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A NOTE ON SOME CLASSES OF FOURIER COEFFICIENTS

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Abstract. It is shown that the class S_p $(p > 1)$ is a subclass of $C \cap BV$, where C is the Garret–Stanojević class and BV the class of sequences of bounded variation. A new direct proof of this theorem is given.

1. Introduction

A sequence $\{a_k\}$ belongs to the class S, if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $|\Delta a_k| \leqslant A_k$ for all k.

Now, we say that a sequence $\{a_k\}$ belongs to the class S_p or $a_k \in S_p$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} A_k < \infty$ and $\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1)$.

Thus in view of the above definitions, it is obvious that $S \subset S_p$.

The class C was defined in [3] as follows: $\{a_n\} \in C$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon,$$

for all n, where

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos kx$$

is the Dirichlet kernel.

2. Main result

THEOREM. $S_p \subseteq C \cap BV$.

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Proof. It suffices to show that

$$C_n = \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx = o(1), \qquad n \to \infty.$$

For each n, let k_n be the least natural number such that $n \leqslant 2^{k_n} - 1$.

Then C_n can be majorized by

$$C_n \leqslant \int_0^{\pi} \left| \sum_{j=n}^{2^{k_n}-1} \Delta a_j D_j(x) \right| dx + \sum_{l=k_n}^{\infty} \int_0^{\pi} \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx = I_1 + I_2.$$

The second term is written as follows:

$$I_2 = \sum_{l=k_n}^{\infty} \left\{ \int_0^{1/2^{l+1}} + \int_{1/2^{l+1}}^{\pi} \right\} \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx = \Sigma_1 + \Sigma_2.$$

For the first term, the uniform estimate $|D_n(x)| \leqslant n + \frac{1}{2}$, is applied, i.e.

$$\begin{aligned} \Sigma_1 &\leqslant \sum_{l=k_n}^{\infty} \frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| \left(j + \frac{1}{2} \right) \\ &= \sum_{i=2^{k_n}}^{\infty} \frac{1}{2i} \sum_{j=i}^{2i-1} |\Delta a_j| \left(j + \frac{1}{2} \right) \leqslant \sum_{i=2^{k_n}}^{\infty} \frac{1}{2i} \sum_{j=i}^{2i-1} |\Delta a_j| 2i \\ &= \sum_{i=2^{k_n}}^{\infty} \sum_{j=i}^{2i-1} |\Delta a_j|. \end{aligned}$$

By summation by parts, we have:

$$\begin{aligned} \sum_{i=2^{k_n}}^{\infty} |\Delta a_i| = \sum_{i=2^{k_n}}^{\infty} \frac{|\Delta a_i|}{A_i} A_i &= \sum_{i=2^{k_n}-1}^{\infty} \Delta A_i \sum_{j=1}^{i} \frac{|\Delta a_j|}{A_j} + A_{2^{k_n}} \sum_{j=1}^{2^{k_n}-1} \frac{|\Delta a_j|}{A_j} \\ &\leqslant \sum_{i=2^{k_n}-1}^{\infty} i(\Delta A_i) \left(\frac{1}{i} \sum_{j=1}^{i} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + 2^{k_n} A_{2^{k_n}} \left(\frac{1}{2^{k_n}} \sum_{j=1}^{2^{k_n}} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \\ &= O(1) \left[\sum_{i=2^{k_n}-1}^{\infty} i(\Delta A_i) + 2^{k_n} A_{2^{k_n}} \right]. \end{aligned}$$

Since $\sum_{n=1}^{\infty} A_n < \infty$, both terms on the right–hand side of the above inequality are $o(1)$, as $n \to \infty$.

Thus $\Sigma_1 = o(1)$, as $n \to \infty$

Let

$$\Sigma_2 = \sum_{l=k_n}^{\infty} \int_{1/2^{l+1}}^{\pi} \left| \sum_{j=2^l}^{2^{l+1}-1} \frac{|\Delta a_j|}{A_j} A_j D_j(x) \right| dx.$$

Applyng Abel's transformation, we get:

$$\int_{1/2^{l+1}}^{\pi} \left| \sum_{j=2^l}^{2^{l+1}-1} \frac{\Delta a_j}{A_j} A_j D_j(x) \right| dx \leqslant \sum_{j=2^l}^{2^{l+1}-2} \Delta A_j \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{j} \frac{\Delta a_r}{A_r} D_r(x) \right| dx + A_{2^l} \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} D_r(x) \right| dx, \quad \text{i.e.}$$

$$\Sigma_2 \leqslant \sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} \Delta A_j \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{j} \frac{\Delta a_r}{A_r} D_r(x) \right| dx + \sum_{l=k_n}^{\infty} A_{2^l} \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} D_r(x) \right| dx.$$

Applyng the Hölder inequality, we get:

$$I_l = \int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} D_r(x) \right| dx = \int_{1/2^{l+1}}^{\pi} \frac{1}{\sin \frac{x}{2}} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right) x \right| dx$$

$$\leqslant \left[\int_{1/2^{l+1}}^{\pi} \frac{dx}{\left(\sin \frac{x}{2}\right)^p} \right]^{1/p} \left[\int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right) x \right|^q dx \right]^{1/q}$$

where $1/p + 1/q = 1$.

Since

$$\int_{1/2^{l+1}}^{\pi} \frac{dx}{\left(\sin \frac{x}{2}\right)^p} \leqslant \pi^p \int_{1/2^{l+1}}^{\pi} \frac{dx}{x^p} \leqslant M_p (2^{l+1})^{p-1},$$

(M_p is absolute constant depending on p), it follows that

$$I_l \leqslant (2^{l+1})^{1/q} (M_p)^{1/p} \left[\int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right) x \right|^q dx \right]^{1/q}.$$

Applying the Hausdorff–Young inequality to the last integral we get:

$$\left(\int_{1/2^{l+1}}^{\pi} \left| \sum_{r=1}^{2^l} \frac{\Delta a_r}{A_r} \sin\left(r + \frac{1}{2}\right) x \right|^q dx \right)^{1/q} \leqslant B_p \left(\sum_{r=1}^{2^l} \frac{|\Delta a_r|^p}{A_r^p} \right)^{1/p}.$$

Thus

$$I_l \leqslant 2^{l+1} C_p \left(\frac{1}{2^{l+1}} \sum_{r=1}^{2^{l+1}} \frac{|\Delta a_r|^p}{A_r^p} \right)^{1/p} \qquad (C_p > 0).$$

Then

$$\Sigma_2 = O(1) \left[\sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} j \Delta A_j + 2 \sum_{l=k_n}^{\infty} 2^l A_{2^l} \right].$$

Applying the Cauchy condensation test,

$$\sum_{l=k_n}^{\infty} 2^l A_{2^l} = o(1), \qquad n \to \infty.$$

But

$$\sum_{j=2^l}^{2^{l+1}-2} j \Delta A_j = \sum_{j=2^l}^{2^{l+1}-1} A_j - (2^{l+1}-1) A_{2^{l+1}-1} \leqslant 2^l A_{2^l} - 2^{l+1} A_{2^{l+1}}.$$

Thus

$$\sum_{l=k_n}^{\infty} \sum_{j=2^l}^{2^{l+1}-2} j \Delta A_j \leqslant \sum_{l=k_n}^{\infty} [2^l A_{2^l} - 2^{l+1} A_{2^{l+1}}] = 2^{k_n} A_{2^{k_n}} = o(1), \qquad n \to \infty,$$

i.e. $\Sigma_2 = o(1)$, $n \to \infty$. Finally $I_2 = o(1)$, $\quad n \to \infty$.

The same method applied to I_1 yields the estimate

$$I_1 \leqslant O(1) \sum_{l=n}^{2^{k_n}-1} |\Delta a_l| + O(1) \left(\sum_{j=n}^{2^{k_n}-1} j \Delta A_j + 2n A_n \right).$$

Letting $n \to \infty$, the proof of the theorem follows.

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