Mathematical
Inequalities
& Applications
Volume 1, Number 4 (1998), 551–557

UPPER AND LOWER SOLUTIONS METHOD FOR HIGHER ORDER DISCRETE BOUNDARY VALUE PROBLEMS

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(*communicated by J. Pečarić*)

Abstract. We shall offer sufficient conditions on the function $f(i, u_1, \cdots, u_{n-1})$, so that the higher order discrete boundary value problem

$$(BVP)\begin{cases} (E)\ \Delta^n u(i-1) + f(i, u(i), \Delta u(i), \cdots, \Delta^{n-2}u(i)) = 0 \text{ for } i \in [1, T+1] \text{ and } n \geqslant 2, \\ (BC)\begin{cases} \Delta^m u(0) = 0, \quad 0 \leqslant m \leqslant n-3, \\ \alpha\Delta^{n-2}u(0) - \beta\Delta^{n-1}u(0) = 0, \\ \gamma\Delta^{n-2}u(T+1) + \delta\Delta^{n-1}u(T+1) = 0, \end{cases} \end{cases}$$

has at least one solution.

1. Introduction

In this paper we shall employ upper and lower solutions method to offer some new existence criteria for the higher order discrete boundary value problem

$$(BVP)\begin{cases} (E)\ \Delta^n u(i-1) + f(i, u(i), \Delta u(i), \cdots, \Delta^{n-2}u(i)) = 0 \text{ for } i \in [1, T+1] \text{ and } n \geqslant 2, \\ (BC)\begin{cases} \Delta^m u(0) = 0, \quad 0 \leqslant m \leqslant n-3, \\ \alpha\Delta^{n-2}u(0) - \beta\Delta^{n-1}u(0) = 0, \\ \gamma\Delta^{n-2}u(T+1) + \delta\Delta^{n-1}u(T+1) = 0, \end{cases} \end{cases}$$

where $T \geqslant 1$ and $n \geqslant 2$ are fixed positive integers, Δ^m denotes the m-th forward difference operator with stepsize 1, and $[a, b] = \{a, a+1, \cdots, b\} \subseteq \mathbf{Z}$ is the set of all integers.

The motivation for the present work stems from many recent investigations in [1–20, and the several references therein] for the above (BVP), and several of its particular cases.

Mathematics subject classification (1991): 34B15, 39A10.

Key words and phrases: Higher order discrete boundary value problem, Green's function, upper and lower solutions.

Paper MIA-01-53

2. Main Results

Let $\alpha, \gamma, \beta, \delta \geqslant 0$, $\rho = \gamma\beta + \alpha\gamma(T+1) + \alpha\delta > 0$, and $\mathbf{B}$ be the Banach space

$$\mathbf{B} \;=\; \{u : [0, T+n] \to \mathbb{R} \mid \Delta^m u(0) = 0, \;\; 0 \leqslant m \leqslant n-3\}$$

with the norm $\|u\| = \max_{i\in[0,T+2]} |\Delta^{n-2}u(i)|$. To abbreviate our discussion, we shall assume that the following conditions are satisfied:
(C_1) $K(i,j)$ defined on $[0, T+n] \times [1, T+1]$ is the Green's function of the difference equation

$$-\Delta^n u(i-1) = 0 \;\text{ for }\; i \in [1, T+1]$$

subject to the boundary conditions (BC).
(C_2) $k(i,j)$ defined on $[0, T+2] \times [1, T+1]$ is the Green's function of the difference equation

$$-\Delta^2 u(i-1) = 0 \;\text{ for }\; i \in [1, T+1]$$

subject to the boundary conditions

$$(BC^*) \quad \begin{cases} \alpha u(0) - \beta\Delta u(0) = 0, \\ \gamma u(T+1) + \delta\Delta u(T+1) = 0. \end{cases}$$

(C_3) $f \in C([1, T+1] \times \mathbb{R}^{n-1}; \mathbb{R})$.
(C_4) $v,\ w : [0, T+n] \to \mathbb{R}$ are lower and upper solutions of (BVP), i.e.,

$$\begin{cases} (1) \quad \Delta^n v(i-1) + f(i, v(i), \Delta v(i), \cdots, \Delta^{n-2}v(i)) \geqslant 0 \;\text{ for }\; i \in [1, T+1], \\ (2) \quad \Delta^n w(i-1) + f(i, w(i), \Delta w(i), \cdots, \Delta^{n-2}w(i)) \leqslant 0 \;\text{ for }\; i \in [1, T+1], \\ (3) \quad \begin{cases} \Delta^m v(0) \leqslant 0, \;\; 0 \leqslant m \leqslant n-3, \\ \alpha\Delta^{n-2}v(0) - \beta\Delta^{n-1}v(0) \leqslant 0, \\ \gamma\Delta^{n-2}v(T+1) + \delta\Delta^{n-1}v(T+1) \leqslant 0, \end{cases} \\ (4) \quad \begin{cases} \Delta^m w(0) \geqslant 0, \;\; 0 \leqslant m \leqslant n-3, \\ \alpha\Delta^{n-2}w(0) - \beta\Delta^{n-1}w(0) \geqslant 0, \\ \gamma\Delta^{n-2}w(T+1) + \delta\Delta^{n-1}w(T+1) \geqslant 0. \end{cases} \end{cases}$$

(C_5) The lower and upper solutions $v(i)$ and $w(i)$, and the function $f(i, u_1, \cdots, u_{n-2}, u_{n-1})$ satisfy the following:
(I) $\Delta^{n-2}v(i) \leqslant \Delta^{n-2}w(i)$ on $[0, T+1]$,
(II) $f(i, v(i), \cdots, \Delta^{n-3}v(i), u_{n-1}) \leqslant f(i, u_1, \cdots, u_{n-2}, u_{n-1})$ $\leqslant f(i, w(i), \cdots, \Delta^{n-3}w(i), u_{n-1})$, for $i \in [1, T+1]$, and $(v(i), \cdots, \Delta^{n-3}v(i))$ $\leqslant (u_1, \cdots, u_{n-2}) \leqslant (w(i), \cdots, \Delta^{n-3}w(i))$, where $(x_1, \cdots, x_{n-2}) \leqslant (y_1, \cdots, y_{n-2})$, if and only if, $x_m \leqslant y_m$ for $m = 1, \cdots, n-2$.

REMARK 2.1. The following hold:
(a) If $f(i, u_1, \cdots, u_{n-2}, u_{n-1})$ is increasing with respect to $(u_1, \cdots, u_{n-2})$ on $\mathbb{R}^{n-2}$ for each fixed $(i, u_{n-1}) \in [1, T+1] \times \mathbb{R}$, then (C_5) (II) holds.

(b) A simple calculation shows that

$$\Delta^{n-2}K(i,j) = k(i,j) = \begin{cases} \frac{1}{\rho}\{\beta+\alpha j\}\{\delta+\gamma(T+1-i)\}, & 1 \leqslant j \leqslant i-1, \\ \frac{1}{\rho}\{\beta+\alpha i\}\{\delta+\gamma(T+1-j)\}, & i \leqslant j \leqslant T+1. \end{cases}$$

(c) There are positive constants M and L such that

$$\begin{cases} (R_1) & \frac{k(i,j)}{k(j,j)} \leqslant L, \quad \text{for all} \quad (i,j) \in [0,T+2]\times[1,T+1], \\ (R_2) & \frac{k(i,j)}{k(j,j)} \geqslant M, \quad \text{for all} \quad (i,j) \in [1,T]\times[1,T+1]. \end{cases}$$

Our main result is embodied in the following:

THEOREM 1. *With respect to the (BVP) assume that conditions $(C_1)-(C_5)$ are satisfied. Then, (BVP) has at least one solution $u \in \mathbf{B}$ such that*

$$\Delta^m v(i) \leqslant \Delta^m u(i) \leqslant \Delta^m w(i) \;\; on \;\; [0,T+n-m-1] \;\; for \;\; m = 0,1,\cdots,n-2.$$

Proof. We separate the proof into the following steps:

Step (1) Consider the modified problem

$$(BVP^*) \quad \begin{cases} (E^*) \;\; \Delta^n u(i-1) + f^*(i,u(i),\Delta u(i),\cdots,\Delta^{n-2}u(i)) = 0 \;\; \text{for} \;\; i \in [1,T+1], \\ (BC) \;\; \begin{cases} \Delta^m u(0) = 0, \quad 0 \leqslant m \leqslant n-3, \\ \alpha\Delta^{n-2}u(0) - \beta\Delta^{n-1}u(0) = 0, \\ \gamma\Delta^{n-2}u(T+1) + \delta\Delta^{n-1}u(T+1) = 0, \end{cases} \end{cases}$$

where

$$f^*(i,u_1,\cdots,u_{n-1}) = f(i,\eta_1,\cdots,\eta_{n-1}) + \rho(\eta_{n-1}-u_{n-1}),$$

and

$$\eta_l = \begin{cases} \Delta^{l-1}w(i) & \text{if} \quad u_l > \Delta^{l-1}w(i), \\ u_l & \text{if} \quad \Delta^{l-1}v(i) \leqslant u_l \leqslant \Delta^{l-1}w(i), \\ \Delta^{l-1}v(i) & \text{if} \quad u_l < \Delta^{l-1}v(i), \end{cases}$$

for all $l = 1,2,\cdots,n-1$, $i \in [1,T+1]$, and $\rho : \mathbf{R} \to [-1,1]$ is the radial retraction defined by

$$\rho(r) = \begin{cases} r & \text{for} \quad |r| \leqslant 1, \\ \frac{r}{|r|} & \text{for} \quad |r| > 1. \end{cases}$$

It is clear that (BVP^*) has a solution $u = u(i)$, if and only if, u is a solution of the operator equation

$$u(i) = \sum_{j=1}^{T+1} K(i,j) f^*(j,u(j),\Delta u(j),\cdots,\Delta^{n-2}u(j)) = (T^*u)(i), \;\; u \in \mathbf{B}$$

for $i \in [0, T+n]$, or equivalently,

$$\Delta^{n-2}u(i) = \sum_{j=1}^{T+1} k(i,j) f^*(j, u(j), \Delta u(j), \cdots, \Delta^{n-2}u(j)) = \Delta^{n-2}(T^*u)(i), \quad u \in \mathbf{B}$$

for $i \in [0, T+2]$.

Since f^* is continuous and bounded on $[1, T+1] \times \mathbb{R}^{n-1}$, $T^* : \mathbf{B} \to \mathbb{R}$ is continuous and compact. Therefore, it follows from Schauder's fixed point theorem that T^* has a fixed point $u \in \mathbf{B}$, i.e., (BVP^*) has a solution $u \in \mathbf{B}$.

Step (2) Let

$$H(i) = \Delta^{n-2}u(i) - \Delta^{n-2}w(i) \ \text{ on } \ [0, T+2],$$

then for $\theta \in [1, T+1]$ such that $H(\theta) > 0$, it follows that

$$\begin{aligned}
\Delta^2 H(\theta-1) \quad &\geqslant \quad -f^*(\theta, u(\theta), \cdots, \Delta^{n-3}u(\theta), \Delta^{n-2}u(\theta)) \\
&\quad\quad +f(\theta, w(\theta), \cdots, \Delta^{n-3}w(\theta), \Delta^{n-2}w(\theta)) \\
&= \quad -f(\theta, \eta_1, \cdots, \eta_{n-2}, \Delta^{n-2}w(\theta)) - \rho(\Delta^{n-2}w(\theta) - \Delta^{n-2}u(\theta)) \\
&\quad\quad +f(\theta, w(\theta), \cdots, \Delta^{n-3}w(\theta), \Delta^{n-2}w(\theta)) \quad (\eta_l \text{ is defined in Step(1)}) \\
&\geqslant \quad -\rho(\Delta^{n-2}w(\theta) - \Delta^{n-2}u(\theta)) > 0, \quad (\text{by using } (C_5)).
\end{aligned}$$

Therefore, there is no $\theta \in [1, T+1]$ such that $H(\theta) > 0$ and $\Delta^2 H(\theta-1) \leqslant 0$.

Step (3) We claim that $H(i) \leqslant 0$ on $[0, T+1]$. Suppose to the contrary that there exists $i_0 \in [0, T+1]$ such that $H(i_0) > 0$. Then, there is a $\theta \in [0, T+1]$ such that

$$H(\theta) = \max_{i \in [0, T+1]} H(i) > 0.$$

If $\theta \in [1, T]$, then we have

$$\Delta^2 H(\theta-1) = [H(\theta+1) - H(\theta)] - [H(\theta) - H(\theta-1)] \leqslant 0,$$

which contradicts the conclusion of Step(2). Hence, $\theta \in \{0, T+1\}$.

Case (1) Suppose that $\theta = T+1$, i.e., $H(T+1) > 0$. We shall first show that $\Delta H(T+1) \leqslant 0$ is impossible. For this, assume that $\Delta H(\theta-1) = \Delta H(T) < 0$, then $H(T+1) < H(T)$. This implies that $H(\theta) = H(T+1)$ cannot be the maximum of $H(i)$. Hence $\Delta H(T) \geqslant 0$. But, then we have

$$\Delta^2 H(\theta-1) = \Delta^2 H(T) = \Delta H(T+1) - \Delta H(T) \leqslant 0,$$

which contradicts the conclusion of Step(2). Thus $\Delta H(T+1) > 0$. Now from (BC) and (4), if $\gamma = 0$ we find $\Delta H(T+1) \leqslant 0$, which we have seen is impossible. Further, if $\gamma > 0$ then in view of $\Delta^{n-1}u(T+1) > \Delta^{n-1}w(T+1)$, we have

$$\Delta^{n-2}w(T+1) \geqslant \frac{-\Delta^{n-1}w(T+1)\delta}{\gamma} \geqslant \frac{-\Delta^{n-1}u(T+1)\delta}{\gamma} = \Delta^{n-2}u(T+1),$$

i.e., $H(T+1) \leqslant 0.$ But, this contradicts $H(T+1) > 0.$ Hence, $\theta \neq T+1.$

Case (2) Suppose that $\theta = 0,$ i.e., $H(0) > 0.$ We shall first show that $\Delta H(0) \geqslant 0$ is impossible. For this, assume that $\Delta H(\theta) = \Delta H(0) > 0,$ then $H(0) < H(1).$ This implies that $H(\theta) = H(0)$ cannot be the maximum of $H(i)$, thus $\Delta H(0) = 0.$ Now, we claim that $\Delta^2 H(\theta) = \Delta^2 H(0) \leqslant 0.$ Assume to the contrary that $\Delta^2 H(0) > 0,$ then $H(2) - H(1) > H(1) - H(0) = \Delta H(0) = 0.$ This leads to $H(2) > H(1) = H(0) = H(\theta).$ Therefore, $H(\theta) = H(0)$ is not the maximum of $H(i),$ which gives a contradiction. But, then from $\Delta^2 H(\theta) = \Delta^2 H(0) = \Delta^2 H(1-1) \leqslant 0,$ and the conclusion of Step(2), we obtain a contradiction. Thus, $\Delta H(0) < 0.$ Now from (BC) and (4), if $\alpha = 0$ we find $\Delta H(0) \geqslant 0,$ which we have seen is impossible. Further, if $\alpha > 0$ then in view of $\Delta^{n-1}u(0) < \Delta^{n-1}w(0),$ we have

$$\Delta^{n-2}w(0) \geqslant \frac{\Delta^{n-1}w(0)\beta}{\alpha} \geqslant \frac{\Delta^{n-1}u(0)\beta}{\alpha} = \Delta^{n-2}u(0),$$

i.e., $H(0) \leqslant 0.$ But, this contradicts $H(0) > 0.$ Hence, $\theta \neq 0.$

Step (4) From Step(3), we have

$$\Delta^{n-2}u(i) \leqslant \Delta^{n-2}w(i) \text{ on } [0, T+1].$$

Similarly, we can show that

$$\Delta^{n-2}v(i) \leqslant \Delta^{n-2}u(i) \text{ on } [0, T+1].$$

Thus, it follows that

$$\Delta^{n-2}v(i) \leqslant \Delta^{n-2}u(i) \leqslant \Delta^{n-2}w(i) \text{ on } [0, T+1].$$

Now the conditions $\Delta^m v(0) \leqslant 0,\ \Delta^m u(0) = 0,$ and $\Delta^m w(0) \geqslant 0,\ 0 \leqslant m \leqslant n-3,$ immediately imply that

$$\Delta^m v(i) \leqslant \Delta^m u(i) \leqslant \Delta^m w(i) \text{ on } [0, T+n-m-1] \text{ for } m = 0, 1, \cdots, n-2.$$

Therefore, from the definition of f^*, we have

$$f^*(i, u(i), \Delta u(i), \cdots, \Delta^{n-2}u(i)) = f(i, u(i), \Delta u(i), \cdots, \Delta^{n-2}u(i)) \text{ on } [1, T+1].$$

This implies that $u(i)$ is infact a solution of (BVP).

In our next result for a given $g(i, u_1, u_2, \cdots, u_{n-1}) \in C([1, T+1] \times [0, \infty)^{n-1}; [0, \infty))$ we shall need the following:

$$\max g_0 = \lim_{u_1, u_2, \cdots, u_{n-1} \to 0^+} \max_{i \in [1, T+1]} \frac{g(i, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\min g_0 = \lim_{u_1, u_2, \cdots, u_{n-1} \to 0^+} \min_{i \in [n-1, T+1]} \frac{g(i, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\max g_\infty = \lim_{u_1, u_2, \cdots, u_{n-1} \to \infty} \max_{i \in [1, T+1]} \frac{g(i, u_1, u_2, \cdots, u_{n-1})}{u_{n-1}},$$

$$\min g_\infty = \lim_{u_1,u_2,\cdots,u_{n-1}\to\infty} \min_{i\in[n-1,T+1]} \frac{g(i,u_1,u_2,\cdots,u_{n-1})}{u_{n-1}}.$$

We shall also need the numbers

$$D_1 = \left(\sum_{j=1}^{T+1} k(j,j)\right)^{-1}, \quad D_2 = \left(\sum_{j=n-1}^{T+1} k(n-1,j)\right)^{-1}, \quad \text{and} \quad M^* = \frac{M}{L}.$$

THEOREM 2. *Suppose that*

(H_1) $f \in C([1,T+1]\times \mathbb{R}^{n-1};\mathbb{R})$ *is such that*

$$f(i,0,0,\cdots,0) \geqslant 0 \ \text{on} \ [1,T+1] \ (f \text{ may be negative for } u_i \neq 0).$$

(H_2) *There exists a function* $g(i,u_1,u_2,\cdots,u_{n-1}) \in C([1,T+1]\times[0,\infty)^{n-1};[0,\infty))$ *which satisfies*

$$g(i,|u_1|,|u_2|,\cdots,|u_{n-1}|) \geqslant f(i,u_1,u_2,\cdots,u_{n-1}) \ \text{on} \ [1,T+1]\times[0,\infty)^{n-1}.$$

(H_3) *One of the following holds:*

(P_1) $\max g_0 = A_1 \in [0,D_1/L)$ *and* $\min g_\infty = A_2 \in (D_2/M^*,\infty]$.

(P_2) $\min g_0 = A_3 \in (D_2/M^*,\infty]$ *and* $\max g_\infty = A_4 \in [0,D_1/L)$.

(P_3) *There exist two functions* $q:[1,T+1]\to[0,\infty)$, *and* $h(u_1,u_2,\cdots,u_{n-1}) \in C([0,\infty)^{n-1};[0,\infty))$, *such that* h *is increasing with respect to* u_{n-1}, *and*

$$\begin{cases} g(i,u_1,u_2,\cdots,u_{n-1}) = q(i)h(u_1,u_2,\cdots,u_{n-1}), \\ \sup\limits_{u_{n-1}\in(0,\infty)} \min\limits_{(u_1,u_2,\cdots,u_{n-2})\in[0,\infty)} \dfrac{u_{n-1}}{Qh(u_1,u_2,\cdots,u_{n-1})} > 1, \end{cases}$$

where $Q = \max_{i\in[1,T+1]} \sum_{j=1}^{T+1} k(i,j)q(j)$.
Then (BVP) *has at least one non–negative solution.*

Proof. Using the same techniques as in Agarwal and Wong [8–10], we can show that

$$(BVP^{**})\begin{cases} (E^{**}) \ \Delta^n w(i-1) + g(i,w(i),\Delta w(i),\cdots,\Delta^{n-2}w(i)) = 0 \ \text{for} \ i\in[1,T+1], \\ (BC^*) \begin{cases} \Delta^m w(0) = 0, \quad 0\leqslant m\leqslant n-3, \\ \alpha\Delta^{n-2}w(0) - \beta\Delta^{n-1}w(0) = 0, \\ \gamma\Delta^{n-2}w(T+1) + \delta\Delta^{n-1}w(T+1) = 0, \end{cases} \end{cases}$$

has at least one solution $w(i)$, $i\in[0,T+n]$ such that $\Delta^m w(i)\geqslant 0$, $i\in[0,T+n-m-1]$ for $m=0,1,\cdots,n-2$. It is clear that this $w(i)$ and $v(i)=0$ are upper and lower solutions of (BVP), respectively. The result now follows from Theorem 1.

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(Received February 16, 1998)

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