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INTEGRAL INEQUALITIES FOR SECOND–ORDER LINEAR OSCILLATION

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Abstract. We present several results related to the classical Lyapunov inequality for the oscillation of second-order linear equations. The first is an improved Lyapunov inequality given in terms of the downswing of the functions $\int_a^t (t-a)q(t)\,dt$ and $\int_t^b (b-t)q(t)\,dt$, extending earlier results of Kwong and Harris and Kong. Nonoscillation criteria are derived as corollaries. A Lyapunov-type inequality for two consecutive zeros of the derivative of a solution is then established and a nonoscillation criterion given as a corollary. An oscillation criterion for positive $q(t)$ is also proved. It extends the known condition $\int t^\gamma q(t)\,dt = \infty$, $\gamma \in [0,1)$.

1. Statement of Main Results

The famous Lyapunov inequality states that if a non-trivial solution of the second-order linear differential equation

$$y''(t) + q(t)y(t) = 0, \tag{1.1}$$

vanishes more than once in the interval $[a,b]$, then

$$\int_a^b q_+(t)\,dt > \frac{4}{b-a}. \tag{1.2}$$

Here, $q(t)$ is an integrable function defined on $[a,b]$ and $q_+(t) = \max\{q(t), 0\}$. We also define $q_-(t) = \max\{-q(t), 0\}$.

Lyapunov's inequality can be derived from the following stronger result. If $y_1(t)$ is a solution of (1.1) such that $y_1(a) = 0$ and $y_1'(c) = 0$ ($c > a$), then

$$\int_a^c q_+(t)\,dt > \frac{1}{c-a}. \tag{1.3}$$

The point a is said to be the left focal point of c, if there are no other zeros of y in (a,c). Likewise, if b is the first zero of y to the right of c, then b is called the right focal point of c.

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In [4], (1.3) has been strengthened to

$$\int_a^c (t-a)q_+(t)\,dt > 1. \tag{1.4}$$

The above inequalities do not take into consideration the compensation that the negative part of $q(t)$ may have on the positive part. Hence, they do not apply very well to those q that oscillate wildly. Harris and Kong [3] proved that a necessary condition for focality is that there exists a subinterval $I \subset [a,c]$, such that

$$\int_I q(t)\,dt > \frac{1}{c-a}. \tag{1.5}$$

They deduced from this an extension of (1.2), namely, there must exist two subintervals I_1 and $I_2 \subset [a,b]$ such that

$$\int_{I_i \cup I_2} q(t)\,dt > \frac{4}{b-a}. \tag{1.6}$$

It is easy to see that neither (1.4) nor (1.5) implies the other.

Recently, Brown and Hinton [1] obtained a similar result, which was then extended to linear Hamiltonian systems by Clark and Hinton [2].

One of the purposes of this paper to point out that a result that encompasses both (1.4) and (1.5), can be obtained from a simple integral identity using just an integration by parts argument.

We first give some definitions which are used to simplify the statement of our criteria. Let F be a continuous function defined on an interval (a,b), where b may be ∞. The (*forward*) *downswing* of F at a point $t \in (a,b)$ is defined to be

$$\mathscr{D}F_b(t) = \mathscr{D}F(t) = F(t) - \inf_{u\in[t,b)} F(u). \tag{1.7}$$

It measures the maximum amount that F can fall below $F(t)$ from the point t onward. If the right endpoint b is obvious from the context, it is sometimes omitted from the notation. The (forward) downswing of F over the entire interval (a,b) is defined as

$$\mathscr{D}F(a,b) = \sup_{t\in(a,b)} \mathscr{D}F(t). \tag{1.8}$$

If F is a monotonically decreasing function, then its downswing is simply the total variation of F in (a,b). It is easy to see that an equivalent definition of $\mathscr{D}F(a,b)$ is

$$\mathscr{D}F(a,b) = \sup_{a<t<u<b} \{F(t) - F(u)\}. \tag{1.9}$$

Furthermore, in the above definition, we can consider only those t and u at which F has local maxima and local minima, respectively.

In a similar way, we can define the *backward downswing* of F at t as

$$\bar{\mathscr{D}}F(t) = F(t) - \inf_{s\in(a,t]} F(s), \tag{1.10}$$

and the backward downswing of F over (a,b) as

$$\bar{\mathscr{D}}F(a,b) = \sup_{t\in(a,b)} \bar{\mathscr{D}}F(t). \tag{1.11}$$

Condition (1.5) can now be restated as

$$\bar{\mathscr{D}}Q(a,c) > \frac{1}{c-a}, \tag{1.12}$$

where $Q(t) = \int_a^t q(s)\,ds$.

Our first result is

THEOREM 1. *Define*

$$K(t) = \int_a^t (s-a)q(s)\,ds. \tag{1.13}$$

A necessary condition for a to be a left focal point of c is that

$$\bar{\mathscr{D}}K(a,c) > 1. \tag{1.14}$$

If, in addition, we know that y_1 is nondecreasing in $[a,c]$, then

$$\bar{\mathscr{D}}K(c) > 1. \tag{1.15}$$

It will be shown in Section 2 that (1.14) implies (1.12), so Theorem 1 contains the result of Harris and Kong.

In order to extend the original Lyapunov inequality for the existence of two zeros of solutions of (1.1) in $[a,b]$, we construct, for the endpoint b, the analogous function

$$\bar{K}(t) = \int_t^b (b-s)q(s)\,ds. \tag{1.16}$$

THEOREM 2. *If there is a non-trivial solution of (1.1) with more than one zero in $[a,b]$, then there exists a point $c \in (a,b)$, such that*

$$\bar{\mathscr{D}}K(a,c) > 1, \tag{1.17}$$

and

$$\mathscr{D}\bar{K}(c,b) > 1, \tag{1.18}$$

where K and $\bar{K}$ are defined as above.

Some applications of Theorem 1 will be given in Section 3. In particular, we have the next two Theorems.

THEOREM 3. *Let (1.1) be defined in $[0,\infty)$. If*

$$\limsup_{T\to\infty} \int_0^T tq(t)\,dt - \liminf_{T\to\infty} \int_0^T tq(t)\,dt < 1, \tag{1.19}$$

then (1.1) is nonoscillatory.

This is an extension of the well-known nonoscillation criterion that requires

$$\lim_{T\to\infty}\int^{T} tq(t)\,dt \qquad \text{to exist.} \tag{1.20}$$

Leighton ([8, p. 71]) has established a rather unconventional nonoscillation criterion: if q is nonincreasing and $\int^{\infty} q^{1/2}(t)\,dt < \infty$, then (1.1) is nonoscillatory. Most oscillation and nonoscillation criteria make use of the integral of q rather than that of $q^{1/2}$. With the help of Theorem 1, we can extend Leighton's criterion to a much wider class of equations. What follows is only a particular case.

THEOREM 4. *If for some* $n > 0$, $q(t)/t^n$ *is nonincreasing (or equivalently,* $tq'(t)/q(t)$ *is bounded above) and*

$$\int_{a}^{\infty} q^{1/2}(t)\,dt < \infty, \tag{1.21}$$

then (1.2) is nonoscillatory in (a,∞).

It is obvious that the above-mentioned nonoscillation criteria are useless for differential equations that have coefficients that are somewhat periodic in nature. In general, only a few very specialized nonoscillation criteria are known for such equations. The following theorem — analogous to Lyapunov's inequality, but for a solution whose derivative has more than one zero — can be used to give one such criterion, albeit rather crude.

Let $[a,b]$ be a given interval. We define

$$P = \int_{a}^{b} q_{+}(t)\,dt, \text{ and} \tag{1.22}$$

$$N = \int_{a}^{b} q_{-}(t)\,dt. \tag{1.23}$$

THEOREM 5. *If (1.1) has a solution* $y_2(t)$ *such that* $y_2'(a) = 0$, $y_2'(b) \leqslant 0$, *and* $y_2(t) > 0$ *in* $[a,b]$, *then*

$$P \geqslant \frac{N}{1 + N(b-a)}, \tag{1.24}$$

or equivalently,

$$\text{either} \quad P(b-a) \geqslant 1 \quad \text{or} \quad N \leqslant \frac{P}{1 - P(b-a)}. \tag{1.25}$$

A simple corollary of Theorem 5 is

THEOREM 6. *If one can decompose* $[0,\infty)$ *into a sequence of consecutive disjoint intervals* $\{I_i, i = 0, 1, \ldots\}$, *of lengths* T_i, *respectively, such that over each* I_i *(* $i > 0$ *) with*

$$P_i = \int_{I_i} q_{+}(t)\,dt, \tag{1.26}$$

and

$$N_i = \int_{I_i} q_-(t)\,dt, \tag{1.27}$$

we have

$$P_i < \frac{N_i}{1+N_iT_i}, \tag{1.28}$$

then equation (1.1) is nonoscillatory in $[0,\infty)$.

We shall prove Theorems 1 and 2 in Section 2. Our starting point is the simple identity

$$y_1(c) = \int_a^c (t-a)q(t)y_1(t)\,dt \tag{1.29}$$

where, as before, y_1 is the solution of (1.1) such that $y_1(a) = y_1'(c) = 0$. This can be easily obtained by integrating (1.1) twice (first over $[s,c]$ and then over $[a,b]$), and then changing the order of integration.

The proof is based on some lemmas relating the downswing of a function to that of another or to some identity of the form (1.29).

In Section 3, we indicate how the results can be extended to the more general equation

$$(p(t)y'(t))' + q(t)y(t) = 0, \tag{1.30}$$

and discuss further applications of the main theorems to the oscillation of (1.1). The following oscillation criterion, based on an identity extending (1.29), will also be established.

THEOREM 7. *If* $q(t) \geqslant 0$ *and*

$$\limsup \frac{1}{T}\int_0^T t^2 q(t)\,dt > 1, \tag{1.31}$$

then (1.1) is oscillatory.

This result extends the well-known criterion that $\int t^\gamma q(t)\,dt = \infty$ for some $\gamma \in [0,1)$.

Section 4 is devoted to some constrained optimization problems that are of interest on their own, and that will be used to prove Theorem 4. Section 5 contains the proofs of Theorems 5 and 6.

2. Proofs of Theorems 1 and 2

Throughout this section, we assume that $g(t)$ is a positive nondecreasing differentiable function on a given interval $[\alpha,\beta]$, and $f(t)$ is a locally integrable function on the same interval. In general, f can change sign arbitrarily often. Let $F(t) = \int_\alpha^t f(s)\,ds$.

LEMMA 1. *Suppose*

$$\int_\alpha^\beta f(t)g(t)\,dt = A > 0. \tag{2.1}$$

Then

$$\bar{\mathscr{D}}F(\alpha,\beta) \geqslant \bar{\mathscr{D}}F(\beta) \geqslant \frac{A - F_m g(\alpha)}{g(\beta)}, \tag{2.2}$$

where $F_m = \min_{t\in[\alpha,\beta]} F(t)$. *In particular, if* $g(\alpha) = 0$, *then*

$$\bar{\mathscr{D}}F(\alpha,\beta) \geqslant \bar{\mathscr{D}}F(\beta) \geqslant \frac{A}{g(\beta)}. \tag{2.3}$$

Furthermore, equality can occur in the second part of these inequalities if and only if at every point $t \in [\alpha,\beta]$, *either* $F(t) = F_m$, *or* $g'(t) = 0$.

Proof. The lemma is a simple consequence of the integration by parts identity

$$\int_\alpha^\beta f(t)g(t)\,dt = F(\beta)g(\beta) - \int_\alpha^\beta F(t)g'(t)\,dt, \tag{2.4}$$

and the facts $F(t) \geqslant F_m$, and $g' \geqslant 0$. □

Proof of Theorem 1. The second part of Theorem 1 follows if we apply Lemma 1 to (1.29), with $g(t) = y_1(t)$, $f(t) = (t-a)q(t)$, and $[\alpha,\beta] = [a,c]$ (and take the effort to exclude the possibility of equality in the resulting inequality).

To obtain the first part of Theorem 1, we let β be the first point in $[a,c]$ with $y_1'(\beta) = 0$, and apply the second conclusion to y_1 in $[a,\beta]$. □

LEMMA 2. *Define* $H(t) = \int_\alpha^t f(t)g(t)\,dt$. *Then*

$$\bar{\mathscr{D}}H(\alpha,\beta) \leqslant g(\beta)\,\bar{\mathscr{D}}F(\alpha,\beta). \tag{2.5}$$

Proof. By definition, there exist two points $\gamma < \delta$ in $[\alpha,\beta]$ such that

$$\begin{aligned} \bar{\mathscr{D}}H(\alpha,\beta) &= \int_\gamma^\delta f(t)g(t)\,dt \\ &= (F(\delta) - F(\gamma))g(\gamma) + \int_\gamma^\delta f(t)(g(t) - g(\gamma))\,dt \\ &\leqslant \bar{\mathscr{D}}F(\gamma,\delta)g(\gamma) + \int_\gamma^\delta f(t)(g(t) - g(\gamma))\,dt. \end{aligned} \tag{2.6}$$

Hence,

$$\int_\gamma^\delta f(t)(g(t) - g(\gamma))\,dt \geqslant \bar{\mathscr{D}}H(\alpha,\beta) - \bar{\mathscr{D}}F(\gamma,\delta)g(\gamma). \tag{2.7}$$

We now apply Lemma 1 to this inequality, with $g(t)$ replaced by $g(t) - g(\gamma)$, to obtain

$$\bar{\mathscr{D}}F(\gamma,\delta)(g(\delta) - g(\gamma)) \geqslant \bar{\mathscr{D}}H(\alpha,\beta) - \bar{\mathscr{D}}F(\gamma,\delta)g(\delta), \tag{2.8}$$

from which (2.5) follows. □

As a corollary of Lemma 2, we see that (1.14) implies (1.5). Hence Theorem 1 includes Harris and Kong's result.

LEMMA 3. *In the interval* $[a,b]$, *define as before* $K(t)=\int_a^t (t-a)q(t)\,dt$. *Take any point* $\alpha\in(a,b)$ *and define* $K_1(t)=\int_\alpha^t (t-\alpha)q(t)\,dt$. *Then*

$$\bar{\mathscr{D}}K_1(\alpha,b)\leqslant\bar{\mathscr{D}}K(\alpha,b)\leqslant\bar{\mathscr{D}}K(a,b). \tag{2.9}$$

Proof. The second inequality is obvious. The first inequality can be obtained by applying Lemma 2, using $f(t)=(t-a)q(t)$, and $g(t)=(t-\alpha)/(t-a)$. □

The significance of this lemma can be explained by the following observation. Given the differential equation on the interval $[a,c]$, suppose it is known that c does not have a left focal in $[a,c]$. In general, one cannot conclude that for all $c_1<c$, there is likewise no left focal point in $[a,c]$. However, as a consequence of Lemma 3, we have

THEOREM 8. *If* $\bar{\mathscr{D}}K(a,c)\leqslant 1$, *with* K *as defined in (1.13), then no point in* $[a,c]$ *can have a left focal point in* $[a,c]$.

We are now ready to prove our second main result.

Proof of Theorem 2. Suppose that a nontrivial solution $y_0(t)$ of equation (1.1) has two consecutive zeros $\alpha<\beta$ in $[a,b]$, and we may assume that $y_0(t)>0$ in (α,β). Let $c\in(\alpha,\beta)$ be a point at which y_0 attains its maximum in (α,β). Then α is the left focal point of c and β is the right focal point of c. By applying Theorem 8 to the two subintervals, we see that

$$\bar{\mathscr{D}}K(a,c)>1, \tag{2.10}$$

and

$$\mathscr{D}\bar{K}(c,b)>1, \tag{2.11}$$

and the conclusion of Theorem 2 follows. □

3. Examples and Further Applications

It is well known how one can transform equation (1.30) to equation (1.1) by a change of the independent variable. Alternatively, one can extend Theorem 1 and its consequences directly by observing that the analog of the basic identity (1.29) for (1.30) is

$$y_1(c)=\int_a^c P(t)q(t)y_1(t)\,dt \tag{3.1}$$

where

$$P(t)=\int_a^t\frac{ds}{p(s)}. \tag{3.2}$$

We also define

$$\bar{P}(t)=\int_t^b\frac{ds}{p(s)}. \tag{3.3}$$

It follows that Theorems 1 and 2 are valid for (1.30) provided that we now define

$$K(t) = \int_a^t P(s)q(s)\,ds \tag{3.4}$$

and

$$\bar{K}(t) = \int_t^b \bar{P}(s)q(s)\,ds. \tag{3.5}$$

A simple consequence of Theorem 1 is

COROLLARY 1. *Let* $K(t) = \int_0^t tq(t)\,dt$. *If*

$$\limsup_{t\to\infty} \bar{\mathscr{D}}K(t,\infty) < 1, \tag{3.6}$$

then (1.1) is nonoscillatory in $[0,\infty)$.

Corollary 1 implies Theorem 3.

EXAMPLE 1. The equation

$$y''(t) + C\frac{\sin t}{t}y(t) = 0 \tag{3.7}$$

has been shown by Willet [5] to be nonoscillatory if $C < 1/\sqrt{2}$, and oscillatory if $C > 1/\sqrt{2}$. Wong [7] shows that the critical case $C = 1/\sqrt{2}$ is nonoscillatory. The results in [5] and [7] are stated for the differential equation $y'' + (\sin\beta/t)y = 0$. A scaling can be used to transform that to (3.7). It is easy to see that Theorem 3 gives nonoscillation only for $C \leqslant 1/2$. This shows that the main results of this paper are far from being optimal. Yet they are still useful because more accurate criteria are often tied to the particular form of the coefficients. For instance, even with the help of a very crude estimate, Theorem 3 shows that

$$y''(t) + \frac{\sum_i C_i \sin\omega_i t}{t}y(t) = 0 \tag{3.8}$$

is nonoscillatory if $\sum_i C_i/\omega_i \leqslant 1/2$. The constant $1/2$ can be sharpened by more carefully estimating the maxima and minima of the expression $\sum_i C_i \cos\omega_i t/\omega_i$. Yet it is not readily obvious how Willet or Wong's method can be applied to (3.8). □

EXAMPLE 2. When we subtract a negative constant from the coefficient in the (3.8), we obtain an example

$$y''(t) + \left(\frac{\sum_i C_i \sin\omega_i t}{t} - \lambda\right)y(t) = 0, \tag{3.9}$$

for which Theorem 3 is no longer applicable because $\lim_{T\to\infty}\int^T tq(t)\,dt = -\infty$, but Corollary 1 is. Again a crude estimate is sufficient to show that the equation is nonoscillatory if $\sum_i C_i/\omega_i \leqslant 1 + \pi\lambda/\omega$, where $\omega = \max\{\omega_i\}$. □

EXAMPLE 3. The differential equation

$$y''(t) + C\sin(\omega t^2)y(t) = 0 \tag{3.10}$$

is nonoscillatory if $C \leqslant \omega$. □

EXAMPLE 4. Note that Theorem 3 is utterly inadequate for differential equations with periodic coefficients. For instance, we cannot conclude from Theorem 3 that the Hill's equation

$$y''(t) + (\sin t - \lambda)y(t) = 0 \tag{3.11}$$

is nonoscillatory, unless $\lambda \geqslant 1$. The Floquet theory provides a way to estimate the critical λ_0 (which is also the infimum of the essential spectrum of the associated second-order differential operator) that demarcate the oscillation and nonoscillation of (3.11). Let y_1 and y_2 be the two fundamental solutions of (3.11), such that $y_1(0) = y_2'(0) = 1$, and $y_1'(0) = y_2(0) = 0$. Then λ_0 is the smallest critical value that gives $y_1(2\pi) + y_2'(2\pi) = 2$. By numerical computation, we find that $0.3784892 < \lambda_0 < 0.3784893$.

On the other hand, Theorem 5 (with the help of numerical computation) shows that (3.11) is nonoscillatory if $\lambda > 0.8126$. We use the decomposition $[0,\infty) = \cup[(i-1)\pi, i\pi]$. One can see that the conclusion from Theorem 5 is not optimal. However, the Floquet method is tied to the periodic nature of the coefficient. For instance, if the coefficient is perturbed by a small nonperiodic function, then the Floquet method no longer works but Theorem 5 may still be useful. □

To conclude this section, we prove Theorem 7 and its corollary.

Proof of Theorem 7. Note that a translation preserves (1.31), namely that

$$\limsup \frac{1}{T-a}\int_0^{T-a} (t-a)^2 q(t)\,dt > 1 \tag{3.12}$$

for all a. Hence we can assume that the solution $y_1(t)$ with initial conditions $y_1(0) = 0$, $y_1'(0) = 1$ has no other zero in $[0,\infty)$. Since $q(t) \geqslant 0$, this implies that $y_1'(t) > 0$ and y_1 is concave in $[0,\infty)$. Let $T > 0$. Concavity gives

$$y_1(t) \geqslant \frac{ty_1(T)}{T}. \tag{3.13}$$

We can obtain the following identity in a way similar to the derivation of (1.29).

$$y_1(T) = Ty_1'(T) + \int_0^T tq(t)y_1(t)\,dt. \tag{3.14}$$

Together with (3.13), this gives

$$y_1(T) > \frac{y_1(T)}{T}\int_0^T t^2 q(t)\,dt, \tag{3.15}$$

which contradicts the hypothesis of the theorem. □

COROLLARY 2. *If* $q(t) \geqslant 0$*, and there exists* $\gamma \in [0,1)$*, such that*

$$\int_0^\infty t^\gamma g(t)\,dt = \infty, \tag{3.16}$$

then

$$\limsup \frac{1}{T}\int_0^T t^2 q(t)\,dt = \infty, \tag{3.17}$$

and hence, by Theorem 7, (1.1) is oscillatory.

Proof. Suppose that (3.17) is false. Then

$$\int_0^T t^2 q(t)\,dt \leqslant MT \tag{3.18}$$

for some $M > 0$ and for all $T > 0$. Define

$$F(t) = \int_0^t s^\gamma q(t)\,ds. \tag{3.19}$$

Integrating (3.18) by parts gives the integral inequality

$$F(T) \leqslant \frac{M}{T^{1-\gamma}} + \frac{2-\gamma}{T^{2-\gamma}}\int_0^T t^{1-\gamma}F(t)\,dt. \tag{3.20}$$

By (3.16), $F(T) \geqslant 0$ for sufficiently large T, say for $T > \alpha$. We rewrite (3.20) as

$$F(T) \leqslant A + \frac{M}{T^{1-\gamma}} + \frac{2-\gamma}{T^{2-\gamma}}\int_\alpha^T t^{1-\gamma}F(t)\,dt, \quad T \geqslant \alpha, \tag{3.21}$$

where $A = \int_0^\alpha t^{1-\gamma}F(t)\,dt$.

The condition $F(T) \geqslant 0$ is necessary to apply the theory of integral inequalities which asserts that if $G(T)$ is the function that satisfies the integral equation

$$G(T) = A + \frac{M}{T^{1-\gamma}} + \frac{2-\gamma}{T^{2-\gamma}}\int_\alpha^T t^{1-\gamma}G(t)\,dt, \quad T \geqslant \alpha, \tag{3.22}$$

then

$$F(T) \leqslant G(T), \quad T \geqslant \alpha. \tag{3.23}$$

It is easy to solve (3.22) to find that

$$G(T) = A + \frac{M}{1-\gamma}\left(\frac{1}{\alpha^{1-\gamma}} - \frac{1}{T^{1-\gamma}}\right) < \infty. \tag{3.24}$$

It follows that $F(T) < \infty$, contradicting (3.16). □

The oscillation criterion (3.16) has been attributed to Wintner, Hille, and Hartman (the case $\gamma = 0$ is the classical Fite criterion), and extended to nonlinear equations by Wong. For references, see [6].

4. Some Optimization Problems and Theorem 4

Theorem 4 is a consequence of the following fact. Suppose that $q(t)/t^n$ is nonincreasing, and $\int_a^T q^{1/2}(t)\,dt$ is sufficiently small; then $\int_a^T tq(t)\,dt$ will also be small. A scaling argument shows that it suffices to establish this fact for $T = 1$.

We are, therefore, interested in the optimization problem:

PROBLEM 1. *Knowing that $k(t)/t^m$ ($m > 0$) is nonincreasing, and $\int_0^1 k(t)\,dt = 1$, find*

$$\max \int_0^1 tk^2(t)\,dt. \tag{4.1}$$

The answer is given by

LEMMA 4. *The maximum as required in the optimization problem above exists and is attained by functions of the form*

$$\bar{k}(t) = \begin{cases} Ct^m & \text{in } [0,\alpha] \\ 0 & \text{in } (\alpha,1] \end{cases} \tag{4.2}$$

for some $\alpha \in (0,1]$, and the constant $C = (m+1)/\alpha^{m+1}$.

Since the proof for the general case is the same as that when $m = 0$, we shall concentrate only on the latter special case and assume that $k(t)$ is nonincreasing. Our method is to approximate $k(t)$ by step functions that are constant in equal subintervals of length $1/N$ of $[0,1]$. We then take the limit when $N \to \infty$ to obtain the conclusion for the continuous case. By rewriting the various integral as a sum, we have a discrete analog of our optimization problem.

PROBLEM 2. *Knowing that the set of real numbers $\{k_i : i = 1 \ldots N\}$ is nonincreasing and $\sum_{i=1}^N k_i = 1$, find*

$$\max \sum_{i=1}^N (2i-1)k_i^2. \tag{4.3}$$

Let us suppose that we have already found the optimal set of numbers, still denoted by k_i for simplicity. We prove by induction that the first j numbers must be equal, provided that $k_j \neq 0$, and this solves the problem.

Since $k_j \neq 0$, $k_{j-1} \neq 0$, and by the induction hypotheses, the first $j-1$ numbers are equal. If we hold $k_{j+1},\ldots,k_N$ constant, then maximizing the entire sum becomes maximizing the partial sum S_j over the first j numbers, which is now a function of k_j, since each of the first $j-1$ numbers are all equal to $(1-s-k_j)/(j-1)$, where $s = \sum_{i=j+1}^N k_i$ is a fixed constant. Direct computation easily shows that S_j is a quadratic function of k_j, with a positive leading coefficient. The maximum of S can only occur at the two endpoints of the interval of definition for k_j, which is $(0,(1-s)/j]$. Again direct computation shows that S_j is the same at either endpoint, but since $k_j \neq 0$, $k_j = (1-s)/j$. So the first j numbers are all equal.

Note that our proof would have failed if the coefficients $(2i-1)$ in (4.3) are replaced by i, because the partial sum S_j will then have a larger value when $k_j = 0$ than when it is at the other endpoint of the interval.

Proof of Theorem 4. We may assume, without loss of generality, that

$$\int_0^\infty q^{1/2}(t)\,dt = A < \left(\frac{4}{n+2}\right)^{1/2}. \tag{4.4}$$

Let us choose any $T > 0$ and try to maximize $\int_0^T tq(t)\,dt$. By Lemma 4, the maximum is attained by the function $q(t) = Ct^n$ on $[0,T]$. Using the constraint $\int_0^T q^{1/2}(t)\,dt \leqslant A$, we find that $C \leqslant (n+2)/T^{2m+2}$. Then

$$\int_0^T tq(t)\,dt = \frac{CT^{2m+2}}{n+2} \leqslant 1. \tag{4.5}$$

Hence by Theorem 3, equation (1.1) is nonoscillatory. □

We conclude this section by noting that the monotonicity of $t^n q(t)$ can be further relaxed. Again we confine ourselves to the simplest case $n = 0$. Instead of requiring that $q(t)$ be nonincreasing, we can require just that at each point t, $q(t)$ will not increase more than a fixed multiple, that is, there exists an $M \geqslant 1$, such that

$$\frac{q(s)}{q(t)} \leqslant M, \quad \text{for all } s > t. \tag{4.6}$$

To see that this is true, we construct the function

$$q^*(t) = \sup_{u>t} q(u). \tag{4.7}$$

Then $q^*(t)$ is nonincreasing and $q(t) \leqslant q^*(t) \leqslant Mq(t)$. Hence,

$$\int_0^\infty q^*(t)\,dt \leqslant M\int_0^\infty q(t)\,dt < \infty. \tag{4.8}$$

By Theorem 4, the equation $z''(t) + q^*(t)z(t) = 0$ is nonoscillatory. By Sturm's Comparison Theorem, (1.1) is also nonoscillatory.

Furthermore, we do not even need (4.6) to hold for every t, but for sufficiently many t. More precisely, we require that there is a number $0 < \delta < 1$ such that for all $T > 0$, the set

$$\left\{t \in [0,T] : \frac{\sup_{s>t} q(s)}{q(t)} > M\right\} \text{ has measure } \leqslant \delta T. \tag{4.9}$$

We omit the simple proof.

5. Proof of Theorems 5 and 6

Proof of Theorem 5. We prove Theorem 5 by using a sequence of reductions to simpler cases. For simplicity, we will leave out the subscript 2 when we write the solution y_2.

Step 1. We can assume without loss of generality that $y'(b) = 0$. Suppose that we originally have $y'(b) < 0$. In a small neighborhood of b, $[b-\varepsilon, b]$, modify $q(t)$ by replacing it by a smaller and sufficiently large negative function, so that the modified solution $\bar{y}'(b) = 0$. Then the conclusion of Theorem 3, (1.24), holds for the modified coefficient. Since the modified coefficient has a smaller P and a larger N than those of the original coefficient, (4.6) holds for the original $q(t)$ as well.

Step 2. We need only prove the theorem for the case when $y(t)$ is increasing in $[a,b]$. By using a reflection, we see that the theorem is also true for the case when $y(t)$ is decreasing in $[a,b]$. Now suppose that $y(t)$ is no longer monotone in $[a,b]$. We decompose $[a,b]$ into subintervals I_i in each of which $y(t)$ is monotone. Then inequality (1.24) applies to each of these intervals. By summing them up, we get (1.24) over the entire interval $[a,b]$. Let us illustrate this in the simplest case when there are only two subintervals $I_1 = [a,c]$ and $I_2 = [c,b]$, then

$$P_1 \geqslant \frac{N_1}{1+N_1(c-a)} > \frac{N_1}{1+N_1(b-a)}. \tag{5.1}$$

Likewise,

$$P_2 \geqslant \frac{N_2}{1+N_2(b-c)} > \frac{N_2}{1+N_2(b-a)}. \tag{5.2}$$

Adding them up gives (1.24).

Step 3. We may assume that $q(t)$ is continuous. If it is not, we can approximate it (in the L^1 norm) by a continuous function for which (1.24) holds. Taking limit gives the general result.

Furthermore, a similar approximation argument shows that we may assume that $q(t)$ changes sign only a finite number of times in $[a,b]$, that is, there is a finite decomposition $[a,b] = \cup I_i$, such that, in each I_i, $q(t)$ is either $\geqslant 0$ or $\leqslant 0$. Since $y(t)$ has a local minimum at a and a local maximum at b, $q(a) \leqslant 0$ and $q(b) \geqslant 0$. As a consequence, $q(t)$ changes sign an even number of times, say $2n$ times ($n \geqslant 1$).

Step 4. Let us first take care of the simplest case of $n = 1$, and we assume $q(t) \leqslant 0$ in $[a,c]$ and $q(t) \geqslant 0$ in $[c,b]$. In the rest of the proof, we freely make use of coefficients that contain delta functions. Such a function introduces a jump in the first derivative of a solution. We need the following lemmas.

LEMMA 5. *Suppose $q(t) \leqslant 0$ in $[\alpha,\beta]$, and $\rho = y'(\alpha)/y(\alpha) \geqslant 0$. Then*

$$\frac{y'(\beta)}{y(\beta)} \geqslant \frac{\rho+N_1}{1+(\rho+N_1)(\beta-\alpha)}, \tag{5.3}$$

where $N_1 = -\int_\alpha^\beta q(t)\,dt$. Equality in the above inequality holds if and only if $q(t)$ is a delta function concentrated at α.

Proof. Essentially, this is another optimization problem. Given a fixed N_1, what $q(t)$ will give the minimum $y'(\beta)/y(\beta)$? The proof again resorts to using the technique of approximation with step functions; it would be interesting to find a direct proof. Instead of starting with equation (1.1), we consider the equivalent Riccati equation

$$r(t) = \rho + N(t) - \int_{\alpha}^{t} r^2(s)\,ds, \tag{5.4}$$

where $r(t) = y'(t)/y(t)$, and $N(t) = -\int_{\alpha}^{t} q(s)\,ds$.

The optimization problem under consideration can now be restated as: minimize $r(\beta)$ under the constraints that $N(\beta) = N$ and $N(t)$ is nondecreasing. In view of the approximation trick, we need only establish the result when $N(t)$ is a step function. Our claim is that $r(\beta)$ can be progressively decreased by raising every step value to be equal to N. Hence, the absolute minimum of $r(\beta)$ must be attained when all the steps have the same value $N(t) = N$. We can prove this claim by induction on the steps of the function, starting with the second to last step and working backwards to the first step.

We have to be a little careful in dealing with (5.4) when $N(t)$ is a step function, because $r(t)$ is no longer a continuous function. In fact, at every point of discontinuity s of $N(t)$, $r(t)$ is also discontinuous and we must be working with $r(s-) = \lim_{t\to s-} r(t)$ and $r(s+) = \lim_{t\to s+} r(t)$, instead of simply $r(s)$. Nonetheless, we only have to observe that

$$r(s+) - r(s-) = N(s+) - N(s-). \tag{5.5}$$

In the j^{th} cycle of our induction process, all the last j steps of $N(t)$ have the same value N, but the current one does not. Let the subinterval corresponding to the current step be $[\gamma,\delta]$. By raising the step value in this interval to N, we actually construct a new Riccati equation. The required conclusion is thus a simple comparison between the two $r(\beta)$ values for the original and the new equations. Since the two functions $N(t)$ have the same steps to the left of the current step, the two solutions are identical to the left of the current step. Hence, $r(\gamma-) = \bar{r}(\gamma-)$, where $\bar{r}(t)$ denotes the solution of (5.4) with the new $N(t)$. However, on the right hand side of γ,

$$r(\gamma+) + N - N(\gamma+) = \bar{r}(\gamma+). \tag{5.6}$$

Direct computation shows that

$$r(\delta+) = r(\delta-) + N - N(\gamma+) = \frac{r(\gamma+)}{1 + r(\gamma+)(\delta-\gamma)} + N - N(\gamma+) \tag{5.7}$$

and

$$\bar{r}(\delta+) = \bar{r}(\delta-) = \frac{\bar{r}(\gamma+)}{1+\bar{r}(\gamma+)(\delta-\gamma)} = \frac{r(\gamma+)}{1+\bar{r}(\gamma+)(\delta-\gamma)} + \frac{N - N(\gamma+)}{1+\bar{r}(\gamma+)(\delta-\gamma)}. \tag{5.8}$$

Obviously, $r(\delta+) \geqslant \bar{r}(\delta+)$. To the right of δ, the two equations become identical again, but r starts with a larger value than $\bar{r}$ at δ. By the classical Sturm Comparison Theorem (use the fact that the Riccati equation corresponds to a second-order linear differential equation), $r(t)$ will remain larger than $\bar{r}(t)$ throughout $[\delta, \beta]$. Hence, $r(\beta) \geqslant \bar{r}(\beta)$. The proof is now complete. □

A similar result holds when we manipulate the positive part of $q(t)$ to minimize $r(t)$. The optimal value is attained when $q(t)$ is all concentrated at the right endpoint β. Since the proof is similar to that of Lemma 5, it is omitted.

LEMMA 6. *Suppose* $q(t) \geqslant 0$ *in* $[\alpha, \beta]$, $y'(t) \geqslant 0$ *in* $[\alpha, \beta]$, *and* $\rho = y'(\alpha)/y(\alpha) \geqslant 0$. *Then*

$$\frac{y'(\beta)}{y(\beta)} \geqslant \frac{\rho - P_1(1+\rho(\beta-\alpha))}{1+\rho(\beta-\alpha)}, \tag{5.9}$$

where $P_1 = \int_\alpha^\beta q(t)\,dt$. *Equality in the above inequality holds if and only if* $q(t)$ *is a delta function concentrated at* β.

Let us return to the two-interval case of Theorem 5. Apply Lemma 5 to the solution in the interval $[a, c]$ to get

$$\rho = \frac{y'(c)}{y(c)} \geqslant \frac{N_1}{1+N_1(c-a).} \tag{5.10}$$

Next we apply Lemma 6 to the solution in the interval $[c, b]$ to get

$$0 = \frac{y'(b)}{y(b)} \geqslant \frac{\rho - P_1(1+\rho(b-c))}{1+\rho(b-c)}, \tag{5.11}$$

from which

$$P_1 \geqslant \frac{\rho}{1+\rho(b-c)}. \tag{5.12}$$

Substituting (5.10) into this inequality give the conclusion of the Theorem.

Step 5. Let us now suppose that, as in Step 3, $q(t)$ changes sign $2n$ times, $n \geqslant 2$. We give below a construction to reduce the case to one in which $q(t)$ change sign $2(n-1)$ times. Then by induction, the conclusion of the general case follows. Since the construction is essentially the same in the case $n = 2$ as in the general case, for simplicity, we just assume $n = 2$.

Let the subintervals of constant sign of $q(t)$ be $q(t) \leqslant 0$ in $[a, c_1]$, $q(t) \geqslant 0$ in $[c_1, c_2]$, $q(t) \leqslant 0$ in $[c_2, c_3]$, and $q(t) \geqslant 0$ in $[c_3, b]$, Denote $N_1 = -\int_a^{c_1} q(t)\,dt$, $P_1 = \int_{c_1}^{c_2} q(t)\,dt$, $N_2 = -\int_{c_2}^{c_3} q(t)\,dt$, and $P_2 = \int_{c_3}^{b} q(t)\,dt$. Using the same proof for the two-interval case as in Step 4, we see easily that

$$P_2 > \frac{N_2}{1+N_2(b-c_2)}. \tag{5.13}$$

Replace $q(t)$ in $[c_1,c_2]$ by a positive delta function concentrated at c_2, while preserving P_1. Retain the same $q(t)$ in the other three intervals. Let us denote by $q_1(t)$ the new function, and by $y_1(t)$ the solution of the new differential equation with the same initial conditions as $y(t)$ at a. We have $y(t)=y_1(t)$ in $[a,c_1]$. Apply Lemma 6 to $[c_1,c_2]$, we have $r(c_2)\geqslant r_1(c_2)$.

If $r_1(c_2)\leqslant 0$, then the two-interval case of the theorem applies to $q_1(t)$ in $[a,c_2]$, and we have

$$P_1 > \frac{N_1}{1+N_1(c_2-a)}. \tag{5.14}$$

Together with (5.13), this implies the theorem and we are done.

So we only need to consider the remaining case $r(c_2)\geqslant r_1(c_2)>0$. By the Sturm Comparison Theorem, $r(t)\geqslant r_1(t)$ in $[c_2,b]$, unless $y_1(t)$ has a zero before b. In the first case, $r_1(b)\leqslant r(b)=0$, giving $y_1'(b)<0$. Hence, there exists a point $\beta\in[c_3,b]$ at which $y_1'(\beta)=0$, and $y(t)$ is increasing in $[c_2,\beta]$. It is easy to see that this fact is also true in the second case. In all cases, β must be in $[c_3,b]$ since $y_1(\beta)$ is a local maximum for y_1. Next we replace $q_1(t)$ in $[\beta,b]$ by 0, to get a new function $q_2(t)$ while the corresponding solution $y_2(t)$ will have vanishing first derivative at a and b. In doing so we have reduced the positive part of the coefficient $q_1(t)$. Hence, if we can prove the required inequality for $q_2(t)$, that will imply the same inequality for $q(t)$.

In an similar way, we replace the part of $q_2(t)$ in $[c_2,c_3]$ by a delta function that is concentrated at c_2, while preserving N_2. By going through a similar sequence of arguments, we arrive at a new function $q_3(t)$ that has less positive part than $q(t)$ but its solution $y_3(t)$ is nondecreasing in $[a,b]$ and $y_3'(a)=y_3'(b)=0$. Notice that we have introduced two delta functions, both concentrated at c_2 and of opposite sign, in the construction of $q_3(t)$ from $q(t)$. Depending on the relative magnitude of these delta functions, they coalesce into one delta function of either positive or negative sign. In any case, $q_3(t)$ is now a function that changes sign only twice, and the result in Step 4 applies. The proof of Theorem 5 is now complete. $\square$

Proof of Theorem 6. Let $I_i=[a_i,a_{i+1}]$ be the subintervals in the hypotheses of Theorem 6, and $y_0(t)$ be the solution of (1.1) with initial condition $y_0(a_1)=1$ and $y_0'(a_1)=0$. Since (1.28) implies that $P_1T_1<1$, the classical Lyapunov inequality for disfocality (1.3) shows that y_0 cannot have a zero in I_1. By applying Theorem 5 to y_0 in the interval I_1, we see that $y_0(a_2)>0$. We claim that y_0 has no zero in I_2 and that $y_0(a_3)>0$. Let $y_1(t)$ be the solution of (1.1) with initial condition $y_1(a_2)=y_0(a_2)$ and $y_1'(a_2)=0$. It follows from the Sturm Comparison Theorem that $y_0'(t)>y_1'(t)$. As before, Lyapunov's inequality and Theorem 5 imply that y_1 has no zero in the interval I_2, and $y_1(a_3)>0$. Hence, y_0 has no zero in I_2 and $y_0(a_3)>y_1(a_3)>0$, as claimed. In a similar way, we can proceed to show that y_0 has no zero in I_i and $y_0(a_i)>0$, for all i. The proof of Theorem 6 is then complete. $\square$

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