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A NOTE OF A ROUGH SINGULAR INTEGRAL OPERATOR

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Abstract. L^p mapping properties will be established in this paper for singular Radon transforms with rough kernels, extending the results of Grafakos and Stefanov.

1. Introduction

Let $\mathbf{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbf{R}^n$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Calderón-Zygmund singular integral operator T is defined by

$$T(f)(x) = \text{p.v.} \int_{\mathbf{R}^n} \Omega(y')|y|^{-n} f(x-y)dy; \tag{2}$$

the truncated maximal operator T^* is defined by

$$T^*(f)(x) = \sup_{\epsilon>0} \Big| \int_{|y|>\epsilon} \Omega(y')|y|^{-n} f(x-y)dy \Big| \tag{3}$$

where $y' = y/|y| \in S^{n-1}$ and $f \in S(\mathbf{R}^n)$, the space of Schwartz functions.

By introducing the method of rotations, Calderón-Zygmund proved that if $\Omega \in L\log^+ L$, then both T and T^* are bounded operators in $L^p(\mathbf{R}^n)$ [2, 3]. Some years later, the condition $\Omega \in L\log^+ L$ was independently improved by Connett [4] and Ricci and Weiss [11] who showed that if

$$\Omega \in H^1(S^{n-1}) \tag{4}$$

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then T maps $L^p(\mathbf{R}^n)$ into itself for all $p \in (1, \infty)$. Here $H^1(S^{n-1})$ denotes the Hardy space on the sphere in the sense of Coifman and Weiss [6]. More details of $H^1(S^{n-1})$ can be found in Colzoni's thesis [5].

Recently Grafakos and Stefanov [10] considered a family of new conditions

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left(\ln \frac{1}{|\langle y', \xi \rangle|} \right)^{1+\alpha} d\sigma(y') < \infty, \quad \alpha > 0, \tag{5}$$

and they proved the following two L^p boundedness theorems:

THEOREM A. *Let* $\alpha > 0$. *Then* T *is bounded in* $L^p(\mathbf{R}^n)$ *for* $(2+\alpha)/(1+\alpha) < p < (2+\alpha)$.

THEOREM B. *Let* $\alpha > 1$. *Then* T^* *is bounded in* $L^p(\mathbf{R}^n)$ *for* $1 + 3/(1+2\alpha) < p < 2(2+\alpha)/3$.

More significantly, Grafakos and Stefanov showed that condition (5) for all $\alpha > 0$ is indeed disjoint from the H^1 condition (4).

The main purpose of this paper is to investigate a more general operator T_P defined by

$$T_P f(x) = \text{p.v.} \int_{\mathbf{R}^n} \Omega(y')|y|^{-n} f(x - P_N(|y|)y')dy \tag{6}$$

and its truncated maximal operator

$$T_P^* f(x) = \sup_{\epsilon > 0} \left| \int_{|y| > \epsilon} \Omega(y')|y|^{-n} f(x - P_N(|y|)y')dy, \right. \tag{7}$$

where $P_N(t)$ is a real polynomial on $\mathbf{R}$ of degree N and satisfies $P_N(0) = 0$. Clearly, if P_N is the identity function I then $T_I = T$.

The operator T_P was first studied by Fan and Pan [8] and its L^p boundedness was known in [1] under condition (4). Thus, naturally, it would be interesting to know the L^p boundedness of T_P and T_P^* under the new condition (5).

The following two theorems are the main results in this paper.

THEOREM 1. *Let* $\alpha > 0$ *and* $\Omega(y')$ *satisfy (5). Then* T_P *is bounded in* $L^p(\mathbf{R}^n)$ *for* $p \in \left(\frac{2+2\alpha}{1+2\alpha},\ 2+2\alpha\right)$. *The bound of* T_P *is independent of the coefficients of* P.

THEOREM 2. *Let* $\alpha > \frac{1}{2}$, *and* $\Omega(y')$ *satisfy (5). Then* T_P^* *is bounded in* $L^p(\mathbf{R}^n)$ *for* $p \in \left(\frac{1+2\alpha}{2\alpha},\ 1+2\alpha\right)$. *The bound of* T_P^* *is independent of the coefficients of* P.

It should be noted that the ranges of p in Theorems 1–2 are strictly larger than those in Theorems A and B, respectively.

The proof of the theorems are based on the technique used in [9].

2. Boundedness of Singular Integrals

For the given polynomial $P_N(t) = \sum_{m=1}^{N} \beta_m t^m$, we denote

$$P_r(t) = \sum_{m=0}^{r} \beta_m t^m \quad \text{for} \quad r = 0, 1, 2, \cdots, N, \quad \text{where} \quad \beta_0 = 0.$$

For all integers $k \in \mathbf{Z}$ and $r = 0, 1, 2, \cdots, N$, we define

$$\hat{\sigma}_{k,r}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} |y|^{-n} \Omega(y') e^{-iPr(|y|)\langle y', \xi \rangle}\, dy, \tag{8}$$

$$\hat{\mu}_{k,r}(\xi) = \int_{2^k \leqslant |y| < 2^{k+1}} |y|^{-n} |\Omega(y')| e^{-iPr(|y|)\langle y', \xi \rangle}\, dy. \tag{9}$$

Also we define

$$\sigma_r^* f(x) = \sup_{k \in \mathbf{Z}} |\mu_{k,r} * f(x)|.$$

Then it is easy to see $T_P f = \sum_{k=-\infty}^{\infty} \sigma_{k,N} * f$.

The following lemma is Theorem 7.4 in [9].

LEMMA 2.1.. *For $r = 1, 2, \cdots, N$ and $p \in (1, \infty)$, if $\Omega \in L^1(S^{n-1})$ then*

$$\|\sigma_r^* f\|_{L^p(\mathbf{R}^n)} \leqslant c \|f\|_{L^p(\mathbf{R}^n)}$$

where c is a constant independent of the coefficients of P_r.

We also need the following lemma:

LEMMA 2.2. *For all integers $k \in \mathbf{Z}$ and $r = 1, 2, \cdots, N$,*
(i) $|\hat{\sigma}_{k,r}(\xi) - \hat{\sigma}_{k,r-1}(\xi)| \leqslant c \left|2^{rk} \beta_r \xi\right|$
(ii) $|\hat{\sigma}_{k,r}(\xi)| \leqslant c \left(\ln |2^{kr} \xi \beta_r|\right)^{-1-\alpha}$ *if* $|2^{kr} \xi \beta_r| > 1$.

Proof. (i) is obvious. To prove (ii), we note

$$\hat{\sigma}_{k,r}(\xi) = \int_{S^{n-1}} \Omega(y') \left[\int_1^2 e^{-iP_r(2^k t)|\xi| \langle \xi', y' \rangle} \frac{dt}{t} \right] d\sigma(y'). \tag{10}$$

By Van der Corput lemma, the integral inside the bracket in (10) is bounded by

$$c \left(|\xi| |\beta_r| 2^{kr} |\langle \xi', y' \rangle| \right)^{-\frac{1}{r}}.$$

On the other hand, it is trivial to see that

$$\left| \int_1^2 e^{-iP_r(2^k t)|\xi| \langle \xi', y' \rangle} \frac{dt}{t} \right| \leqslant \log 2.$$

Thus it must satisfy the estimate

$$\left|\int_1^2 e^{-iP_r(2^k t)|\xi|\langle \xi',y'\rangle}\,\frac{dt}{t}\right| \leqslant c\frac{\left(\ln\left(\frac{3}{2}|\langle \xi',y'\rangle|^{-1}\right)\right)^{1+\alpha}}{\left(\ln 2^{kr}|\xi||\beta_r|\right)^{1+\alpha}}.$$

Therefore, by (5), (ii) is proved. We now choose and fix a function $\phi\in C_0^\infty(\mathbf{R}^n)$ such that $\phi(t)\equiv 1$ for $|t|\leqslant 1$ and $\phi(t)\equiv 0$ for $|t|>2$.

Let $\varphi(t)=\phi(t^2)$ and define the measures $\{\tau_{k,N-\lambda}\}$ by

$$\begin{aligned}\hat\tau_{k,N-\lambda}(\xi) &= \hat\sigma_{k,N-\lambda}(\xi)\prod_{N-\lambda<l\leqslant N}\varphi(2^{lk}\beta_{N-\lambda}\xi)\\ &\quad-\hat\sigma_{k,N-\lambda-1}(\xi)\prod_{N-\lambda-1<l\leqslant N}\varphi(2^{lk}\beta_{N-\lambda}\xi)\end{aligned}$$

for $k\in\mathbf{Z}$ and $\lambda=0,1,\cdots,N-1$, where we use the convention $\prod_{j\in\phi}a_j=1$.

Noting $\sum_{\lambda=0}^{N-1}\tau_{k,N-\lambda}=\sigma_{k,N}$, we have

$$T_pf=\sum_{k=-\infty}^{\infty}\sigma_{k,N}*f=\sum_{\lambda=0}^{N-1}\sum_{k=-\infty}^{\infty}\tau_{k,N-\lambda}*f.$$

Thus,

$$\|T_pf\|_{L^p(\mathbf{R}^n)}\leqslant\sum_{\lambda=0}^{N-1}\left\|\sum_{k=-\infty}^{\infty}\tau_{k,N-\lambda}*f\right\|_{L^p(\mathbf{R}^n)}.$$

Therefore, to prove Theorem 1, it suffices to show

$$\left\|\sum_{k=-\infty}^{\infty}\tau_{k,N-\lambda}*f\right\|_{L^p(\mathbf{R}^n)}\leqslant c\|f\|_{L^p(\mathbf{R}^n)} \tag{11}$$

for $\lambda=0,1,2,\cdots,N-1$ and $p\in\left(\frac{2+2\alpha}{1+2\alpha},2+2\alpha\right)$.

It is easy to see that $\tau_{k,N-\lambda}=0$ if $\beta_{N-\lambda}=0$. Thus without loss of generality, we assume $\beta_{N-\lambda}\neq 0$ for $\lambda=0,1,\cdots,N-1$.

By the definition of $\tau_{k,N-\lambda}$ and Lemma 2.2, it is easy to see

$$|\hat\tau_{k,N-\lambda}(\xi)|\leqslant c2^{(N-\lambda)k}|\beta_{N-\lambda}\xi|, \tag{12}$$

$$|\hat\tau_{k,N-\lambda}(\xi)|\leqslant c\left(\ln\left|2^{k(N-\lambda)}\xi\beta_{N-\lambda}\right|\right)^{-1-\alpha}, \tag{13}$$

$\lambda=0,1,\cdots,N-1$.

Also, by Lemma 2.1 and the definition of $\tau_{k,N-\lambda}$, we find for $\lambda=0,1,\cdots,N-1$,

$$\left\|\sup_{k\in\mathbf{Z}}|\tau_{k,N-\lambda}*f|\right\|_{L^p(\mathbf{R}^n)}\leqslant c\|f\|_{L^p(\mathbf{R}^n)} \tag{14}$$

for all $p \in (1,\infty)$.

It should be pointed out that the constants c in (12)–(14) are independent of the coefficients of P.

By (14), we obtain the following lemma in [9].

LEMMA 2.3. *([9] Theorem 7.5) Let* $p \in (1,\infty)$. *For arbitrary functions* q_k, *and* $\lambda = 0,1,\cdots,N-1$,

$$\left\| \left(\sum_{k=-\infty}^{\infty} |\tau_{k,N-\lambda} * q_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)} \leqslant c \left\| \left(\sum_{k=-\infty}^{\infty} |q_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{R}^n)}$$

where the constant c *is independent of the coefficients of the polynomial* P.

Now we are in the position to prove Theorem 1. As we mentioned before, it suffices to show (11). By duality, we may assume $p \in (2, 2+2\alpha)$.

Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $(2^{-(N-\lambda)j}\beta_{N-\lambda}^{-1},\ 2^{-(N-\lambda)j+1}\beta_{N-\lambda}^{-1})$. To be precise, we require the following:

$$\Phi_j \in C^\infty,\ \ 0 \leqslant \Phi_j \leqslant 1,\ \ \sum_j \Phi_j(t)^2 = 1,$$

$$\operatorname{supp}(\Phi_j) \subseteq \left(2^{-(N-\lambda)j-1}\beta_{N-\lambda}^{-1},\ 2^{-(N-\lambda)j+1}\beta_{N-\lambda}^{-1}\right).$$

Define the multiplier operators S_j in $\mathbf{R}^n$ by

$$(S_jf)^\wedge(\xi) = f^\wedge(\xi)\Phi_j(|\xi|).$$

We have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \tau_{k,N-\lambda} * f &= \sum_k \tau_{k,N-\lambda} * \left(\sum_j S_{j+k}S_{j+k}f \right) \\ &= \sum_j \left(\sum_k S_{j+k}(\tau_{k,N-\lambda} * S_{j+k}f) \right) = \sum_j I_jf.\end{aligned}$$

Thus

$$\left\| \sum_{k=-\infty}^{\infty} \tau_{k,N-\lambda} * f \right\|_p \leqslant \sum_{j=-\infty}^{\infty} \|I_jf\|_p.$$

From classical Littlewood-Paley theory and Lemma 2.3, we know that $\|I_jf\|_{p_o} \leqslant C\|f\|_{p_o}$ with C independent of j, for any $p_o \in (1,\infty)$. By the Plancherel theorem

$$\|I_jf\|_2^2 \leqslant C \sum_k \int_{E_{j+k,N-\lambda}} |\hat{f}(\xi)|^2\ |\hat{\tau}_{k,N-\lambda}(\xi)|^2 d\xi$$

where

$$E_{j+k,N-\lambda} = \left\{ \xi : 2^{-(N-\lambda)(j+k+1)}\beta_{N-\lambda}^{-1} \leqslant |\xi| < 2^{-(N-\lambda)(j+k-1)}\beta_{N-\lambda}^{-1} \right\}.$$

Thus by the estimate (ii) in Lemma 2.2, we obtain that for $j < 0$

$$\|I_j f\|_2^2 \leqslant c|j|^{-(\alpha+1)}\|\hat{f}\|_2^2,$$

which implies

$$\|I_j\|_{L^2\to L^2} \leqslant c|j|^{-\alpha-1}.$$

Noting $2 \leqslant p < 2+\alpha$, by interpolation between $\|I_j\|_{L^2\to L^2} \leqslant c|j|^{-\alpha-1}$ and $\|I_j\|_{L^{p_o}\to L^{p_o}} \leqslant c$ for any $p_o \in (1,\infty)$, we obtain a $\beta > 1$ such that

$$\|I_j f\|_{L^p(\mathbf{R}^n)} \leqslant c|j|^{-\beta}\|f\|_{L^p(\mathbf{R}^n)}.$$

Similarly, using (i) in Lemma 2.2, we can obtain a $\theta > 0$ such that

$$\|I_j f\|_{L^p(\mathbf{R}^n)} \leqslant c2^{-j\theta}\|f\|_{L^p(\mathbf{R}^n)}, \quad \text{if } j \geqslant 0.$$

Therefore, we have

$$\sum_{j=-\infty}^{\infty} \|I_j f\|_{L^p(\mathbf{R}^n)} \leqslant c\|f\|_{L^p(\mathbf{R}^n)}.$$

The theorem is proved.

3. Proof of Theorem 2

For any $\epsilon > 0$ there is an integer k such that $2^{k-1} \leqslant \epsilon < 2^k$. So we have

$$T_p^* f \leqslant \sigma_N^*(f) + \sup_{k\in\mathbf{Z}} \left| \sum_{j=k}^{\infty} \sigma_{j,N} * f \right|$$

Let $\{\tau_{k,N-\lambda}\}$ be the family defined in § 2, we know

$$\sup_{k\in\mathbf{Z}} \left| \sum_{j=k}^{\infty} \sigma_{j,N} * f \right| = \sup_{k\in\mathbf{Z}} \left| \sum_{\lambda=0}^{N-1} \sum_{j=k}^{\infty} \tau_{j,N-\lambda} * f \right|$$
$$\leqslant \sum_{\lambda=0}^{N-1} \sup_{k\in\mathbf{Z}} \left| \sum_{j=k}^{\infty} \tau_{j,N-\lambda} * f \right|.$$

By Lemma 2.1, $\|\sigma_N^* f\|_{L^p(\mathbf{R}^n)} \leqslant c\|f\|_{L^p(\mathbf{R}^n)}$ for any $p \in (1,\infty)$. So it suffices to show for any $\lambda = 1, 2, \cdots, N$

$$\left\| \sup_{k\in\mathbf{Z}} \left| \sum_{j=k}^{\infty} \tau_{j,\lambda} * f \right| \right\|_{L^p(\mathbf{R}^n)} \leqslant c\|f\|_{L^p(\mathbf{R}^n)}. \tag{15}$$

Take a radial function $\varphi \in S(\mathbf{R}^n)$ such that $\varphi(\xi) = 1$ when $|\xi| < 1$ and $\varphi(\xi) = 0$ when $|\xi| > 2^{\lambda}$. Let $\varphi_k(\xi) = \varphi(2^{k\lambda}|\beta_r\xi|)$ and let $\hat{\Phi}_k(\xi) = \varphi_k(\xi)$. Now let $J_k(f) = \sum_{j=k}^{\infty} \tau_{j,\lambda} * f$. Then

$$\begin{aligned} J_k(f) &= (\delta - \Phi_k) * \sum_{j=k}^{\infty} \tau_{j,\lambda} * f + \Phi_k * T_p(f) - \Phi_k * \sum_{j=-\infty}^{k-1} \tau_{j,\lambda} * f \\ &= J_{k,1}(f) + J_{k,2}(f) + J_{k,3}(f), \end{aligned}$$

where δ is the Dirac delta function. Let M be the standard Hardy-Littlewood maximal operator. By Theorem 1,

$$\left\| \sup_{k\in\mathbf{Z}} |J_{k,2}(f)| \right\|_p \leqslant c\, \|M(T_p f)\|_p \leqslant c\|f\|_p.$$

Next,

$$\begin{aligned} \sup_{k\in\mathbf{Z}} |J_{k,3}(f)| &= \sup_{k\in\mathbf{Z}} \left| \sum_{j=1}^{\infty} \tau_{k-j,\lambda} * \Phi_k * f \right| \\ &\leqslant \sum_{j=1}^{\infty} \left(\sup_{k\in\mathbf{Z}} |\tau_{k-j,\lambda} * \Phi_k * f| \right) = \sum_{j=1}^{\infty} G_j(f). \end{aligned}$$

By (14), we have $\|G_j(f)\|_p \leqslant c\|Mf\|_p \leqslant c\|f\|_p$ for any $p \in (1,\infty)$. On the other hand,

$$G_j(f) \leqslant \left\{ \sum_{k=-\infty}^{\infty} \left| \tau_{k-j,\lambda} * \Phi_k * f \right|^2 \right\}^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} \|G_j(f)\|_2^2 &\leqslant \sum_{k=-\infty}^{\infty} \int_{\mathbf{R}^n} |\hat{\tau}_{k-j,\lambda}(\xi)|^2 |\varphi_k(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\ &\leqslant c \sup_{\xi\neq 0} \sum_{k=-\infty}^{\infty} |\hat{\tau}_{k-j,\lambda}(\xi)|^2 \|f\|_2^2 \\ &\leqslant c \sup_{\xi\neq 0} \left(\sum_{2^{k\lambda} \leqslant |\xi|^{-1}|\beta_\lambda|^{-1} 2^{\lambda}} 2^{k\lambda} |\beta_r \xi| \right) 2^{-2j\lambda} \|f\|_2^2 \leqslant c 2^{-2j\lambda} \|f\|_2^2. \end{aligned}$$

Thus, by interpolation, we have

$$\left\| \sup_{k\in\mathbf{Z}} |J_{k,3}(f)| \right\|_p \leqslant c\|f\|_p.$$

Finally, we estimate $\sup_{k\in\mathbf{Z}}|J_{k,1}(f)|$. Similar to $J_{k,3}(f)$, we have

$$\sup_{k\in\mathbf{Z}}|J_{k,1}(f)| \leqslant \sum_{j=1}^{\infty}\left(\sup_{k\in\mathbf{Z}}|\tau_{k+j,\lambda}*(\delta-\Phi_k)*f|\right)$$
$$=\sum_{j=1}^{\infty}\Delta_j(f), \quad \text{and} \quad \|\Delta_j(f)\|_p \leqslant c\|f\|_p \text{ for any } p\in(1,\infty).$$

By Plancherel Theorem and the choice of Φ_k,

$$\begin{aligned}
\|\Delta_j(f)\|_2^2 &\leqslant \sum_{k=-\infty}^{\infty}\int_{|\xi|\geqslant|\beta_\lambda|^{-1}2^{-\lambda k}}|\hat{\tau}_{k+j,\lambda}(\xi)|^2|\hat{f}(\xi)|^2 d\xi\\
&= \sum_{k=-\infty}^{\infty}\sum_{i=-k}^{\infty}\int_{2^{i\lambda}\leqslant|\beta_\lambda\xi|\leqslant 2^{\lambda(i+1)}}|\hat{\tau}_{k+j,\lambda}(\xi)|^2|\hat{f}(\xi)|^2 d\xi\\
&\leqslant c\sum_{k=-\infty}^{\infty}\sum_{i=-k}^{\infty}\int_{2^{i\lambda}\leqslant|\beta_\lambda\xi|\leqslant 2^{\lambda(i+1)}}|\hat{f}(\xi)|^2\left(\ln|2^{(k+j)\lambda}\xi\beta_r|\right)^{-2-2\alpha}d\xi\\
&\leqslant c\sum_{k=-\infty}^{\infty}\sum_{i=-k}^{\infty}\left(\frac{1}{k+j+i}\right)^{2+2\alpha}\int_{2^{i\lambda}\leqslant|\beta_r\xi|\leqslant 2^{\lambda(i+1)}}|\hat{f}(\xi)|^2 d\xi\\
&= c\sum_{i=0}^{\infty}\sum_{k=-\infty}^{\infty}\left(\frac{1}{i+j}\right)^{2+2\alpha}\int_{2^{\lambda(i-k)}\leqslant|\beta_r\xi|\leqslant 2^{\lambda(i+1-k)}}|\hat{f}(\xi)|^2 d\xi\\
&\leqslant c\sum_{i=0}^{\infty}\left(\frac{1}{i+j}\right)^{2+2\alpha}\|f\|_2^2\\
&\leqslant cj^{-1-2\alpha}\|f\|_2^2.
\end{aligned}$$

Thus,

$$\|\Delta_j(f)\|_2 \leqslant cj^{-\alpha-\frac{1}{2}}\|f\|_2.$$

Since $p\in\left(\frac{1+2\alpha}{2\alpha}, 1+2\alpha\right)$ and $\|\Delta_j(f)\|_{p_o}\leqslant c\|f\|_{p_o}$ for any $p_o\in(1,\infty)$, one can find a $\theta>1$ via interpolation such that

$$\|\Delta_j(f)\|_p \leqslant cj^{-\theta}\|f\|_p,$$

which implies $\left\|\sup_k|J_{k,1}(f)|\right\|_p \leqslant c\|f\|_p$. This finishes the proof of Theorem 2.

REFERENCES

[1] A. J. AL-HASAN AND D. FAN, *L^p boundedness of a singular integral operator,* Canadian Math. Bulletin, to appear.
[2] A. P. CALDERÓN AND A. ZYGMUND, *On existence of certain singular integrals,* Acta. Math. **88** (1952), 85-129.
[3] A. P. CALDERÓN AND A. ZYGMUND, *On singular integrals,* Amer. J. Math. **18** (1956) 289-309.
[4] W. C. CONNETT, *Singular integrals near L^1*, Proc. Sympos. Pure Math. of Amer. Math. Soc. (S. Wainger and G. Weiss, eds.) **35**, I (1979), 163-165.
[5] L. COLZANI, *Hardy Spaces on Sphere,* Ph.D. Thesis, Washington University, St. Louis, MO, 1982.
[6] R. R. COIFMAN AND G. WEISS, *Extensions of Hardy spaces and their use in analysis,* Bull. Amer. Math. Soc. **83** (1977), 569-645.
[7] J. DUOANDIKOETXEA AND J. L. RUBIO DE FRANCIA, *Maximal and singular integral operators via Fourier transform estimates,* Invent. Math. **84** (1986), 541-561.
[8] D. FAN AND Y. PAN, *L^2 boundedness of a singular integral operator,* Publicacions Matemàtiques, **41** (1997), 317-333.
[9] D. FAN AND Y. PAN, *Singular integral with rough kernels supported by subvarieties,* Amer. J. Math. **119** (1997), 799-839.
[10] L. GRAFAKOS AND A. STEFANOV, *L^p bounds for singular integrals and maximal singular integrals with rough kernels,* Indiana Univ. Math. J., to appear.
[11] F. RICCI AND G. WEISS, *A characterization of $H^1(\sum_{n-1})$*, Proc. Sympos. Pure Math. of Amer. Math. Soc. (S. Wainger and G. Weiss, eds.), Vol. **35**, I (1979), 289-294.

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