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CACCIOPPOLI'S INEQUALITY FOR QUASILINEAR ELLIPTIC OPERATORS

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Abstract. We generalize Caccioppoli's inequality to the setting of quasilinear operators of Leray - Lions type, including arbitrary open subsets in $\mathbf{R}^N$, $N \geqslant 1$, instead of balls, and we obtain an explicit value of the constant on the right-hand side. The best Caccioppoli constant for the p-Laplacian is $\leqslant p^p$.

1. Introduction

Caccioppoli's inequality represents an important tool in the study of qualitative properties of solutions of elliptic partial differential equations and elliptic systems, see e.g. [3], [6], [5], [7]. We generalize Caccioppoli's inequality in three directions: first, we formulate it for quasilinear elliptic operators of divergence type, second, we deal with arbitrary open subsets of Ω instead of balls, and third, we obtain an explicit value of the constant appearing on the right hand side of the inequality.

We first formulate a variant of the classical Caccioppoli inequality, as in [6] (see also [3], Proposition III.2.1, and Remark 2.1 there). Let Ω be an open subset of $\mathbf{R}^N$, $N \geqslant 1$, L a linear differential operator of the second order defined by

$$Lu = -\sum_{i,j=1}^{N} D_i\,[a_{ij}(x)\,D_j u], \tag{1}$$

where $D_i = \frac{\partial}{\partial x_i}$, $u \in W^{1,2}_{loc}(\Omega)$, $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, and

$$W^{1,2}_{loc}(\Omega) = \{f \in L^2_{loc}(\Omega)\ \varphi f \in W^{1,2}(\Omega),\ \forall \varphi \in C_0^\infty(\Omega)\}.$$

For general definition of Sobolev spaces see e.g. [4] or [10]. We assume L to be uniformly elliptic, that is, there exist two positive constants α and β such that for a.e. $x \in \Omega$,

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leqslant \beta|\xi|^2. \tag{2}$$

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This inequality is equivalent to saying that the spectrum of the symmetric matrix $(a_{ij}(x))_{i,j=1,\dots,n}$ lies in the interval $[\alpha,\beta]$ for a.e. $x \in \Omega$.

We say that a function $u \in W^{1,2}_{loc}(\Omega)$ is an L-*subsolution*, or subsolution of the problem

$$Lv = 0 \quad \text{in } \mathscr{D}'(\Omega), \tag{3}$$

if $Lu \leqslant 0$ in the weak sense, i.e.

$$\sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x)\, D_i u\, D_j \varphi\, dx \leqslant 0 \quad \forall \varphi \in \mathscr{D}^{+}(\Omega), \tag{4}$$

where $\mathscr{D}'(\Omega)$ is the space of distributions, and $\mathscr{D}^{+}(\Omega)$ is the cone of all nonnegative test functions defined on Ω. Now we formulate Caccioppoli's result, see e.g. [6]:

THEOREM 1. *(Caccioppoli) Under the above conditions we have that each nonnegative subsolution u of (3) satisfies the following inequality:*

$$\int_{B_R(x_1)} |\nabla u|^2 dx \leqslant \frac{C}{R^2} \int_{B_{2R}(x_1)} u^2\, dx, \tag{5}$$

where x_1 and $R > 0$ are such that $\overline{B_{2R}(x_1)} \subset \Omega$, and C is a constant independent of x_1, R, and u.

We call the constant C the Caccioppoli constant. Our aim is to find its explicit value. The proof of this result presented in [6] requires in essential way that the matrix $A(x) = (a_{ij}(x))$ be symmetric. As we shall see in Theorem 3, it is possible to drop this condition.

2. Caccioppoli's inequality for Leray-Lions operators

Let Ω is an open, possibly unbounded set in $\mathbf{R}^N$, $N \geqslant 1$, $1 < p < \infty$, $p' = p/(p-1)$, and let $a(x,\eta,\xi) : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory vector function (i.e. measurable with respect to x for all ξ, η, and continuous with respect to η and ξ for a.e. x) satisfying the conditions of Leray - Lions type:

$$\exists \alpha > 0,\ a(x,\eta,\xi)\cdot\xi \geqslant \alpha|\xi|^p,\ \text{a.e. } x \in \Omega,\ \eta \in \mathbf{R},\ \xi \in \mathbf{R}^N, \tag{6}$$

$$\begin{cases} \exists c_1 \geqslant 0,\ \exists c_2 > 0,\ \exists h \in L^{p'}_{loc}(\Omega),\ \forall \eta \in \mathbf{R},\ \forall \xi \in \mathbf{R}^N, \\ |a(x,\eta,\xi)| \leqslant h(x) + c_1|\eta|^{p-1} + c_2|\xi|^{p-1} \text{ a.e. in } \Omega. \end{cases} \tag{7}$$

Let us define the quasilinear operator of divergence type, or Leray-Lions operator, by $Lu = -\operatorname{div} a(x,u,\nabla u)$, $u \in W^{1,p}_{loc}(\Omega)$. We say that $u \in W^{1,p}_{loc}(\Omega)$ is an L-subsolution if $Lu \leqslant 0$ in the weak sense, that is,

$$\int_{\Omega} a(x,u,\nabla u)\cdot\nabla\varphi\, dx \leqslant 0 \quad \forall \varphi \in \mathscr{D}^{+}(\Omega). \tag{8}$$

We also define the nonnegative part of u by $u^{+} = \max\{u,0\}$. As is well known, if u is an L-subsolution, so is u^{+} (see Stampacchia [8], Theorem II.6.6, whose result can easily be seen to hold for Leray-Lions operators too). Here is the main result.

THEOREM 2. *Let A be an open subset such that $A \subset\subset \Omega$, $0 < r < d(A, \partial\Omega)$, $m =$ the number of nonzero elements in the set $\{h|_{A_r}, c_1, c_2\}$, $0 < \theta < \frac{\alpha p'}{pm^{p'-1}c_2^{p'}}$. Then for any L-subsolution $u \in W^{1,p}_{loc}(\Omega)$ we have the following Caccioppoli type inequality:*

$$\int_A |\nabla u^+|^p \, dx \leqslant C(\theta) \int_{A_r} [h(x)^{p'} + c_1^{p'} |u^+|^p] \, dx + \frac{D(\theta)}{r^p} \int_{A_r \setminus A} |u^+|^p \, dx, \tag{9}$$

where the constants $C(\theta)$ and $D(\theta)$ have explicit values:

$$C(\theta) = \frac{\theta p m^{p'-1}}{\alpha p' - \theta p m^{p'-1} c_2^{p'}}, \quad D(\theta) = \frac{p' \theta^{p-1}}{\alpha p' - \theta p m^{p'-1} c_2^{p'}}. \tag{10}$$

The crucial role in the proof of Theorem 2 is played by the following localization result, see [9].

LEMMA 1. *(smooth localization of measurable subsets) Let Ω be an open subset of $\mathbf{R}^N$. Assume that A is a measurable subset of Ω and $r > 0$ such that $A_r \subseteq \Omega$, where A_r is r-neighbourhood of A. Then for any $c_0 > 1$ there exists a function $\Phi \in C^\infty(\Omega)$ such that*

$$0 \leqslant \Phi \leqslant 1, \tag{11}$$

$$\Phi = 0 \quad \text{on } \Omega \setminus A_r, \quad \Phi = 1 \quad \text{on } A, \tag{12}$$

$$|\nabla\Phi| \leqslant \tfrac{c_0}{r}. \tag{13}$$

The novelty in the above lemma is that we can take the constant c_0 arbitrarily close to 1. It is easy to see by an example that the condition $c_0 > 1$ cannot be improved.

Proof of Theorem 2. Since u^+ is also a subsolution, we can assume without loss of generality that $u \geqslant 0$ on Ω. Let $c_0 > 1$ be given. Assume that Φ is as in Lemma 1. Then we have $\varphi = u\Phi^p \in W^{1,p}_0(\Omega)$, and $\varphi \geqslant 0$. It will be convenient to denote $M = m^{p'-1}$. Then $Lu \leqslant 0$ implies

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla u \, \Phi^p \, dx \leqslant -p \int_\Omega a(x, u, \nabla u) u \Phi^{p-1} \nabla\Phi \, dx. \tag{14}$$

Using ellipticity of L, see (6), and Young's inequality we obtain that

$$\begin{aligned} \alpha \int_{A_r} |\nabla u|^p \Phi^p \, dx &\leqslant p \int_\Omega [h(x) + c_1|u|^{p-1} + c_2|\nabla u|^{p-1}] \Phi^{p-1} \theta^{\frac{1}{p'}} \cdot \theta^{-\frac{1}{p'}} u |\nabla\Phi| \, dx \\ &\leqslant \frac{p\theta M}{p'} \int_{A_r} [h(x)^{p'} + c_1|u|^p + c_2^{p'} |\nabla u|^p] \Phi^p \, dx \\ &\quad + \theta^{-p/p'} \int_{A_r} |u|^p |\nabla\Phi|^p dx. \end{aligned}$$

Exploiting properties of Φ from Lemma 1 we arrive to

$$\begin{aligned} \left(\alpha - \frac{p\theta M c_2^{p'}}{p'}\right) \int_A |\nabla u|^p dx &\leqslant \frac{p\theta M}{p'} \int_{A_r} [h(x)^{p'} + c_1|u|^p] \, dx + \\ &\quad + \frac{\theta^{-p/p'} c_0^p}{r^p} \int_{A_r \setminus A} |u|^p dx. \end{aligned}$$

Since this inequality is independent of Φ, we can send $c_0 \to 1$, and the claim follows.
Q.E.D.

COROLLARY 1. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a $-\Delta_p$-subsolution, where*

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$$

is the p-Laplacian. If A is an open subset of Ω such that $A \subset\subset \Omega$, $0 < r < d(A, \partial\Omega)$, then

$$\int_A |\nabla u^+|^p dx \leqslant \frac{p^p}{r^p} \int_{A_r \setminus A} |u^+|^p dx. \tag{15}$$

Proof. We apply Theorem 2 with $h(x) \equiv 0$, $c_1 = 0$, $\alpha = c_2 = m = 1$. Using simple calculus we see that in this case the minimal value of $D(\theta)$ is attained for $\theta = 1/p$, and the claim follows since $D(1/p) = p^p$. Q.E.D.

Remark. In particular, if $p = 2$ we obtain that the Caccioppoli constant for $-\Delta$-subsolutions (i.e. subharmonic functions) is equal to 4. Note that we do not claim that the Caccioppoli constant p^p in this corollary is optimal. In fact, we shall see that in one-dimensional case the best Caccioppoli constant related to subsolutions of $-v'' = 0$ is equal to $2\sqrt{3} - 3$, see Theorem 4.

It is interesting to compare the above result with that due to Heinonen, Kilpeläinen, and Martio [5], see also [7]. They consider the case of the Leray-Lions operator L of the form $Lu = -\,\mathrm{div}\, a(x, \nabla u)$ satisfying $h(x) \equiv 0$ and $c_1 = 0$ in (7), which also includes p-Laplacian as a special case. They showed that if $u \in W^{1,p}_{loc}(\Omega)$ is an L-subsolution, and $\Phi \in C_0^\infty(\Omega)$, then

$$\int_\Omega |\nabla u^+|^p |\Phi|^p dx \leqslant C \int_\Omega |u^+|^p |\nabla \Phi|^p dx. \tag{16}$$

Using Lemma 1, it is easy to see that for any L-subsolution $u \in W^{1,p}_{loc}(\Omega)$, and with A, r as in the above corollary, we have that with the same C

$$\int_A |\nabla u^+|^p dx \leqslant \frac{C}{r^p} \int_{A_r \setminus A} |u^+|^p dx. \tag{17}$$

COROLLARY 2. *Let $a > 0$, $r > 0$, and assume that $u \in W^{1,p}(B_{a+r}(0))$ is nonnegative and radially symmetric. If u is an $-\Delta_p$-subsolution, then for $v(\rho) = u(x)$, $\rho = |x|$, we have*

$$\int_0^a |v'(\rho)|^p \rho^{N-1}\, d\rho \leqslant \frac{p^p}{r^p} \int_a^{a+r} v(\rho)^p \rho^{N-1}\, d\rho.$$

It is possible to obtain a variant of Theorem 2 for quasilinear elliptic systems of divergence type, see [2] and [3] for similar problems. Assume that $a^j(x, \eta, \xi) : \Omega \times \mathbf{R}^n \times \mathbf{R}^{Nn} \to \mathbf{R}^N$ are Carathéodory vector functions, $j = 1, \ldots, n$, where $\eta = (\eta_1, \ldots, \eta_n)$, $\xi = (\xi^1, \ldots, \xi^n)$, $\xi^j \in \mathbf{R}^N$. We denote $u = (u_1, \ldots, u_n)$, $Du = (\nabla u_1, \ldots, \nabla u_n)$. Assume that $1 < p_j < \infty$ for all j, and let the ellipticity condition be fulfilled, i.e. there exists $\alpha > 0$ such that:

$$\sum_{j=1}^n a^j(x, \eta, \xi) \cdot \xi_j \geqslant \alpha \sum_{j=1}^n |\xi^j|^{p_j}, \ \text{a.e. } x \in \Omega,\ \eta \in \mathbf{R}^n,\ \xi \in \mathbf{R}^{Nn}, \tag{18}$$

$$\left\{\begin{array}{l}\exists c_{jk} \geqslant 0,\ \exists d_{jk} \geqslant 0,\ \exists h_j \in L^{p_j'}(\Omega),\ \forall j,\ \forall \eta \in \mathbf{R}^n,\ \forall \xi \in \mathbf{R}^{Nn}, \\ |a^j(x,\eta,\xi)| \leqslant h_j(x) + \sum_{k=1}^n c_{jk}|\eta_k|^{p_k/p_j'} + d_{jk}|\xi_k|^{p_k/p_j'} \text{ a.e. in } \Omega.\end{array}\right. \tag{19}$$

Consider the following quasilinear elliptic system of n equations, with unknwons $u_j \in W^{1,p_j}(\Omega)$, analogous to the system (3.1) in [2]:

$$-\operatorname{div} a^j(x,u,Du) = -\operatorname{div} f_j \quad \text{in } \mathscr{D}'(\Omega),\ j = 1,\dots,n, \tag{20}$$

where $f_j \in L^{p_j'}(\Omega)$ are given. Then similarly as in the proof of Theorem 2 we can conclude that for any solution $u = (u_j)$, $u_j \in W^{1,p_j}(\Omega)$, $A \subset\subset \Omega$ open, $0 < r < d(A,\partial\Omega)$, the following Caccioppoli type estimate holds:

$$\begin{aligned}\sum_{j=1}^n \int_A |\nabla u_j|^{p_j} dx \ \leqslant\ & \sum_{j=1}^n \left(\int_{A_r} C_j[h_j(x)^{p_j'} + |u_j|^{p_j}]\, dx + \frac{D_j}{r^{p_j}} \int_{A_r\setminus A} |u_j|^{p_j} dx \right.\\ & \left. + E_j \int_{A_r} |f_j|^{p_j'} dx \right).\end{aligned}$$

Here C_j, D_j, E_j are nonnegative constants depending on the coefficients that describe the behaviour of vector functions $a_1,\dots,a_n$, and on an auxilliary constant $\theta > 0$, analogously as in Theorem 2. All these constants can be expressed explicitly. Furthermore, if $h_j \equiv 0$ and $c_{jk} = 0$ for all j,k, then we can take $C_j = 0$.

3. The case of uniformly elliptic operators

Now we would like to obtain the value of the Caccioppoli constant when L is a quasilinear differential operator of the second order defined by

$$Lu = -\sum_{i,j=1}^N D_i\,[a_{ij}(x,u,\nabla u)\,D_j u]. \tag{21}$$

We assume L to be uniformly elliptic, that is, $a_{ij}(x,\eta,\xi)$ are Carathéodory functions defined on $\Omega \times \mathbf{R} \times \mathbf{R}^N$ (measurable with respect to x for all (η,ξ) and continuous with respect to (ξ,η) for a.e. x), essentially bounded with respect to (x,η,ξ), and there exists a positive constant α such that for a.e. $x \in \Omega$ and for all $(\eta,\xi) \in \mathbf{R}\times\mathbf{R}^N$,

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^N a_{ij}(x,\eta,\xi)\xi_i\xi_j. \tag{22}$$

THEOREM 3. *Let L be uniformly elliptic, quasilinear differential operator of second order, u an L-subsolution, and A an arbitrary open subset of Ω such that $A \subset\subset \Omega$, and $0 < r < d(A,\partial\Omega)$.*

(a) If $a_0 > 0$ is such that $|a_{ij}(x,\eta,\xi)| \leqslant a_0$ for all i,j,η,ξ, and a.e. x, then the following inequality holds:

$$\int_A |\nabla u^+|^2 dx \leqslant \frac{4N^2a_0^2}{\alpha r^2} \int_{A_r\setminus A} |u^+|^2 dx. \tag{23}$$

(b) If the matrix $\mathbf{A}(x,\eta,\xi)=(a_{ij}(x,\eta,\xi))$ *is symmetric for a.e.* x *and all* η *and* ξ*, and the right hand side of (22) is* $\leqslant \beta|\xi|^2$*, then*

$$\int_A |\nabla u^+|^2 dx \leqslant \frac{4\beta}{\alpha r^2}\int_{A_r\setminus A}|u^+|^2 dx. \tag{24}$$

Remark. Note that with $A=B_R(x_1)$ and $r=R<\frac{1}{2}d(x_1,\partial\Omega)$ we obtain an improvement of the classical Caccioppoli inequality (5).

Proof. (a) We have that the operator L is of divergence type with $a_i(x,\eta,\xi)=\sum_{j=1}^N a_{ij}(x,\eta,\xi)\xi_j$. It is easy to see that $|a(x,\eta,\xi)|\leqslant Na_0|\xi|^2$, and the claim follows from Theorem 2 with $p=2$, $h(x)\equiv 0$, $c_1=0$, and $c_2=Na_0$.

(b) In the case when the matrix $\mathbf{A}=\mathbf{A}(x,\eta,\xi)$ is symmetric we can proceed as in the proof of the classical Cacciopoli inequality, using also Lemma 1. We include the proof for the sake of completeness. Let $c_0>1$ be arbitrary, and let Φ be a smooth localization of A from our Lemma 1. Since Φ vanishes on the boundary of Ω, we have that $\varphi=u\Phi^2\in W_0^{1,2}(\Omega)$, $\varphi\geqslant 0$. As u is subsolution, using (4) we obtain

$$\sum_{i,j=1}^N\int_\Omega a_{ij}(x,u,\nabla u)\,D_iu\,D_j(u\Phi^2)\,dx\leqslant 0,$$

that is,

$$K=\sum_{i,j=1}^N\int_\Omega a_{ij}\,D_iu\,D_ju\,\Phi^2\,dx\leqslant -2\sum_{i,j=1}^N\int_\Omega a_{ij}\,(\Phi\,D_iu)\,(u\,D_j\Phi)\,dx.$$

Since the matrix $\mathbf{A}$ is positively semidefinite, we can use the following well known variant of the Cauchy inequality (here we need $\mathbf{A}$ to be symmetric):

$$|\langle\mathbf{A}\xi(x),\zeta(x)\rangle|\leqslant\langle\mathbf{A}\xi(x),\xi(x)\rangle^{1/2}\langle\mathbf{A}\zeta(x),\zeta(x)\rangle^{1/2}$$

with $\mathbf{A}=(a_{ij}(x,u,\nabla u))_{i,j}$, $\xi(x)=(\Phi\,D_iu)_i$, $\zeta(x)=(u\,D_j\Phi)_j$, so that

$$\begin{aligned}
K &\leqslant 2\int_\Omega\langle\mathbf{A}\xi(x),\xi(x)\rangle^{1/2}\langle\mathbf{A}\zeta(x),\zeta(x)\rangle^{1/2}\,dx\\
&\leqslant 2\left(\int_\Omega\langle\mathbf{A}\xi(x),\xi(x)\rangle\,dx\right)^{1/2}\left(\int_\Omega\langle\mathbf{A}\zeta(x),\zeta(x)\rangle\,dx\right)^{1/2}\\
&= 2\left(\sum_{i,j=1}^N\int_\Omega a_{ij}\,D_iu\,D_ju\,\Phi^2\,dx\right)^{1/2}\left(\sum_{i,j=1}^N\int_\Omega a_{ij}\,D_i\Phi\,D_j\Phi\,u^2\,dx\right)^{1/2}.
\end{aligned}$$

Taking into account the condition of uniform ellipticity and properties of Φ in Lemma 1 we obtain

$$\begin{aligned}\alpha \int_A |\nabla u|^2 dx &\leqslant \sum_{i,j} \int_\Omega a_{ij}\, D_i u\, D_j u\, \Phi^2\, dx \\ &\leqslant 4 \sum_{i,j} \int_\Omega a_{ij}\, D_i\Phi\, D_j\Phi\, u^2\, dx \\ &\leqslant 4\beta \int_\Omega |\nabla\Phi|^2 u^2\, dx \\ &\leqslant \frac{4\beta c_0^2}{r^2} \int_{A_r\setminus A} u^2 dx, \quad \forall c_0 > 1.\end{aligned}$$

We can let $c_0 \to 1$, and the claim is proved. Q.E.D.

Remark. It is interesting to note that, while the p-Laplacian is elliptic in the sense of Leray-Lions for any $p > 1$, it is not elliptic in the sense of (22), except for $p = 2$. However it is elliptic in the sense of ellipticity introduced in Gilbarg and Trudinger [4] (see relation 10.2 there). Indeed, we have that $\Delta_p u = \sum_{i,j} \overline{a}_{ij}(\nabla u) D_{ij} u$, where $\overline{a}_{ij}(\xi) = |\xi|^{p-2}\delta_{ij} + (p-2)|\xi|^{p-4}\xi_i\xi_j$. Therefore

$$S = \sum_{i,j=1}^{N} \overline{a}_{ij}(\xi)\, z_i z_j = |\xi|^{p-4}[|\xi|^2|z|^2 + (p-2)(\xi\cdot z)^2] \geqslant 0. \tag{25}$$

For $p \geqslant 2$ we have $|\xi|^{p-2}|z|^2 \leqslant S \leqslant (p-1)|\xi|^{p-2}|z|^2$, i.e. $-\Delta_p$ is elliptic in the sense of Gilbarg and Trudinger in the region $U = \Omega\times\mathbf{R}\times\mathbf{R}^N$, and for $1 < p < 2$ we have $(p-1)|\xi|^{p-2}|z|^2 \leqslant S \leqslant |\xi|^{p-2}|z|^2$, i.e. we have ellipticity condition in the region $U = \Omega\times\mathbf{R}\times(\mathbf{R}^N\setminus\{0\})$.

Now let us consider the second order operator Q defined on $W^{1,p}_{loc}(\Omega)$ by

$$Qu = -\operatorname{div}[(1+|\nabla u|^2)^{\frac{p}{2}-1}\nabla u], \quad 1 < p < \infty. \tag{26}$$

Then Q is elliptic for all p in the sense of (22), see also Chapter 10 in [4]. However, Q is elliptic in the sense of Leray-Lions only for $p \geqslant 2$. Indeed, if $p \geqslant 2$ then it is easy to see that $a(\xi)\cdot\xi \geqslant |\xi|^p$, $|a(\xi)| \leqslant 2^{p/2-2} + 2^{p/2-1}|\xi|^{p-1}$, where $a(\xi) = (1+|\xi|^2)^{\frac{p}{2}-1}\xi$.

On the other hand, if $p \in (1,2)$ then the ellipticity condition for Q in the sense of Leray-Lions is not fulfilled. To see this, note that

$$a(\xi)\cdot\xi = \left(\frac{t^{\frac{2}{2-p}}}{1+t}\right)^{1-p/2},$$

where we denote $t = |\xi|^2$. If we had $\alpha, d > 0$ such that $t^{\frac{2}{2-p}}/(1+t) \geqslant \alpha t^d$ for all $t > 0$, then having in mind that the order of the left-hand side is $t^{\frac{2}{2-p}}$ near $t = 0$ and $t^{\frac{p}{2-p}}$ near $t = \infty$, this would imply $d \geqslant \frac{2}{2-p}$ and $d \leqslant \frac{p}{2-p}$, which is impossible for $1 < p < 2$.

The operator Q appears in many physical situations, see [4], Chapter 10. Since for $p \geqslant 2$ we have $h(x) \equiv 2^{\frac{p}{2}-2}$, $c_1 = 0$, $c_2 = 2^{\frac{p}{2}-1}$, Theorem 2 implies

COROLLARY 3. *Let $A \subset\subset \Omega$, $0 < r < d(A, \partial\Omega)$. If $p \geqslant 2$ and $u \in W^{1,p}_{loc}(\Omega)$ is a Q-subsolution, then for any θ, $0 < \theta < \frac{p}{2p'} 2^{\frac{1}{2}pp'}$ we have the following Caccioppoli type inequality*

$$\int_\Omega |\nabla u^+|^p dx \leqslant \overline{C}(\theta)|A_r| + \frac{D(\theta)}{r^p} \int_{A_r \setminus A} |u^+|^p dx, \tag{27}$$

where

$$\overline{C}(\theta) = \frac{p\theta 2^{p'-1}}{p' - p\theta 2^{\frac{1}{2}pp'-1}}, \quad D(\theta) = \frac{p\theta^{-p/p'}}{p' - p\theta 2^{\frac{1}{2}pp'-1}}. \tag{28}$$

4. One-dimensional case

In one-dimensional case we have that a function $u \in W^{1,2}_{loc}(a,b)$ is subharmonic if and only if u is convex, see [1]. In this case it is possible to obtain the best Caccioppoli constant.

THEOREM 4. *Let $u \in W^{1,2}(a-r, b+r)$ be a nonnegative convex function, where (a,b) is bounded and $r > 0$. Then*

$$\int_a^b u'^2 dx \leqslant \frac{2\sqrt{3}-3}{r^2} \left[\int_{a-r}^a u^2\, dx + \int_b^{b+r} u^2\, dx \right], \tag{29}$$

and the inequality is sharp.

Proof. Since $C^\infty([a-r, b+r])$ is dense in $W^{1,2}(a-r, b+r)$, it suffices to assume $u \in C^\infty([a-r, b+r])$. Using the identity $(uu')' = (u')^2 + uu''$ and $uu'' \geqslant 0$ we obtain

$$\int_a^b u'^2 dx \leqslant \int_a^b (uu')' dx = u(b)u'(b) - u(a)u'(a).$$

The proof will follow if we estimate the right-hand side of (29) from below, using tangents of u at a and b on the corresponding intervals.

(a) Suppose first that $u'(a) \leqslant 0$ or $0 < r \leqslant \frac{u(a)}{u'(a)}$ if $u'(a) > 0$, and $u'(b) \geqslant 0$ or $0 < r \leqslant -\frac{u(b)}{u'(b)}$ if $u'(b) < 0$. Then the right-hand side of (29) is bounded from below by

$$\begin{aligned} f(r) &:= \frac{1}{r^2} \left[\int_{a-r}^a (u'(a)(x-a) + u(a))^2 dx + \int_b^{b+r} (u'(b)(x-b) + u(b))^2 dx \right] \\ &= \frac{1}{3}(r + \frac{3}{r})[u'(a)^2 + u'(b)^2] + [u(b)u'(b) - u(a)u'(a)]. \end{aligned}$$

Since $f(r)$ attains its minimum at $r_0 = \sqrt{3} \cdot \sqrt{\frac{u(a)^2+u(b)^2}{u'(a)^2+u'(b)^2}}$, then

$$f(r_0) = \frac{2}{\sqrt{3}} \sqrt{[u(a)^2 + u(b)^2] \cdot [u'(a)^2 + u'(b)^2]}.$$

The desired inequality $u(b)u'(b) - u(a)u'(a) \leqslant f(r_0)$ is equivalent to

$$u(b)u'(b) - u(a)u'(a) \leqslant \sqrt{[u(a)^2 + u(b)^2] \cdot [u'(a)^2 + u'(b)^2]}, \tag{30}$$

and the claim follows from the Cauchy inequality.

(b) Now suppose that $u'(b) > 0$ and $r > \frac{u(a)}{u(a)}$, or $u'(b) < 0$ and $r > -\frac{u(b)}{u'(b)}$. Since only one alternative is possible, assume that the first one holds. In this case it suffices to prove that

$$u(b)u'(b) \leqslant \frac{2\sqrt{3}-3}{r^2} \int_b^{b+r} [u'(b)(x-b) + u(b)]^2 \, dx =: g(r)$$

It is easy to see that $g(r)$ attains its minimum for $r_1 = \sqrt{3} \cdot \frac{u(b)}{u'(b)}$, and $g(r_1) = u(b)u'(b)$. The second alternative can be treated in the same way.

(c) The equality in (29) holds for a convex nonnegative function $u \in W^{1,2}(a - r, b+r)$ if we assume that $u'' = 0$, i.e. if u is piecewise linear, and if we have equality in (30), i.e. $-\frac{u'(a)}{u(a)} = \frac{u'(b)}{u(b)}$. Therefore to have equality in (29) it suffices to take $c \in (a,b)$ and define $u(x) = k_1(x-c) + d$ for $x \in [c,b)$, $u(x) = k_2(x-c) + d$ for $x \in (a,c]$, with $k_1 \geqslant 0$, $k_2 \leqslant 0$, $d \geqslant 0$, $k_1 k_2(a+b-2c) + d(k_1+k_2) = 0$ and $r = r_0$ as in (a). If we take for example $a = -b$, $b > 0$, and $r = \sqrt{3}\, b$, then for $u(x) = |x|$ we have equality in (29). Q.E.D.

COROLLARY 4. *Assuming $a \in \mathbf{R}$, $r > 0$, let $u \in W^{1,2}(a-r, \infty)$ be a nonnegative, convex function. Then*

$$\int_a^\infty u'^2 dx \leqslant \frac{2\sqrt{3}-3}{r^2} \int_{a-r}^a u^2 dx. \tag{31}$$

Proof. The claim follows from the preceding theorem by letting $b \to \infty$. The second integral in (29) vanishes as $b \to \infty$, because u is convex and quadratically integrable over $(a-r, \infty)$, so that $u(x) \to 0$ as $x \to \infty$. Q.E.D.

THEOREM 5. *Let $u \in W^{1,p}(a-r, b+r)$, $1 \leqslant p < \infty$, be a nonnegative convex function, where (a,b) is bounded and $r > 0$. Then*

$$\int_a^b |u'|^p dx \leqslant \frac{C_p}{r^p} \left[\int_{a-r}^a u^p \, dx + \int_b^{b+r} u^p \, dx \right], \tag{32}$$

where

$$C_p = \max_{\lambda > 0} \frac{(p+1)\lambda^p}{(1+\lambda)^{p+1} - 1} \tag{33}$$

is the best constant, and $C_p < 1$ for $p > 1$, $C_1 = 1$.

Sketch of the proof. We proceed in the same way as in Theorem 4 for $p = 2$, but starting with the identity $|u'|^p = \operatorname{sgn}(u') \cdot (u|u'|^{p-1})' - (p-1)u|u'|^{p-2}u''$ a.e. In the final step we have only to check the following inequality for $u'(b) > 0$:

$$u(b)u'(b)^{p-1} \leqslant \frac{C_p}{r^p(p+1)} \cdot \frac{u(b)^{p+1}}{u'(b)} \left[\left(1 + \frac{u'(b)}{u(b)} r\right)^{p+1} - 1 \right],$$

and the same for $x = a$, with $u'(a) > 0$. Setting $\lambda = \frac{u'(b)}{u(b)} r$ we reduce the preceding inequality to $1 \leqslant \frac{C_p}{\lambda^p(p+1)}[(1+\lambda)^{p+1} - 1]$ which is equivalent to the definition of C_p. Q.E.D.

Remark. If $a = -b$, $b > 0$, the equality in (32) is achieved for $u(x) = |x|$ and $r = \lambda_p b$, where λ_p is the point of maximum in (33). It is easy to see that we have $C_2 = 2\sqrt{3} - 3$, and C_3 is obtained using $\lambda_3 = \sqrt[3]{4 + 2\sqrt{2}} + \sqrt[3]{4 - 2\sqrt{2}}$.

Numerical experiments confirm our conjecture that

$$\lim_{p \to \infty} (\lambda_p - p) = 0, \quad \lim_{p \to \infty} C_p = e^{-1}.$$

5. Concluding remarks

Remark. Note that, contrary to the case of $N = 1$, for $N \geqslant 2$ subharmonic functions defined on a convex open set Ω do not have to be convex functions. For example, if we define

$$E_N(x) = \begin{cases} \log |x|, & \text{for } N = 2, \\ \frac{1}{|x|^{N-2}}, & \text{for } N \geqslant 3, \end{cases} \tag{34}$$

and Ω a convex domain such that $0 \notin \Omega$, it is well known that $-\Delta E_N \equiv 0$ on Ω, while the functions $E_N(x)$, $N \geqslant 2$, are neither convex nor concave on Ω. Furthermore, for all functions $u(x) = E_N(x)$, $N \geqslant 2$, there holds inequality (15) with $p = 2$. Many other examples of nonconvex subharmonic functions can be found in [1].

Remark. We want to discuss the possibility of taking subsets A of Ω in Theorem 3 having the largest possible measure for a given $r > 0$. To this end we introduce the following regularity condition on Ω. We say that an open set Ω in $\mathbf{R}^N$ has the *uniform inner ball property* if there exists $r_0 > 0$ such that for every $x_0 \in \partial\Omega$ there exists a (not necessarily unique) ball $B \subseteq \Omega$ of radius r_0 such that $x_0 \in \partial B$. This is equivalent to saying that the r-neighborhood of the set

$$\Omega_{-r} = \{x \in \Omega : d(x, \partial\Omega) > r\} \tag{35}$$

is equal to Ω for every $r \in (r, r_0]$, that is, $(\Omega_{-r})_r = \Omega$. Our Theorem 3(b) implies that for every nonnegative subsolution $u \in W^{1,2}(\Omega)$ of the quasilinear elliptic problem (3) we have that for every $r \in (0, r_0]$

$$\int_{\Omega_{-r}} |\nabla u|^2 dx \geqslant \frac{4\beta}{\alpha r^2} \int_{\Omega \setminus \Omega_{-r}} u^2 dx. \tag{36}$$

It is easy to see that rectangles in $\mathbf{R}^2$ do not have uniform inner ball property. The same holds for domains in $\mathbf{R}^2$ having at least one acute cusp. However, there exist domains having uniform inner ball property that are not even of class C^1. Such is the case with some domains in $\mathbf{R}^2$ whose boundary has obtuse cusps, like the complement of the rectangle.

As another example, this time in $\mathbf{R}^N$, take $\Omega = B_1(0) \setminus D$, where D is a closed drop-like subset of $B_1(0)$ obtained as a convex hull $D = \operatorname{conv}\{x_0, B\}$, x_1 being a point in $B_1(0)$ and B being another ball contined in $B_1(0)$, such that $x_0 \notin \overline{B}$. Then x_0 is the acute cuspidal point on the boundary of Ω, and Ω has the uniform inner ball property. One can easly modify this example to obtain also contractible domains with cuspidal points, having uniform inner ball property.

Remark. All results in this article involving L-subsolutions u can also be formulated for L-supersolutions, with u^- instead of u^+, where the negative part of u is defined by $u^- = (-u)^+$.

It is easy to see that if we assume $u \in W^{1,p}(\Omega)$ instead of $u \in W^{1,p}_{loc}(\Omega)$, then our results hold also for unbounded subsets A, provided $d(A, \partial\Omega) > 0$. Theorem 2 and its consequences can be extended to Sobolev spaces with weights.

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