\int_x^\infty t^{\alpha-1}f(t)dt\right]^p dx$$
$$\leqslant \left\{ b^{\frac{\gamma}{p}-\alpha}\int_b^\infty x^{\alpha-\frac{\gamma}{p}-1}\left[\int_x^\infty t^{\gamma-1}f^p(t)dt\right]^{\frac{1}{p}} dx\right\}^p. \tag{26}$$

Let

$$J_b=\int_b^\infty t^{\gamma-1}f^p(t)dt.$$

Since $\int_x^\infty t^{\gamma-1}f^p(t)dt \leqslant J_b$, $x \geqslant b$, and $\alpha p-\gamma<0$, an upper bound for the right side of (26) is

$$\left\{ b^{\frac{\gamma}{p}-\alpha}J_b^{\frac{1}{p}}\int_b^\infty x^{\alpha-\frac{\gamma}{p}-1}dx\right\}^p$$
$$=\left(-\frac{1}{\alpha-\frac{\gamma}{p}}\right)^p J_b=\left(\frac{p}{\gamma-\alpha p}\right)^p\int_b^\infty t^{\gamma-1}f^p(t)dt.$$

Finally,

$$\int_b^\infty x^{\gamma-\alpha p-1}\left[\int_x^\infty t^{\alpha-1}f(t)dt\right]^p dx \leqslant \left(\frac{p}{\gamma-\alpha p}\right)^p\int_b^\infty t^{\gamma-1}f^p(t)dt$$

or, considering the notation from the begining of the proof,

$$\int_b^\infty x^{-r}\left[\int_x^\infty t^{\alpha-1}f(t)dt\right]^p dx \leqslant \left(\frac{p}{1-r}\right)^p\int_b^\infty t^{-r}\left[t^\alpha f(t)\right]^p dt,$$

and we reach (25) by taking $\lim_{b\to 0}$. □

REMARK 2. Theorem 11 can also be proved directly from Theorem 10, using the method described in the proof of Lemma 2.

REMARK 3. The inequality that occurs in [6], Theorem 330, is in fact

$$\int_0^\infty x^{-r}F^p dx < \left(\frac{p}{|r-1|}\right)^p\int_0^\infty t^{-r}(tf)^p dt, \tag{27}$$

where

$$F(x)=\begin{cases}\displaystyle\int_0^x f(t)dt &, \quad r>1\\ \displaystyle\int_x^\infty f(t)dt &, \quad r<1\end{cases}$$

and $p>1$, that differs a little bit from our Theorem 10 and Theorem 11. But, (27) is easily obtained from these results by writing (22) and (25) for the function $t^{\alpha-1}f$ instead of f.

4. Carleman's inequality and inequalities of Levin–Cochran–Lee type

The well–known Carleman inequality (see [6], Theorem 334, or [13], Chapter IV, Theorem 3) is the subject of

THEOREM 12.

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty}a_n, \tag{28}$$

unless (a_n) *is null. The constant* e *is the best possible.*

Inequality (28) can be proved by using Nanjundiah's idea, i.e. by applying the mixed arithmetic–geometric mean inequality, as it is shown in [3] and [4]. Bullen derived (28) by taking the limit as $k\to\infty$ of

$$\sum_{n=1}^{k}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{k}a_n. \tag{29}$$

Using the same method, but more carefully, here we improve (29) by

THEOREM 13. *Let* $a_n \geqslant 0$. *Then*

$$\sum_{n=1}^{k}(a_1a_2\cdots a_n)^{\frac{1}{n}} < \frac{k}{\sqrt[k]{k!}}\sum_{n=1}^{k}a_n, \tag{30}$$

for all $k\in\mathbf{N}$, *unless* (a_n) *is null.*

Proof. By Theorem 1 we have

$$\frac{1}{k}\sum_{n=1}^{k}(a_1a_2\cdots a_n)^{\frac{1}{n}} \leqslant \left[\prod_{i=1}^{k}\left(\frac{1}{i}\sum_{n=1}^{i}a_n\right)\right]^{\frac{1}{k}}. \tag{31}$$

Since $\sum_{n=1}^{i}a_n < \sum_{n=1}^{k}a_n = A_k$, for all $1\leqslant i\leqslant k$, unless all a_n are zero, the right side of (31) is less than $\sqrt[k]{\prod_{i=1}^{k}\frac{A_k}{i}} = \frac{A_k}{\sqrt[k]{k!}}$. Hence,

$$\frac{1}{k}\sum_{n=1}^{k}(a_1a_2\cdots a_n)^{\frac{1}{n}} < \frac{1}{\sqrt[k]{k!}}\sum_{n=1}^{k}a_n$$

and (30) follows by multiplying by k. □

This result is really an improvement, since our constant satisfies

$$\frac{k}{\sqrt[k]{k!}} < e^{1-\frac{1}{k}} < e,$$

that was proved by A. Lupaş in [11]. Moreover, Carleman's inequality follows from (30) by taking $\lim_{k\to\infty}$, since $\lim_{k\to\infty}\frac{k}{\sqrt[k]{k!}} = e$.

REMARK 4. The constant e is the best possible for the infinite series, although there is no convergent series for which equality in (28) holds. If, however, the series in (28) are restricted to a finite number of terms, we obtained an inequality with the smaller constant, $\frac{k}{\sqrt[k]{k!}}$, dependent on the number of terms in the sum in (30), and given explicitly. The best possible constant for

$$\sum_{n=1}^{k}(a_1a_2\cdots a_n)^{\frac{1}{n}} < \mu_k \sum_{n=1}^{k} a_n$$

was given by N. G. De Bruijn in [2] (see also [13], Chapter IV) only by its asymptotic behavior as $k \to \infty$. He established that

$$\mu_k = e - \frac{2\pi^2 e}{(\ln k)^2} + O\left(\frac{1}{(\ln k)^3}\right).$$

J. A. Cochran and C.–S. Lee in [5] proved the following result:

THEOREM 14. *Let α and γ be real numbers with $\alpha > 0$, $f(t)$ a positive function such that $t^{\alpha-1}\ln f(t)$ is locally integrable in $[0,\infty\rangle$. Then*

$$\int_0^\infty x^{\gamma-1}\exp\left\{\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\ln f(t)dt\right\}dx \leqslant e^{\frac{\gamma}{\alpha}}\int_0^\infty x^{\gamma-1}f(x)dx. \tag{32}$$

The constant $e^{\frac{\gamma}{\alpha}}$ is the best possible.

Originally, inequality (32) was discovered by V. Levin in [9] and rediscovered by Cochran and Lee, as it is mentioned by M. Alić and J. Pečarić in [1]. So, (32) will be called the Levin–Cochran–Lee inequality. Here we give a new approach to this inequality. It holds as the limiting case of Lemma 3.

Proof. In (14), let $s = 1$ and $a = 0$. Then

$$\frac{1}{b^\gamma}\int_0^b x^{\gamma-1}\exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\ln f(t)dt\right)dx \leqslant \exp\left\{\frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\ln\left[\frac{1}{x^\gamma}\int_0^x t^{\gamma-1}f(t)dt\right]dx\right\},$$

or

$$\int_0^b x^{\gamma-1}\exp\left(\frac{\alpha}{x^\alpha}\int_0^x t^{\alpha-1}\ln f(t)dt\right)dx \leqslant e^{\gamma\ln b}\exp\left\{\frac{\alpha}{b^\alpha}\int_0^b x^{\alpha-1}\ln\left[\frac{1}{x^\gamma}\int_0^x t^{\gamma-1}f(t)dt\right]dx\right\}. \tag{33}$$

Denote

$$I_b^* = \int_0^b t^{\gamma-1} f(t)dt.$$

It is obvious that $\int_0^x t^{\gamma-1} f(t)dt \leqslant I_b^*$ holds for $x \in [0,b]$, and hence the right side of (33) is not greater than

$$\begin{aligned}
&e^{\gamma \ln b} \exp\left\{ \frac{\alpha}{b^\alpha} \int_0^b x^{\alpha-1} \ln\left(\frac{I_b^*}{x^\gamma}\right) dx \right\} \\
&= \exp\left\{ \gamma \ln b + \frac{\alpha}{b^\alpha} \left[\ln(I_b^*) \int_0^b x^{\alpha-1} dx - \int_0^b x^{\alpha-1} \ln(x^\gamma) dx \right] \right\} \\
&= \exp\left\{ \gamma \ln b + \ln(I_b^*) - \frac{\alpha}{b^\alpha} \gamma \int_0^b x^{\alpha-1} \ln x dx \right\}. \qquad (34)
\end{aligned}$$

Since

$$\int_0^b x^{\alpha-1} \ln x dx = \frac{b^\alpha}{\alpha} \left(\ln b - \frac{1}{\alpha} \right),$$

the last row in (34) is equal to

$$\exp\left\{ \gamma \ln b + \ln(I_b^*) - \gamma \left(\ln b - \frac{1}{\alpha} \right) \right\} = e^{\frac{\gamma}{\alpha}} I_b^*.$$

Hence,

$$\int_0^b x^{\gamma-1} \exp\left(\frac{\alpha}{x^\alpha} \int_0^x t^{\alpha-1} \ln f(t)dt \right) dx \leqslant e^{\frac{\gamma}{\alpha}} \int_0^b t^{\gamma-1} f(t)dt$$

and inequality (32) follows by taking $\lim_{b\to\infty}$. □

The companion result of Cochran–Lee's is due to E. R. Love, given in [10]:

THEOREM 15. *If* α *and* γ *are real constants with* $\alpha < 0$ *and* $f(x)$ *is a measurable and non-negative function on* $\langle 0, \infty \rangle$*, then*

$$\int_0^\infty x^{\gamma-1} \exp\left\{ \frac{-\alpha}{x^\alpha} \int_x^\infty t^{\alpha-1} \ln f(t)dt \right\} dx \leqslant e^{\frac{\gamma}{\alpha}} \int_0^\infty x^{\gamma-1} f(x)dx. \qquad (35)$$

Our proof is direct consequence of Lemma 4.

Proof. Put $s = 1$ in (17). Then

$$\begin{aligned}
&\frac{1}{b^\gamma} \int_b^\infty x^{\gamma-1} \exp\left(\frac{-\alpha}{x^\alpha} \int_x^\infty t^{\alpha-1} \ln f(t)dt \right) dx \\
&\leqslant \exp\left\{ \frac{-\alpha}{b^\alpha} \int_b^\infty x^{\alpha-1} \ln\left[\frac{1}{x^\gamma} \int_x^\infty t^{\gamma-1} f(t)dt \right] dx \right\},
\end{aligned}$$

or equivalently,

$$\int_b^{\infty} x^{\gamma-1} \exp\left(\frac{-\alpha}{x^{\alpha}} \int_x^{\infty} t^{\alpha-1} \ln f(t)dt\right) dx \leqslant e^{\gamma \ln b} \exp\left\{\frac{-\alpha}{b^{\alpha}} \int_b^{\infty} x^{\alpha-1} \ln\left[\frac{1}{x^{\gamma}} \int_x^{\infty} t^{\gamma-1} f(t)dt\right] dx\right\}. \tag{36}$$

Since

$$J_b^* = \int_b^{\infty} t^{\gamma-1} f(t)dt \geqslant \int_x^{\infty} t^{\gamma-1} f(t)dt,$$

for all $x \geqslant b$, the right side of (36) is not greater than

$$\begin{aligned} &e^{\gamma \ln b} \exp\left\{\frac{-\alpha}{b^{\alpha}} \int_b^{\infty} x^{\alpha-1} \ln\left(\frac{J_b^*}{x^{\gamma}}\right) dx\right\} \\ &= \exp\left\{\gamma \ln b + \frac{\alpha}{b^{\alpha}}\left[\int_b^{\infty} x^{\alpha-1} \ln(x^{\gamma})dx - \ln(J_b^*) \int_b^{\infty} x^{\alpha-1} dx\right]\right\} \\ &= \exp\left\{\gamma \ln b + \frac{\alpha}{b^{\alpha}}\gamma \int_b^{\infty} x^{\alpha-1} \ln x dx + \ln(J_b^*)\right\}. \end{aligned} \tag{37}$$

Elementary calculus gives

$$\int_b^{\infty} x^{\alpha-1} \ln x dx = \frac{b^{\alpha}}{\alpha}\left(\frac{1}{\alpha} - \ln b\right).$$

Hence, the last equality in (37) is equal to

$$\exp\left\{\gamma \ln b + \gamma\left(\frac{1}{\alpha} - \ln b\right) + \ln(J_b^*)\right\} = e^{\frac{\gamma}{\alpha}} J_b^*$$

and, finally, we have

$$\int_b^{\infty} x^{\gamma-1} \exp\left\{\frac{-\alpha}{x^{\alpha}} \int_x^{\infty} t^{\alpha-1} \ln f(t)dt\right\} dx \leqslant e^{\frac{\gamma}{\alpha}} \int_b^{\infty} x^{\gamma-1} f(x)dx.$$

So, (35) holds by taking $\lim_{b\to 0}$. □

Note that the most recent proof of Theorem 15 is given by G.–S. Yang and Y.–J. Lin in [19].

REMARK 5. Inequality (35) can also be derived directly from Theorem 14 using the same sequence of substitutions mentioned in the proof of Lemma 2.

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(Received December 18, 1997)

A. Čižmešija
Department of Mathematics
University of Zagreb
Bijenička cesta 30
10 000 Zagreb
Croatia
e-mail: cizmesij@cromath.math.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10 000 Zagreb
Croatia
e-mail: pecaric@hazu.hr