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SOME EXPONENTIAL OPERATOR INEQUALITIES

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Abstract. We will simply show Ando's exponential inequality by making use of an approximation method and Furuta inequality, and then we will give some results about exponential inequalities.

Let A and B be bounded selfadjoint operators on a Hilbert space. The following celebrated inequality was found by Furuta in [4] and simply proved in [5].

If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

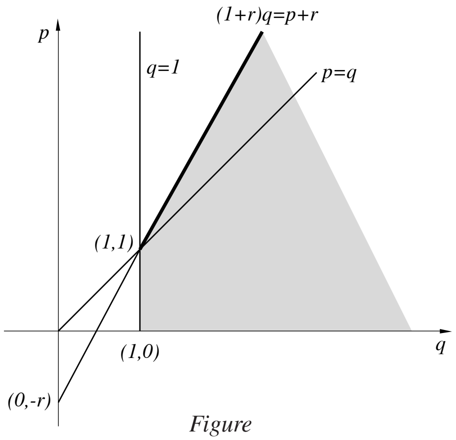
(*i*) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(*ii*) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with*

$$(1+r)q \geqslant p+q.$$



Figure

Tanahashi[6] proved that the domain drawn for p, q, and r in Figure is the best possible one for (i) and (ii).

We cite the following:

THEOREM A. $A \geqslant B$ *implies that for* $p \geqslant 0$, $r \geqslant s \geqslant 0$

$$e^{sA} \geqslant \left(e^{\frac{r}{2}A}e^{pB}e^{\frac{r}{2}A}\right)^{\frac{s}{r+p}}. \tag{1}$$

Ando [1] showed Theorem A in the case of $s = p = r$ with his splendid idea. Then by making use of Ando's result [1] and Furuta inequalty, Theorem A has been proved in [2] and [3]. In [1] Ando also showed the converse:

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THEOREM B. *If*

$$e^{tA} \geqslant \left(e^{\frac{t}{2}A}e^{tB}e^{\frac{t}{2}A}\right)^{\frac{1}{2}} \quad \textit{for every } t > 0,$$

then $A \geqslant B$.

The aim of this note is to give a new way to get exponential inequalities from operator inequalities like (i), (ii), and to extend Theorem A and Theorem B. We start with a quite simple proof of Theorem A. This technique seems to be very effective to study operator inequality.

Simplified proof of Theorem A. For sufficiently large n we have $1+\frac{A}{n} \geqslant 1+\frac{B}{n} \geqslant 0$. By substituting np and nr to p and r of (ii), respectively, we get,

$$\left(1+\frac{A}{n}\right)^{\frac{n(p+r)}{q}} \geqslant \left\{\left(1+\frac{A}{n}\right)^{n\frac{r}{2}}\left(1+\frac{B}{n}\right)^{np}\left(1+\frac{A}{n}\right)^{n\frac{r}{2}}\right\}^{1/q}, \quad (rq \geqslant p+r).$$

Since, for selfadjoint operator X, $(1+\frac{X}{n})^n$ converges to e^X in the operator norm as $n \to \infty$, we gain (1) by setting $s = \frac{p+r}{q}$. □

We slightly extend Theorem A by using itself.

THEOREM 1. $A \geqslant B$ *implies*

$$e^{sA} \geqslant \left\{e^{\frac{r}{2}A}e^{(qA+pB)}e^{\frac{r}{2}A}\right\}^{\frac{s}{(p+q+r)}} \tag{2}$$

for $p,\ q,\ r,\ s$ *with* $r \geqslant s \geqslant 0$, $p,\ p+q \geqslant 0$ *and* $p+q+r > 0$.

Proof. If $p+q=0$, then $e^{(qA+pB)}$ is contractive, so that the above inequality follows. Therefore we assume that $p+q>0$. Since

$$\frac{qA+pB}{q+p} \leqslant A,$$

by using (1), we gain (2). □

Now we extend Theorem B:

THEOREM 2. *If there are* p,q,r,s *with* $p>0, p+q \geqslant 0, r \geqslant s > 0$ *such that*

$$e^{stA} \geqslant \left\{e^{\frac{rt}{2}A}e^{t(qA+pB)}e^{\frac{rt}{2}A}\right\}^{\frac{s}{(p+q+r)}} \qquad \textit{for every } t > 0,$$

then $A \geqslant B$.

Proof. If $p+q+r=s$, then the above inequality implies that $e^{t(qA+pB)}$ is contractive, for $p+q=0$ and $r=s$. Hence $A \geqslant B$. Suppose $p+q+r>s$. Set

$$f(t) = e^{\frac{-rt}{2}A}e^{-t(qA+pB)}e^{\frac{-rt}{2}A}, \quad g(t) = e^{-stA}.$$

Then we get

$$\left(f(t)^{\frac{s}{(p+q+r)}}x,x\right) \geqslant (g(t)x,x) \quad (||x||=1, \quad t>0),$$

from which it follows that

$$(f(t)x,x)^{\frac{s}{(p+q+r)}} \geqslant (g(t)x,x) \quad (t>0)$$

because of Jensen's inequality. Since the values of both sides of the inequality above at $t=0$ are 1, the right derivative of the left hand side at $t=0$ is greater than or equal to that of the right hand side. Since the norm derivative of e^{tT} at $t=0$ is generally T, we have

$$\frac{s}{(p+q+r)}\left(\left(-\frac{r}{2}A-(qA+pB)-\frac{r}{2}A\right)x,x\right) \geqslant (-sAx,x).$$

Hence we gain $A \geqslant B$. □

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