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EXTENSIONS OF THE GEOMETRIC–ARITHMETIC MEANS INEQUALITY TO A DISC OF THE COMPLEX PLANE

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Abstract. For complex numbers z_j with $|z_j - 1| \leqslant r$, $r < 1$, we consider the arithmetic mean $A_n := (1/n)\sum_{j=1}^{n} z_j$ and the geometric mean $G_n := \prod_{j=1}^{n} z_j^{1/n}$ and prove, amongst other results, that

$$\frac{1-r^2}{|2-A_n|} \leqslant |G_n| \leqslant \frac{|A_n|}{\sqrt{1-r^2}}.$$

Introduction and Statement of the Results

The inequality

$$\left(\prod_{j=1}^{n} z_j\right)^{1/n} \leqslant \frac{1}{n}\sum_{j=1}^{n} z_j, \quad 0 < z_j,\ n = 1, 2, \ldots,$$

is a most important result and in any case one of the keystones of the general theory of inequalities. Numerous proofs and generalizations (see [1, 3, 4]) are known; for example,

$$\prod_{j=1}^{n} z_j^{\lambda_j} \leqslant \sum_{j=1}^{n} \lambda_j z_j, \quad 0 \leqslant \lambda_j,\ \sum_{j=1}^{n} \lambda_j = 1. \tag{1}$$

In this note we shall be concerned by extensions of (1) to the case where the quantities z_j are assumed to represent complex numbers. Our point of view is the following: let $G := \prod_{j=1}^{n} z_j^{\lambda_j}$, $G_n := \left(\prod_{j=1}^{n} z_j\right)^{1/n}$, $A := \sum_{j=1}^{n} \lambda_j z_j$ and $A_n := 1/n\sum_{j=1}^{n} z_j$, where the complex numbers z_j are assumed to satisfy $|z_j - 1| \leqslant r < 1$; we wish to obtain bounds for $|G|$ (resp. $|G_n|$) in terms of A (resp. A_n) and r. Our main results are

THEOREM 1. $\dfrac{1-r^2}{|2-A|} \leqslant |G| \leqslant \dfrac{|A|}{\sqrt{1-r^2}}$.

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THEOREM 2. $(1-r)^{(r+|1-A|)/2r}(1+r)^{(r-|1-A|)/2r} \leqslant |G| \leqslant \sqrt{1+2|1-A|+r^2}$.

THEOREM 3. $|A_n|^{1/n}(1-r)^{1-1/n} \leqslant |G_n| \leqslant |A_n|^{1/n}(1+r)^{1-1/n}$.

All our results are sharp and it should be noted that no upper (resp. lower) bound given for $|G|$ in our Theorems is universally better (that is, for all eligible complex numbers z_j and all reals r and λ_j) than any other one. All our inequalities but one also sharpen the trivial estimates $1-r \leqslant |G| \leqslant 1+r$. Our method is based on well-known results concerning starlike univalent functions (we refer to the book of Duren [2] for basic facts concerning geometric function theory).

Proof of Theorem 1

Let $\mathbb{D}$ denote the open unit disc $\{z \mid |z|<1\}$ of the complex plane. We consider the class S_α of functions $f(z) := z + \sum_{n=2}^{\infty} a_n(f) z^n$ analytic in $\mathbb{D}$ and starlike of order α, i.e., $\mathrm{Re}\big(zf'(z)/f(z)\big) > \alpha$ if $z \in \mathbb{D}$. It is a result of Sheil-Small [6] that $f \in S_{1/2}$ if and only if, for each $x \in \mathbb{D}$, the function F defined by

$$F(z) := \frac{x}{f(x)}\frac{f(z)-f(x)}{z-x} = 1 + \frac{f(x)/x-1}{f(x)}z + \frac{f(x)/x-1-a_2(f)x}{xf(x)}z^2 + \cdots,$$

satisfies $\mathrm{Re}\big(F(z)\big) > 1/2$ if $z \in \mathbb{D}$. It then follows from Schwarz Lemma that for $x \in \mathbb{D}$,

$$\left|\frac{f(x)/x-1-a_2(f)x}{xf(x)} - \left(\frac{f(x)/x-1}{f(x)}\right)^2\right| \leqslant 1 - \left|\frac{f(x)/x-1}{f(x)}\right|^2$$

i.e., $\big|\big(1+a_2(f)x\big)f(x)/x + \big(1-2f(x)/x\big)\big| \leqslant -\big(1-|x|^2\big)\big|f(x)/x\big|^2 + 2\,\mathrm{Re}\big(f(x)/x\big) - 1$ and

$$\mathrm{Re}\left(\big(1+a_2(f)x\big)\frac{f(x)}{x}\right) \geqslant \big(1-|x|^2\big)\left|\frac{f(x)}{x}\right|^2 > 0, \quad x \in \mathbb{D}. \tag{2}$$

In particular

$$\left|\frac{f(x)}{x}\right| \leqslant \frac{|1+a_2(f)x|}{1-|x|^2}, \quad x \in \mathbb{D}. \tag{3}$$

The class S_α^0 of functions f defined as

$$f(z) = z\prod_{j=1}^{n}(1+x_j z)^{-2(1-\alpha)\lambda_j}, \quad |x_j| \leqslant 1$$

is known to be a dense subset of S_α (endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$). For such functions f, with $\alpha = 1/2$, we have $a_2(f) = -\sum_{j=1}^{n}\lambda_j x_j$ and an application of (3) with $x = r \in (0,1)$ yields

$$\frac{1-r^2}{|2-A|} \leqslant |G|. \tag{4}$$

Clearly equality shall hold in (4) if we choose $z_j = 1 + r$ for all $1 \leqslant j \leqslant n$ or if $z_j = 1 - r$ for all $1 \leqslant j \leqslant n$.

The proof of the right-hand side inequality in Theorem 1 is rather elementary. Indeed

$$\begin{aligned} |A| &= \left| \sum_{j=1}^{n} \lambda_j (1 + x_j r) \right| \\ &\geqslant \sum_{j=1}^{n} \lambda_j |1 + x_j r| \cos\bigl(\arg(1 + x_j r)\bigr) \\ &\geqslant \sqrt{1 - r^2} \sum_{j=1}^{n} \lambda_j |1 + x_j r| \\ &\geqslant \sqrt{1 - r^2} \prod_{j=1}^{n} |1 + x_j r|^{\lambda_j}, \end{aligned}$$

where we have made use of the "real" geometric-arithmetic means inequality. We obtain

$$|G| \leqslant \frac{|A|}{\sqrt{1 - r^2}}. \tag{5}$$

For values of $|A|$ larger than $(1+r)\sqrt{1 - r^2}$, the estimate (5) is rather crude. Nevertheless equality may hold in (5) for each fixed $r \in (0,1)$; indeed if $n = 2$, $z_1 = 1 + ie^{i\theta} r$, $z_2 = \bar{z}_1$, $\lambda_1 = \lambda_2 = 1/2$ and finally θ is an acute angle with $\cos(\theta) = \sqrt{1 - r^2}$, we obtain $|G| = \sqrt{1 - r^2}$ and $|A| = 1 - r^2$.

The following corollary can be obtained by a density argument and should be compared with (3):

COROLLARY 1. *The sharp inequality*

$$\frac{\sqrt{1 - |z|^2}}{|1 - a_2(f) z|} \leqslant \left| \frac{f(z)}{z} \right|, \quad z \in \mathbb{D},$$

holds for functions $f \in S_{1/2}$.

We shall conclude this section by a remark. By applying (2) to functions $f(z) = z \prod_{j=1}^{n} (1 + x_j z)^{-\lambda_j} \in S^0_{1/2}$, we obtain that $F(z) := \sum_{j=1}^{n} \lambda_j (1 - x_j z) \big/ \prod_{j=1}^{n} (1 + x_j z)^{\lambda_j}$ is an analytic function with positive real part in $\mathbb{D}$; as in the proof of Theorem 2 (see below), we obtain

COROLLARY 2. $|G| \leqslant \dfrac{|2 - A|}{1 - r^2} \bigl(1 + 2|1 - A| + r^2\bigr)$.

This last inequality is stronger than (5) when A is close enough to $1 + r$. It can also seen to be sharp if $A = G = 1 + r$.

Proof of Theorem 2

We consider functions $f(z) = z + a_2(f)z^2 + \cdots \in S_0$. We have the representation

$$z\frac{f'(z)}{f(z)} = \frac{1+m(z)}{1-m(z)}, \quad z \in \mathbb{D},\ m'(0) = \frac{a_2(f)}{2}, \tag{6}$$

where m is analytic in $\mathbb{D}$ and, by Schwarz Lemma,

$$\left|m(z)\right| \leqslant |z|\frac{|z| + |m'(0)|}{1 + |m'(0)||z|} < 1 \quad \text{if } z \in \mathbb{D}. \tag{7}$$

By (6) and (7),

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) \leqslant \frac{1+|m(z)|}{1-|m(z)|} \leqslant \frac{1 + |a_2(f)||z| + |z|^2}{1-|z|^2} \tag{8}$$

and

$$\operatorname{Re}\left(z\frac{f'(z)}{f(z)}\right) \geqslant \frac{1-|m(z)|}{1+|m(z)|} \geqslant \frac{1-|z|^2}{1 + |a_2(f)||z| + |z|^2}. \tag{9}$$

Since

$$\frac{d}{dr}\ln\left(\frac{|f(re^{i\theta})|}{r}\right) = \frac{1}{r}\operatorname{Re}\left(re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})} - 1\right), \quad \theta \in \mathbb{R},$$

we obtain upon integration of the last identity, for $z \in \mathbb{D}$,

$$\frac{|z|}{1 + |a_2(f)||z| + |z|^2} \leqslant \left|f(z)\right| \leqslant \frac{|z|}{(1-|z|)^{1+|a_2(f)|/2}(1+|z|)^{1-|a_2(f)|/2}}, \tag{10}$$

We now choose $f(z) := z\prod_{j=1}^{n}(1+x_j z)^{-2\lambda_j} \in S_0^0$; clearly $a_2(f) = -2\sum_{j=1}^{n}\lambda_j x_j$ and an application of (10) with $z = r \in (0,1)$ yields

$$(1-r)^{1/2+|1-A|/2r}(1+r)^{1/2-|1-A|/2r} \leqslant |G| \tag{11}$$

and

$$|G| \leqslant \sqrt{1 + 2|1-A| + r^2}. \tag{12}$$

Equality shall hold in (11) if $n = 2$, $z_1 = 1 + r$, $z_2 = 1 - r$, $\lambda_1 = \lambda \in [0, 1/2]$ and $\lambda_2 = 1 - \lambda$. Equality shall hold in (12) if $n = 2$, $z_1 = 1 + e^{i\theta}r$, $\lambda_1 = 1/2 = \lambda_2$ and $z_2 = 1 + e^{-i\theta}r$, where $\cos(\theta) \geqslant 0$. We finally mention

COROLLARY 3. $\sqrt{1-r^2} \leqslant |G| \leqslant \sqrt{1+r^2}$ *if* $A = 1$.

Proof of Theorem 3

Let us consider $H(\mathbb{D})$, the set of analytic functions in $\mathbb{D}$, endowed with the topology of uniform convergence over compact subset of $\mathbb{D}$. Let $n \geqslant 1$ and

$$\mathscr{P}_n = \{p \mid p \text{ is a polynomial of degree } \leqslant n,\ p(0) = 1, \text{ with no zeroes in } \mathbb{D}\}.$$

$\mathscr{P}_n$ is a compact and rotationally invariant subset of $H(\mathbb{D})$. Let us also consider ℓ_1 and ℓ_2, two continuous linear functionals over $H(\mathbb{D})$ with $0 \notin \ell_2(\mathscr{P}_n)$. It follows from Ruscheweyh's Duality Theory [5] that

$$\left\{ \frac{\ell_1(p)}{\ell_2(p)} \,\middle|\, p \in \mathscr{P}_n \right\} = \left\{ \frac{\ell_1(p)}{\ell_2(p)} \,\middle|\, p \in \mathscr{P}'_n \right\} \tag{13}$$

where $\mathscr{P}'_n$ denotes the subset of $\mathscr{P}_n$ consisting of the polynomials $p(z) = (1+xz)^n$, $|x| \leqslant 1$.

We choose, for fixed $r \in (0,1)$,

$$\ell_1(f) := f(r) \quad \text{and} \quad \ell_2(f) := f(0) + \frac{f'(0)}{n} r.$$

Any $p \in \mathscr{P}_n$ can be represented as

$$p(z) = \prod_{j=1}^{n} (1 + x_j z), \quad |x_j| \leqslant 1.$$

For any p as above,

$$\ell_2(p) = 1 + \frac{1}{n} \sum_{j=1}^{n} x_j r = \frac{1}{n} \sum_{j=1}^{n} (1 + x_j r) \neq 0$$

and by (13), for $n \geqslant 1$,

$$\left\{ \frac{\prod_{j=1}^{n}(1+x_j r)}{1/n \sum_{j=1}^{n}(1+x_j r)} \,\middle|\, |x_j| \leqslant 1 \right\} = \{(1+xr)^{n-1} \mid |x| \leqslant 1\}.$$

It follows easily that

$$|A_n|(1-r)^{n-1} \leqslant |G_n|^n \leqslant |A_n|(1+r)^{n-1},$$

with obvious cases of equality.

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