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ON A NEW EQUIVALENCE OF COEFFICIENT CONDITIONS AND APPLICATIONS

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Dedicated to my Friend,
Professor Ferenc Gécseg
on his 60th *birthday*

(*communicated by J. Pečarić*)

Abstract. We show that if the sequence $\{\kappa_n\}$ is quasi geometrically increasing, then a so-called block-condition

$$\sum_{m=0}^{\infty} \kappa_m \Bigl(\sum_{n=\nu_m+1}^{\nu_{m+1}} |c_n|^q \Bigr)^{p/q} < \infty, \quad 0 < p < q,$$

for every $\{\nu_m\}$ is equivalent to the following two conditions

$$\sum_{n=1}^{\infty} |c_n|^q \mu_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \kappa_n \Bigl(\frac{\kappa_n}{\mu_{\nu_n+1}} \Bigr)^{\frac{p}{q-p}} < \infty,$$

where $\{\mu_n\}$ is a nondecreasing sequence.

Applications to absolute $|C,\alpha|$-summability of general orthogonal series are also presented.

1. Introduction. In the theory of orthogonal series several families of coefficient conditions are being utilized. Among them the three primarily used have the following structure:

$$\sum_{n=1}^{\infty} c_n^2 \rho_n < \infty, \tag{1.1}$$

$$\sum_{m=1}^{\infty} \kappa_m \Bigl(\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2 \Bigr)^{p/2} < \infty \tag{1.2}$$

and

$$\sum_{m=1}^{\infty} \alpha_m \Bigl(\sum_{n=m}^{\infty} c_n^2 \Bigr)^{p/2} < \infty, \tag{1.3}$$

where $p > 0$, $\rho := \{\rho_n\}$, $\alpha := \{\alpha_n\}$ and $\kappa := \{\kappa_n\}$ are certain monotone sequences of real numbers, $\nu := \{\nu_m\}$ is a subsequence of natural numbers and $c := \{c_n\}$ is a real

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coefficient sequence. We do believe that the reader knows plenty of results using one of the above conditions, but in any case in [5] there are cited several different theorems incorporating conditions (1.i). In the same paper we studied the relations between these conditions. Among others, we gave sufficient conditions for the equivalence of (1.2) and (1.3), moreover, we have analyzed the relation between (1.1) and (1.2).

V. Totik and I. Vincze [10] continued our investigations replacing the exponent 2 by a positive number q in (1.i), and gave necessary and sufficient conditions for the equivalences of the conditions generalized in that way.

In [8] Y. Okuyama and T. Tsuchikura proved that for a specific sequence κ and $p=1$ the condition (1.2) is equivalent to a condition of the type

$$\sum_{m=1}^{\infty} \beta_m \Big(\sum_{n=1}^{m} \gamma_n c_n^2\Big)^{1/2} < \infty \quad (\beta_n, \gamma_n > 0). \tag{1.4}$$

As we know, this was the first result verifying the equivalence between conditions of the type (1.2) and (1.4).

In [6] we proved a general equivalence theorem pertaining to the following conditions:

$$\sigma_1 := \sum_{m=0}^{\infty} \kappa_m \Big(\sum_{n=\nu_m+1}^{\nu_{m+1}} |c_n|^q\Big)^{p/q} < \infty \tag{1.5}$$

and

$$\sigma_2 := \sum_{m=1}^{\infty} \beta_m \Big(\sum_{n=1}^{m} \gamma_n |c_n|^q\Big)^{p/q} < \infty. \tag{1.6}$$

The equivalence of conditions (1.5) and (1.6) means that there exists a constant $K := K(\alpha,\beta,\gamma,\nu,p,q) > 0$ such that $K^{-1}\sigma_2 \leqslant \sigma_1 \leqslant K\sigma_2$ for any sequence $\{c_n\}$. In what follows K, K_i will denote absolute constants or constants depending only on parameters being irrelevant to the problem in question. The constants are not necessarily the same ones at which are different occurrences.

Since the equivalence of (1.5) with the conditions

$$\sum_{n=1}^{\infty} |c_n|^q \rho_n < \infty \tag{1.7}$$

and

$$\sum_{m=1}^{\infty} \alpha_m \Big(\sum_{n=m}^{\infty} |c_n|^q\Big)^{p/q} < \infty \tag{1.8}$$

is settled by Totik and Vincze, it follows that all the equivalences of conditions (1.5) – (1.8) are analyzed.

In [10], among others, we can read

THEOREM A. *If $p \neq q$ then (1.7) and (1.8) are equivalent if and only if the three sequences*

$$\{\rho_n\}, \quad \{1/\rho_n\} \quad \textit{and} \quad \Big\{\sum_{k=1}^{n} \alpha_k\Big\}$$

are bounded.

If $p = q$ then (1.7) and (1.8) are equivalent if and only if

$$K^{-1}\rho_n \leqslant \sum_{k=1}^{n} \alpha_k \leqslant K\rho_n \quad (n = 1, 2, \ldots)$$

is satisfied.

It is easy to see that these restrictions on the sequences $\{\rho_n\}$ and $\{\alpha_n\}$ are very rigorous, they claim nearly that the sequence ρ is "almost a positive constant", furthermore then both conditions (1.7) and (1.8) demand only that the sequence $\{c_n\}$ belongs to the space ℓ^q.

But in an old paper [3], improving a celebrated theorem of W. Orlicz [9] relating to the unconditional convergence of orthogonal series, we proved that the condition (1.8) with $p = 1$, $q = 2$ and $\alpha_m = 1/m$ is equivalent to the pair of following conditions:

$$\sum_{n=1}^{\infty} c_n^2 \rho_n < \infty \tag{1.9}$$

and

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{\rho_{2^{2^n}}} < \infty,$$

where $\rho := \{\rho_n\}$ is a nondecreasing sequence of positive numbers.

We emphasize that then the sequence ρ is not a nearly constant sequence as in Theorem A; on the contrary it tends to infinity.

Later we [4] generalized this equivalence statement as follows:

THEOREM B. *Condition*

$$\sum_{m=1}^{\infty} \frac{1}{\lambda_m} \Big(\sum_{n=m}^{\infty} c_n^2\Big)^{1/2} < \infty$$

holds if and only if there exists a nondecreasing sequence $\mu := \{\mu_n\}$ of positive numbers satisfying the conditions (1.9) with μ_n in place of ρ_n and

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n \mu_n} < \infty,$$

where $\Lambda_n := \sum_{k=1}^{n} \lambda_k^{-1}$ and $\lambda := \{\lambda_n\}$ is a monotone sequence of positive numbers.

This result was also utilized for problems in connection with orthogonal series.

Recently, in [7], we generalized Theorem B and proved the analogous pair of the new theorem in connection with the condition (1.6).

These theorems read as follows:

THEOREM C. *Let* $0 < p < q$, $\lambda := \{\lambda_n\}$, *and* $c := \{c_n\}$ *be a sequence of nonnegative numbers, furthermore let* $\Lambda_n := \sum_{k=1}^{n} \lambda_k$. *The condition*

$$\sum_{m=1}^{\infty} \lambda_m \Big(\sum_{n=m}^{\infty} c_n^q\Big)^{p/q} < \infty \tag{1.10}$$

holds if and only if there exists a nondecreasing sequence $\mu := \{\mu_n\}$ *of positive numbers satisfying conditions*

$$\sum_{n=1}^{\infty} c_n^q \mu_n < \infty \tag{1.11}$$

and

$$\sum_{n=1}^{\infty} \lambda_n \Big(\frac{\Lambda_n}{\mu_n}\Big)^{p/(q-p)} < \infty. \tag{1.12}$$

THEOREM D. *Let* $0 < p < q$, $\beta := \{\beta_n\}$ *and* $c := \{c_n\}$ *be a sequence of nonnegative numbers,* $\sum_{n=1}^{\infty} \beta_n < \infty$, *furthermore let* $B_n := \sum_{k=n}^{\infty} \beta_k$. *Then condition (1.6) with* $\gamma_n \equiv 1$ *holds if and only if there exists a nonincreasing sequence* $\mu := \{\mu_n\}$ *of positive numbers satisfying conditions (1.11) and*

$$\sum_{n=1}^{\infty} \beta_n \Big(\frac{B_n}{\mu_n}\Big)^{p/(q-p)} < \infty. \tag{1.13}$$

We underline that if $p > q$ then Theorems C and D are not valid universally.

The aim of the present paper is to establish a theorem similar to the last two theorems, but replacing the so-called rest-condition (1.10), and the head-condition (1.6), respectively, by the block-condition (1.5).

Considering the very strict restrictions on the sequences ρ and α appearing in Theorem A for the case $p \neq q$, we cannot expect that such a theorem can be given easily for arbitrary sequences κ and ν.

Now we can prove an equivalence theorem for quasi geometrically increasing sequence κ and for arbitrary ν. A monotone sequence $\omega := \{\omega_n\}$ of positive terms will be called quasi geometrically increasing if there exists a natural number N such that $\omega_{n+N} \geqslant 2\omega_n$ holds for every natural number n.

2. Our result reads as follows:

THEOREM 1. *Let* $0 < p < q$, $\kappa := \{\kappa_n\}$ *be a quasi geometrically increasing sequence,* $\nu := \{\nu_m\}$ *be a subsequence of natural numbers, and* $c := \{c_n\}$ *be a sequence of nonnegative numbers. The condition (1.5) holds if and only if there exists a nondecreasing sequence* $\mu := \{\mu_n\}$ *of positive numbers satisfying the conditions*

$$\sum_{n=1}^{\infty} c_n^q \mu_n < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \kappa_n \Big(\frac{\kappa_n}{\mu_{\nu_n+1}}\Big)^{\frac{p}{q-p}} < \infty. \tag{2.2}$$

REMARKS.

1. Our proof will show that the conditions (2.1) and (2.2) jointly imply (1.5) for any sequence κ of positive terms. Here we shall not use the assumption that κ is quasi geometrically increasing.
2. We also stress that without any additional requirement on the sequence κ the equivalence given in Theorem 1 does not hold. This can be demonstrated by the following simple example.

Let $p = 1$, $q = 2$, $\kappa_m := \log m$, $\nu_m := m$ and

$$c_n := \begin{cases} m^{-3} & \text{if} \quad n = 2^m, \\ 0 & \text{otherwise.} \end{cases}$$

Then (1.5) is satisfied, but (2.1) and (2.2) cannot be fulfilled simultaneously. Namely, then with a nondecreasing sequence $\{\mu_n\}$ the conditions

$$\sum_{m=1}^{\infty} m^{-6}\mu_{2^m} < \infty$$

and

$$\sum_{m=2}^{\infty} \frac{2^m m^2}{\mu_{2^m+1}} \leqslant \sum_{n=2}^{\infty} \frac{\log^2 n}{\mu_{n+1}} < \infty$$

yield a trivial contradiction.

3. Proof. First we show that the conditions (2.1) and (2.2) jointly imply (1.5). Namely, using Hölder's inequality, we have

$$\sum_{m=0}^{\infty} \kappa_m \Big(\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^q\Big)^{p/q} \leqslant$$

$$\leqslant \Big\{\sum_{m=0}^{\infty}\Big(\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^q\Big)\mu_{\nu_m+1}\Big\}^{p/q}\Big\{\sum_{m=0}^{\infty} \kappa_m^{\frac{q}{q-p}} \mu_{\nu_m+1}^{\frac{p}{p-q}}\Big\}^{1-\frac{p}{q}} \leqslant$$

$$\leqslant \Big\{\sum_{n=1}^{\infty} c_n^q \mu_n\Big\}^{p/q}\Big\{\sum_{m=0}^{\infty} \kappa_m \Big(\frac{\kappa_m}{\mu_{\nu_m+1}}\Big)^{\frac{p}{q-p}}\Big\}^{1-\frac{p}{q}}.$$

Hence it is obvious that (2.1) and (2.2) imply (1.5).

As we have stated in our Remark 1, we underline that this part of the proof does not require the assumption that κ is quasi geometrically increasing, it holds for any positive sequence.

Before starting the opposite part of the proof we note that the following inequality

$$\sum_{n=1}^{m} \kappa_n \leqslant K\kappa_m \tag{3.1}$$

holds for all m, subsequent to the fact that κ is a quasi geometrically increasing sequence (see e.g. [6, Lemma 1]).

In order to prove that if (1.5) is satisfied then there exists a monotone sequence μ satisfying (2.1) and (2.2) simultaneously we distinguish two cases. If only a finite number of c_n is positive, the assertion (2.1) is trivial for any μ, and it is enough and easy to find a monotone increasing sequence μ such that (2.2) hold, too.

Therefore, we can assume that the terms

$$C_{\nu_k} := \Big(\sum_{n=\nu_k+1}^{\infty} c_n^q\Big)^{\frac{p-q}{q}}$$

are all positive, creating thus a nondecreasing sequence.

Now, we define μ as follows:

For $\nu_m < n \leqslant \nu_{m+1}$, let

$$\mu_n := \kappa_m C_{\nu_m}, \quad m = 0, 1, \dots$$

It is plain that this sequence $\mu := \{\mu_n\}$ is nondecreasing.

Next, we show that with such μ (2.1) holds if (1.5) is satisfied. Namely, by $p < q$ we can use the so-called power-sum inequality (see e.g. [1], p. 28), later the inequality (3.1), and finally (1.5), and thus we have

$$\begin{aligned}\sum_{n=\nu_0+1}^{\infty} c_n^q \mu_n &= \sum_{m=0}^{\infty} \sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^q \mu_n = \\ &= \sum_{m=0}^{\infty} \kappa_m C_{\nu_m} \sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^q \leqslant \\ &\leqslant \sum_{m=0}^{\infty} \kappa_m \Big(\sum_{n=\nu_m+1}^{\infty} c_n^q\Big)^{p/q} \leqslant \\ &\leqslant \sum_{m=0}^{\infty} \kappa_m \sum_{i=m}^{\infty} \Big(\sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^q\Big)^{p/q} = \\ &= \sum_{i=0}^{\infty} \Big(\sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^q\Big)^{p/q} \sum_{m=0}^{i} \kappa_m \leqslant \\ &\leqslant K \sum_{i=0}^{\infty} \kappa_i \Big(\sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^q\Big)^{p/q} < \infty.\end{aligned} \tag{3.2}$$

Herewith the implication (1.5) $\Rightarrow$ (2.1) is verified.

Similar arguing delivers the implication (1.5) $\Rightarrow$ (2.2). Using the definition of μ we get

$$\sum_{n=1}^{\infty} \kappa_n \Big(\frac{\kappa_n}{\mu_{\nu_n+1}}\Big)^{\frac{p}{q-p}} = \sum_{n=1}^{\infty} \kappa_n \Big(\sum_{n=\nu_n+1}^{\infty} c_n^q\Big)^{p/q} =: S_1,$$

and under (3.2) we have

$$S_1 \leqslant K \sum_{i=0}^{\infty} \kappa_i \Big(\sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^q\Big)^{p/q}.$$

Thus the proof of $(1.5) \Rightarrow (2.2)$ is also complete.

We remark that our proof also shows that the sum in (1.5) can be majorized by the product of the sums appearing in (2.1) and (2.2) endowed with exponents p/q and $1-\frac{p}{q}$, respectively.

Furthermore, at least in the elaborated case, the sums in (2.1) and (2.2) do not exceed the sum of (1.5) multiplied by a suitable constant which depends only on the sequence κ.

4. Applications. Utilizing our equivalence theorem, we could present several new sufficient conditions in pair for general orthogonal series, since there exist a lot of block-type conditions implying desired properties for the general orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x), \quad x \in (0,1). \tag{4.1}$$

We present and prove only three sample results.

THEOREM 2. *If* $0 \leqslant \alpha < \frac{1}{2}$ *and there exists a monotone sequence* $\mu := \{\mu_n\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty} c_n^2 \mu_n < \infty \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} \frac{2^{n(1-2\alpha)}}{\mu_{2^n}} < \infty, \tag{4.3}$$

then the orthogonal series (4.1) is $|C,\alpha|$*-summable almost everywhere in (0,1).*

Proof. Theorem 1 with $p=1$, $q=2$, $\nu_n = 2^n$ and $\kappa_n = 2^{\frac{n}{2}(1-2\alpha)}(0 \leqslant \alpha < \frac{1}{2})$ implies that the conditions (4.2) and (4.3) are equivalent to

$$\sum_{m=0}^{\infty} 2^{\frac{m}{2}(1-2\alpha)} \Big\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \Big\}^{1/2} < \infty. \tag{4.4}$$

In [2] (see Satz II) we have proved that (4.4) implies the $|C,\alpha|$-summability of the series (4.1) almost everywhere in (0,1) $(0 \leqslant \alpha < 1/2)$.

Thus Theorem 2 is proved.

Considering our Remark 1 we can present two more theorems using only the first part of Theorem 1.

THEOREM 3. *If there exists a monotone sequence* $\mu := \{\mu_n\}$ *of positive numbers such that (4.2) holds and*

$$\sum_{m=1}^{\infty} \frac{m}{\mu_{2^m}} < \infty, \tag{4.5}$$

then the orthogonal series (4.1) is $|C,\frac{1}{2}|$ *-summable almost everywhere in (0,1).*

Proof. The conditions (4.2) and (4.5) imply

$$\sum_{m=1}^{\infty}\sqrt{m}\Big\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2\Big\}^{1/2}<\infty \tag{4.6}$$

by Theorem 1 with $p=1$, $q=2$, $\nu_m=2^m$ and $\kappa_m=\sqrt{m}$ (see also our Remark 1).

In [2] it is proved that (4.6) implies the $|C,\frac{1}{2}|$-summability of (4.1) almost everywhere (see Satz II), thus we have the proof.

THEOREM 4. *If* $\alpha>1/2$ *and there exists a monotone sequence* μ *of positive numbers such that (4.2) is fulfilled and*

$$\sum_{m=1}^{\infty}\frac{1}{\mu_{2^m}}<\infty, \tag{4.7}$$

then the orthogonal series (4.1) is $|C,\alpha|$ *-summable almost everywhere in (0,1).*

We remark that Theorem 4 is a slight improvement of Theorem 3 given in [7].

We leave out the proof because it follows the line of our previous proof with the modifications that $\kappa_m=1$; the sufficiency of the condition

$$\sum_{m=1}^{\infty}\Big\{\sum_{n=2^m+1}^{2^{m+1}}c_n^2\Big\}^{1/2}<\infty$$

for the $|C,\alpha|$-summability is proved by "Satz I" in [2].

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