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ON HEINZ–KATO TYPE CHARACTERIZATIONS OF THE FURUTA INEQUALITY II

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Abstract. This paper is a continuation of the previous one [8]. Here we show that it is still possible to have several characterizations of the Furuta inequality in terms of different types of the Heinz-Kato inequality. Some application are given.

1. Notation and introduction

It is to be understood throughout the paper that the capital letters present bounded linear operators acting on a Hilbert space H. T is positive (written $T \geqslant O$) in case $(Tx, x) \geqslant 0$ for all $x \in$ H. If S and T are Hermitian, we write $T \geqslant S$ in case $T - S \geqslant O$. $T = U|T|$ is the polar decomposition of T with U the partial isometry, and $|T|$ the positive sqare root of the positive operator T^*T. I will denote the identity operator.

We recall first the remarkable two inequalities of Furuta : If $A \geqslant B \geqslant O$, then both inequalities

$$(i)\quad (B^rA^pB^r)^{1/q} \geqslant (B^rB^pB^r)^{1/q}$$

and

$$(ii)\quad (A^rA^pA^r)^{1/q} \geqslant (A^rB^pA^r)^{1/q}$$

hold and they are equivalent for $p, r \geqslant 0$, and $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$ [1, p. 85, or 2, p. 126]. The inequalities are outstanding in a sense that the conditions on p, q, r, and the inequality $(1+2r)q \geqslant p+2r$ are the best possible with respect to inequalities themselves [9]. If r is replaced by $r/2$ in the statement above, then the Figure shows that every point (q, p) in the shaded area satisfies inequalities (i) and (ii). The two inequalities may be easily converted into altered expressions, viz.

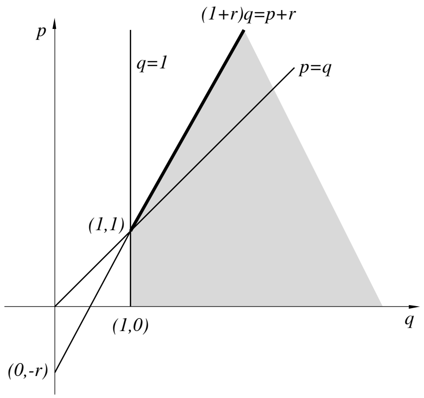

Figure 1

$(B^rA^pB^r)^{(1+2r)\alpha/(p+2r)} \geqslant B^{(1+2r)\alpha}$ and $A^{(1+2r)\alpha} \geqslant (A^rB^pA^r)^{(1+2r)\alpha/(p+2r)}$, respectively, for $A \geqslant B \geqslant O$, $p \geqslant 1$, $r \geqslant 0$, and $\alpha \in [0, 1]$. In our previous paper [8] we gave

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some characterizations of these altered inequalities by means of the Heinz-Kato type inequality. In this note we shall give more characterizations of inequalities (i) and (ii) in terms of different types of the Heinz-Kato inequality. In particular, we obtain new characterizations of the Löwner-Heinz inequality, and some results about hyponormal operators.

2. Characterizations of the Furuta inequality

The Furuta inequality implies the next lemma, and we shall use it as a basic tool.

LEMMA. *Let $A,B \geqslant O$, and let $T = U|T|$ be the polar decomposition. If $|T|^2 \leqslant A^2$ and $|T^*|^2 \leqslant B^2$, then, for $p,r \geqslant 0$ and $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$, the following operator inequalities hold.*

(1) $(|T|^{2r}A^{2p}|T|^{2r})^{1/q} \geqslant |T|^{2(p+2r)/q}$;

(2) $A^{2(p+2r)/q} \geqslant (A^{2r}|T|^{2p}A^{2r})^{1/q}$;

(3) $(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/q} \geqslant |T^*|^{2(p+2r)/q}$;

(4) $B^{2(p+2r)/q} \geqslant (B^{2r}|T^*|^{2p}B^{2r})^{1/q}$.

Proof. In inequalities (i) and (ii) above replace A and B by A^2 and $|T|^2$ (B^2 and $|T^*|^2$), respectively. Then inequalities (1) and (2) ((3) and (4)) follow immediately.

THEOREM. *Let $A,B \geqslant O$, and let $T = U|T|$ be the polar decomposition. Then the following are equivalent for all $x,y \in H$, $k,p,r,s \geqslant 0$, and $j,q \geqslant 1$ with $(1+2r)q \geqslant p+2r$ and $(1+2s)j \geqslant k+2s$, and if $|T|^2 \leqslant A^2$ and $|T^*|^2 \leqslant B^2$ for each one of the inequalities (2) through (9).*

(1) $(B^rA^pB^r)^{1/q} \geqslant (B^rB^pB^r)^{1/q}$, *or* $(A^rA^pA^r)^{1/q} \geqslant (A^rB^pA^r)^{1/q}$ *if* $A \geqslant B$ *(Inequalities of Furuta);*

(2) $|(|T|^{(p+2r)/q}U(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x,y)| \leqslant \|B^{(k+2s)/j}x\|\|(|T|^{2r}A^{2p}|T|^{2r})^{1/2q}y\|$;

(3) $|(|T|^{(p+2r)/q}U(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x,y)| \leqslant \|(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x\|\|(|T|^{2r}A^{2p}|T|^{2r})^{1/2q}y\|$;

(4) $|(|T|^{(p+2r)/q}(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x,y)| \leqslant \|B^{(k+2s)/j}x\|\|(|T|^{2r}A^{2p}|T|^{2r})^{1/2q}y\|$;

(5) $|(|T|^{(p+2r)/q}(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x,y)| \leqslant \|(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x\|\|(|T|^{2r}A^{2p}|T|^{2r})^{1/2q}y\|$;

(6) $|(|T^*|^{(p+2r)/q}U(A^{2s}|T|^{2k}A^{2s})^{1/2j}x,y)| \leqslant \|A^{(k+2s)/j}x\|\|(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/2q}y\|$;

(7) $|(|T^*|^{(p+2r)/q}U(A^{2s}|T|^{2k}A^{2s})^{1/2j}x,y)| \leqslant \|(A^{2s}|T|^{2k}A^{2s})^{1/2j}x\|\|(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/2q}y\|$;

(8) $|(|T^*|^{(p+2r)/q}(A^{2s}|T|^{2k}A^{2s})^{1/2j}x,y)| \leqslant \|A^{(k+2s)/j}x\|\|(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/2q}y\|$;

(9) $|(|T^*|^{(p+2r)/q}(A^{2s}|T|^{2k}A^{2s})^{1/2j}x,y)| \leqslant \|(A^{2s}|T|^{2k}A^{2s})^{1/2j}x\|\|(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/2q}y\|$.

Proof. (1) implies (2). Due to (1) and (4) in Lemma and since U is partial isometry, we have

$$\begin{aligned}
&|(|T|^{(p+2r)/q}U(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x,y)| \\
&\quad \leqslant \|U(B^{2s}|T^*|^{2k}B^{2s})^{1/2j}x\|\,\||T|^{(p+2r)/q}y\| \\
&\quad \leqslant ((B^{2s}|T^*|^{2k}B^{2s})^{1/j}x,x)^{1/2}(|T|^{2(p+2r)/q}y,y)^{1/2} \\
&\quad \leqslant (B^{2(k+2s)/j}x,x)^{1/2}((|T|^{2r}A^{2p}|T|^{2r})^{1/q}y,y)^{1/2} \\
&\quad = \|B^{(k+2s)/j}x\|\|(|T|^{2r}A^{2p}|T|^{2r})^{1/2q}y\|.
\end{aligned}$$

(1) implies (3). This is included in the proof "(1) implies (2)".

(2) or (3) implies (1). In (2) or (3) let $T=B$ so that $U=I$. Also let $s=r$, $k=p$, $j=q$, and $y=x$. Then

$$(B^{2(p+2r)/q}x,x) \leqslant (B^{2(p+2r)/q}x,x)^{1/2}((B^{2r}A^{2p}B^{2r})^{1/q}x,x)^{1/2}.$$

Thus, $(B^{2r}A^{2p}B^{2r})^{1/q} \geqslant (B^{2r}B^{2p}B^{2r})^{1/q}$ under the condition that $A^2 \geqslant B^2$, and so the Furuta inequality follows.

It is readily seen from the proof above that in (2) and (3) the operator U may be omitted without affecting equivalence of inequalities. Thus, we have inequalities (4) and (5) which are all equivalent to (1).

(1) implies (6). We shall use (2) and (3) in lemma this time and since U is partial isometry, we have

$$\begin{aligned}
&|(|T^*|^{(p+2r)/q}U(A^{2s}|T|^{2k}A^{2s})^{1/2j}x,y)| \\
&\quad \leqslant \|U(A^{2s}|T|^{2k}A^{2s})^{1/2j}x\|\||T^*|^{(p+2r)/q}y\| \\
&\quad \leqslant ((A^{2s}|T|^{2k}A^{2s})^{1/j}x,x)^{1/2}(|T^*|^{2(p+2r)/q}y,y)^{1/2} \\
&\quad \leqslant (A^{2(k+2s)/j}x,x)^{1/2}((|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/q}y,y)^{1/2} \\
&\quad = \|A^{(k+2s)/j}x\|\|(|T^*|^{2r}B^{2p}|T^*|^{2r})^{1/2q}y\|.
\end{aligned}$$

(1) implies (7). It is clear from the proof above.

(6) or (7) implies (1). In (6) or (7) let $T=A$ so that $U=I$. Also let $s=r$, $k=p$, $j=q$, and $y=x$. Then

$$(A^{2(p+2r)/q}x,x) \leqslant (A^{2(p+2r)/q}x,x)^{1/2}((A^{2r}B^{2p}A^{2r})^{1/q}x,x)^{1/2}.$$

Thus, $(A^{2r}B^{2p}A^{2r})^{1/q} \geqslant (A^{2r}A^{2p}A^{2r})^{1/q}$ under the condition that $B^2 \geqslant A^2$, and so the Furuta inequality follows.

Here, again, the operator U in inequalities (6) and (7) may be omitted, so that we obtain (8) and (9), respectively, and this completes the proof.

3. Applications

The next result is some new characterizations of the Löwner-Heinz inequality, so called in the literature.

COROLLARY 1. *Let $A, B \geqslant O$, and let $T = U|T|$ be the polar decomposition. Then the following are equivalent for all $x, y \in H$, $\alpha, \beta \in [0,1]$, and if $|T|^2 \leqslant A^2$ and $|T^*|^2 \leqslant B^2$ for each one of the inequalities (2) through (9).*

(1) $A^\alpha \geqslant B^\alpha$ *if* $A \geqslant B$ *(Löwner-Heinz inequality);*

(2) $|(|T|^\alpha U|T^*|^\beta x, y)| \leqslant \|B^\beta x\| \|A^\alpha y\|$;

(3) $|(|T|^\alpha U|T^*|^\beta x, y)| \leqslant \||T^*|^\beta x\| \|A^\alpha y\|$;

(4) $|(|T|^\alpha |T^*|^\beta x, y)| \leqslant \|B^\beta x\| \|A^\alpha y\|$;

(5) $|(|T|^\alpha |T^*|^\beta x, y)| \leqslant \||T^*|^\beta x\| \|A^\alpha y\|$;

(6) $|(|T^*|^\alpha U|T|^\beta x, y)| \leqslant \|A^\beta x\| \|B^\alpha y\|$;

(7) $|(|T^*|^\alpha U|T|^\beta x, y)| \leqslant \||T|^\beta x\| \|B^\alpha y\|$;

(8) $|(|T^*|^\alpha |T|^\beta x, y)| \leqslant \|A^\beta x\| \|B^\alpha y\|$;

(9) $|(|T^*|^\alpha |T|^\beta x, y)| \leqslant \||T|^\beta x\| \|B^\alpha y\|$.

Proof. Let $r = s = 0$, $j = q = 1$ in Theorem (hence, $p, k \in [0,1]$), and let $p = \alpha$ and $k = \beta$. Then, we have corresponding inequalities (1) through (9).

Note that the so called Heinz-Kato inequality is $|(Tx, y)| \leqslant \|A^{1-\alpha} x\| \|B^\alpha y\|$ if $A, B \geqslant O$, $|T|^2 \leqslant A^2$ and $|T^*|^2 \leqslant B^2$, which can be obtained from (6) in Corollary 1 when $\alpha + \beta = 1$. Because $|T^*|^\alpha U|T|^{1-\alpha} = U|T|^\alpha U^* U|T|^{1-\alpha} = U|T| = T$. This should justify our title of the paper. Also note that the main result in [4] is to prove the validity of the inequality $|(T|T|^{\alpha+\beta-1} x, y)| \leqslant \|A^\beta x\| \|B^\alpha y\|$, which is nothing but the inequality (6) in Corollary 1.

REMARK 1. Recall T is a p-hyponormal operator if $(T^*T)^p \geqslant (TT^*)^p$ for $0 < p \leqslant 1$. It is hyponormal when $p = 1$, and every hyponormal operator is p-hyponormal by the Löwner-Heinz inequality. It was proved in [5,Corollary 7] that $T = U|T|$ is p-hyponormal if and only if $|(Tx, y)| \leqslant \||T|^{1-p} x\| \||T|^p y\|$ for all $x, y \in$ H. The next result is some new characterizations of a hyponormal operator.

COROLLARY 2. *Let $T = U|T|$ be a hyponormal operator. Then the following are equivalent for all $x, y \in H$, $k, p, r, s \geqslant 0$, and $j, q \geqslant 1$ with $(1+2r)q \geqslant p + 2r$ and $(1+2s)j \geqslant k + 2s$. And each inequality holds true.*

(1) $|(|T|^{(p+2r)/q} U(|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x, y)| \leqslant \||T|^{(k+2s)/j} x\| \||T|^{(p+2r)/q} y\|$;

(2) $$|(|T|^{(p+2r)/q} U(|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x, y)| \leqslant \|(|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x\| \||T|^{(p+2r/q} y\|;$$

(3) $|(|T|^{(p+2r)/q} (|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x, y)| \leqslant \||T|^{(k+2s)/j} x\| \||T|^{(p+2r)/q} y\|$;

(4) $$|(|T|^{(p+2r)/q} (|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x, y)| \leqslant \|(|T|^{2s}|T^*|^{2k}|T|^{2s})^{1/2j} x\| \||T|^{(p+2r/q)} y\|;$$

(5) $|(|T^*|^{(p+2r)/q} U|T|^{(k+2s)/j} x, y)| \leqslant \||T|^{(k+2s)/j} x\| \|(|T^*|^{2r}|T|^{2p}|T^*|^{2r})^{1/2q} y\|$;

(6) $|(|T^*|^{(p+2r)/q} |T|^{(k+2s)/j} x, y)| \leqslant \||T|^{(k+2s)/j} x\| \|(|T^*|^{2r}|T|^{2p}|T^*|^{2r})^{1/2q} y\|$.

Moreover, each one of the above inequalities implies that T is hyponormal.

Proof. In Theorem let $A = B = |T|$. Then the conditions that $|T|^2 \leqslant A^2$ and $|T^*|^2 \leqslant A^2$ in Theorem become $|T^*|^2 \leqslant |T|^2$, which means that T is hyponormal. Moreover, (1) in Theorem holds trivially. Therefore, all inequalities in above are equivalent to one another, and they hold true. Now, let, in particular, $r = s = 0$, $j = q = 1$ in the inequality (5) (hence, $p,k \in [0,1]$), and let $k = 0$ and $p = 1$. Then it becomes

$$|(Tx,y)| \leqslant \|x\|\||T|y\|,$$

which means that T is hyponormal by Remark 1. This takes care of the last statement.

REMARK 2. If we put $r = s = 0$, $j = q = 1$ and $k = 1-p$ in (5) of Corollary 2, then it becomes

$$|(Tx,y)| \leqslant \||T|^{1-p}x\|\||T|^p y\|$$

for all $x,y \in \mathrm{H}$ and $0 < p \leqslant 1$, which means T is p-hyponormal by Remark 1.

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