Mathematical
Inequalities
& Applications
Volume 2, Number 4 (1999), 517–528

SOME INEQUALITIES CONSTRUCTED BY TCHEBYSHEFF'S INTEGRAL INEQUALITY

FENG QI, LI-HONG CUI AND SEN-LIN XU

(*communicated by Á. Elbert*)

Abstract. In the article, using Tchebysheff's integral inequality, the authors refine Conte's inequality and some estimates concerning the probability integral which are related to the Mills' ratio, form several inequalities of trigonometric functions, obtain some integral inequalities and estimates of definite integrals related to $\sin x/x$, $x \in [0, \pi/2]$, and construct many miscellaneous inequalities.

1. Introduction

Let $f, g : [a, b] \to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \to R$ be a positive and integrable function. Then

$$\int_a^b p(x)f(x)\,\mathrm{d}x \int_a^b p(x)g(x)\,\mathrm{d}x \leqslant \int_a^b p(x)\,\mathrm{d}x \int_a^b p(x)f(x)g(x)\,\mathrm{d}x. \tag{1}$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then the inequality in (1) is reversed.

The inequality (1) is called the Tchebysheff's integral inequality. For details, see [7, 9, 10].

In this paper, we apply inequality (1) to create new inequalities. Using inequality (1), the Conte's inequality and some estimates concerning the probability integral which are related to the Mills' ratio are refined, several inequalities of trigonometric functions and some integral inequalities and estimates of definite integrals related to $\sin x/x, x \in [0, \pi/2]$ are formed, a lot of miscellaneous inequalities are constructed.

It is pointed out that, although the method used in this paper is not typical but trivial, the results obtained below are of importance and of interest theoretically.

Mathematics subject classification (1991): Primary 26D15; Secondary 33B20, 60E15.

Key words and phrases: Probability integral, Conte's inequality, Jordan's inequality, Fejér-Jackson's inequality, Tchebysheff's integral inequality, trigonometric function, integral inequality, Wallis' inequality.

The first author was supported in part by NSF of Henan Province, The People's Republic of China.

Paper MIA-02-42

2. Refinements of Conte's inequality and probability integral

2.1. Introduction

It is well-known that

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}\,t,\quad x\geqslant 0 \tag{2}$$

is refered to the probability integral or the error function.

The function R defined by

$$R(x)=\frac{\int_x^{+\infty}\mathrm{e}^{-t^2/2}\,\mathrm{d}\,t}{\mathrm{e}^{-t^2/2}} \tag{3}$$

is usually called Mills' ratio.

J. T. Chu proved in [2] and [9, p. 385] the following result: If inequalities

$$\frac{1}{2}\left(1-\mathrm{e}^{-ax^2}\right)^{1/2}\leqslant\frac{1}{\sqrt{2\pi}}\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}\,t\leqslant\frac{1}{2}\left(1-\mathrm{e}^{-bx^2}\right)^{1/2} \tag{4}$$

hold for $x\geqslant 0$, then it is necessary and sufficient that $0\leqslant a\leqslant\frac{1}{2}$ and $b\geqslant\frac{2}{\pi}$.

G. Pólya and J. D. Williams have earlier proved independently in [2] that

$$\frac{1}{\sqrt{2\pi}}\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}\,t\leqslant\frac{1}{2}\left(1-\mathrm{e}^{-2x^2/\pi}\right),\quad x\geqslant 0. \tag{5}$$

The above results can be connected with Mills' ratio, see problem 3.9.77 in [9].

In [10, p. 529] and [11], the following estimates were given: If $a>0$ and $b>0$, then with $t>0$

$$0<\frac{\sqrt{\pi}}{2}-\sqrt{bt}\int_0^a \exp(-btx^2)\,\mathrm{d}\,x<\frac{\exp(-a^2bt)}{\sqrt{bt}}. \tag{6}$$

For all $x>0$, we have, in [3, p. 229] and [9, pp. 177–181]

$$\left(x+\frac{x^2}{24}+\frac{x^3}{12}\right)\mathrm{e}^{-3x^2/4}<\mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}\,t\leqslant\frac{\pi^2}{8x}\left(1-\mathrm{e}^{-x^2}\right),\quad x>0. \tag{7}$$

Inequality (7) is called Conte's inequality. It is also related to Mills' ratio, see section 2.26 of [9].

In [7, pp. 591–593], [9, pp. 291–292] and other references, more related inequalities were given, for instances

$$\int_0^{\pi/2}\mathrm{e}^{-x^2\sin^2 t}\sin t\,\mathrm{d}\,t\leqslant\frac{\pi^2}{8x^2}\left(1-\mathrm{e}^{-x^2}\right),\quad x>0; \tag{8}$$

$$\int_0^x \mathrm{e}^{-t^2/2}\,\mathrm{d}\,t>\sqrt{\frac{\pi}{2}}\left(1-\mathrm{e}^{-x^2}-\frac{1}{2}\mathrm{e}^{-x^2/2}\right),\quad x>0; \tag{9}$$

$$\int_a^{\infty}\mathrm{e}^{-x^2}\,\mathrm{d}\,x<\min\left(\frac{\sqrt{\pi}}{2}\mathrm{e}^{-a^2},\frac{1}{2a}\mathrm{e}^{-a^2}\right),\quad a>0; \tag{10}$$

$$\operatorname{erf}(x)\operatorname{erf}(y)\geqslant\operatorname{erf}(x)+\operatorname{erf}(y)-\operatorname{erf}(x+y),\quad x\geqslant 0,\quad y\geqslant 0. \tag{11}$$

Equality of (11) holds if and only if x or y equals to an end point of the closed interval $[0,+\infty]$.

In this section, using Tchebysheff's integral inequality repeatedly, we will refine and extend inequalities (7)–(10) explicitly, and so on.

2.2. Estimates of probability integral and refinements of Conte's inequality

2.2.1. Taking $p(x)=1$, $f(x)=e^{x^2}$, $g(x)=x$, $[a,b]=[0,t]$, $t>0$ in (1) yields

$$\int_0^t e^{x^2}\,dx\int_0^t x\,dx<\int_0^t dx\int_0^t xe^{x^2}\,dx,$$

that is

$$\int_0^t e^{x^2}\,dx<\frac{1}{t}(e^{t^2}-1),\quad t>0. \tag{12}$$

This inequality (12) refines the right-hand side of (7).

2.2.2. Letting $p(x)=\sin x$, $f(x)=e^{-t^2\sin^2 x}$, $g(x)=\cos x$, $[a,b]=[0,\frac{\pi}{2}]$ in (1) arrives at

$$\int_0^{\pi/2}\sin xe^{-t^2\sin^2 x}\,dx\int_0^{\pi/2}\sin x\cos x\,dx\leqslant\int_0^{\pi/2}\sin x\,dx\int_0^{\pi/2}\sin x\cos xe^{-t^2\sin^2 x}\,dx.$$

By direct calculation we obtain

$$\int_0^{\pi/2}\sin xe^{-t^2\sin^2 x}\,dx\leqslant\frac{1-e^{-t^2}}{t^2}. \tag{13}$$

Inequality (13) improves the inequality (8).

2.2.3. Substituting $p(x)=\cos x$, $f(x)=e^{-t^2\sin^2 x}$, $g(x)=\sin x$, and $[a,b]=[0,\frac{\pi}{2}]$ into (1) gives us

$$\int_0^{\pi/2}\cos xe^{-t^2\sin^2 x}\,dx\int_0^{\pi/2}\cos x\sin x\,dx\geqslant\int_0^{\pi/2}\cos x\,dx\int_0^{\pi/2}\cos x\sin xe^{-t^2\sin^2 x}\,dx,$$

therefore

$$\int_0^{\pi/2}\cos xe^{-t^2\sin^2 x}\,dx=\int_0^1 e^{-(tx)^2}\,dx\geqslant\frac{1-e^{-t^2}}{t^2}. \tag{14}$$

Inequality (14) refines (8) and generalizes (13). In fact, the inequality (14) implies the following inequality:

$$\int_0^x e^{-t^2}\,dt\geqslant\frac{1-e^{-x^2}}{x}, \tag{15}$$

which is equivalent to

$$\int_0^x e^{-t^2/2}\,dt\geqslant\frac{2\left(1-e^{-x^2/2}\right)}{x}. \tag{16}$$

Inequality (16) partially improves (9), since the lower bound of (16) is sharper than that of (9) when $x>0$ is smaller.

2.2.4. By the same arguments, let $g(x) = e^{-t^\alpha}$ or e^{t^α}, $f(t) = t^{\alpha-1}$, and $p(t) = 1$, then we can easily find

$$\int_0^x e^{-t^\alpha}\, dt \geqslant \frac{1 - e^{-x^\alpha}}{x^{\alpha-1}}, \quad x > 0, \quad \alpha \geqslant 1, \tag{17}$$

and

$$\int_0^x e^{t^\alpha}\, dt \leqslant \frac{e^{x^\alpha} - 1}{x^{\alpha-1}}, \quad x > 0, \quad \alpha \geqslant 1. \tag{18}$$

These two inequalities (17) and (18) extend the others above.

Remark 1. From the estimates and inequalities obtained in this section, we can get inequalities regarding the probability integral $\operatorname{erf}(x)$ which refine the related results in [7, pp. 591–593] and [9, pp. 171–181, 291–293, 385].

Remark 2. Recently, the first author in [15, 17, 20] further investigates the integrals $\int_0^x e^{-t^\alpha}\, dt$, $\int_0^x e^{t^\alpha}\, dt$ and $\int_x^\infty e^{-t^\alpha}\, dt$ for $\alpha > 0$ which are related to the incomplete gamma function.

3. Inequalities related to function $\sin x/x$

3.1. Introduction

In [7, 9, 10] the following result is given

$$0 < S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} < \int_0^\pi \frac{\sin x}{x}\, dx \tag{19}$$

where $n = 1, 2, \cdots$, $x \in (0, \pi)$.

In the literature, (19) is known as the Fejér-Jackson's inequality. This inequality is used in a discussion of the J. W. Gibb's phenomena in Fourier series theory. Thus, it is interesting to give the bounds for the integrals $\int_0^\pi \sin x/x\, dx$, $\int_0^{\pi/2} \sin x/x\, dx$, and the like.

Now we have had some estimates in [6, pp. 420–422, 435–437], [7, 10], and [9, p. 354] as follows:

$$\int_0^\pi \frac{\sin x}{x}\, dx < 1.851; \qquad \int_0^\pi \frac{\sin x}{\sqrt[4]{x}}\, dx < \pi^{3/4}; \tag{20}$$

$$\frac{1}{2} < \int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx < \frac{\sqrt{2}}{2}; \qquad \frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{-}\sin x\, dx < \frac{\pi^2}{6}; \tag{21}$$

$$1 < \int_0^{\pi/2} \frac{\sin x}{x}\, dx < \frac{\pi}{2}; \tag{22}$$

$$\int_x^{ax} \left(\frac{\sin t}{t}\right)^2 dt \leqslant \frac{(a-1)\pi}{2a}, \quad a > 1, \quad x \geqslant 0; \tag{23}$$

and so on.

Recently the first author [18] improved (21) and (22) to

$$\frac{4}{3} < \int_0^{\pi/2} \frac{\sin x}{x}\,\mathrm{d}x < \frac{\pi+1}{3}, \quad \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,\mathrm{d}x < \frac{2\pi^3}{27\sqrt{3}}, \tag{24}$$

$$\int_{\pi/4}^{\pi/2} \frac{\sin t}{t}\,\mathrm{d}t > \frac{4\sqrt{2}}{3\pi}. \tag{25}$$

In this section, by Tchebysheff's integral inequality, some inequalities of trigonometric functions and integral inequalities are obtained, the values of some definite integrals are estimated. The main results are as follows

$$\frac{\sin x}{x} \geqslant \frac{1+\cos x}{2}, \quad x \in \Big[0, \frac{\pi}{2}\Big]; \tag{26}$$

$$\frac{\sin t}{t} \geqslant \frac{1+2\cos t}{3} + \frac{t\sin t}{6}, \quad t \in \Big[0, \frac{\pi}{2}\Big]; \tag{27}$$

$$\Big(\frac{\sin t}{t}\Big)^2 \leqslant 2\Big(\frac{\sin t}{t} + \frac{\cos t - 1}{t^2}\Big), \quad t \in \Big[0, \frac{\pi}{2}\Big]; \tag{28}$$

$$\Big(\frac{\sin t}{t}\Big)^2 + 2\Big(\frac{\sin t}{t}\Big) \geqslant 4\Big(\frac{1-\cos t}{t^2}\Big) + \cos t, \quad t \in [0, \pi]; \tag{29}$$

$$\frac{5}{9} + \frac{57\pi}{288} + \frac{\pi^3}{1728} < \int_0^{\pi/2} \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t < \frac{\pi}{6} + \frac{2}{3} + \frac{(\pi-2)^2}{10\pi}; \tag{30}$$

$$\frac{\pi}{12} + \frac{41}{288} + \frac{(20-\pi)\sqrt{2}}{72} - \frac{(6-\pi)\pi^2}{13824} < \int_0^{\pi/4} \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t < \frac{5\pi+2}{24} + \frac{(\pi-2)^2}{320\pi}; \tag{31}$$

$$\int_{\pi/4}^{\pi/2} \frac{\sin t}{t}\,\mathrm{d}t < \frac{5\pi}{48} + \frac{7}{24}; \quad \int_0^{\pi/2} \frac{\sin t}{t}\,\mathrm{d}t > \frac{\pi+5}{6}; \tag{32}$$

$$\int_0^x \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t + 2\int_0^{x/2} \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t < 2\int_0^x \frac{\sin t}{t}\,\mathrm{d}t, \quad x \in \Big[0, \frac{\pi}{2}\Big]; \tag{33}$$

$$\int_0^x \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t + 2\int_0^x \frac{\sin t}{t}\,\mathrm{d}t > 4\int_0^{x/2} \Big(\frac{\sin t}{t}\Big)^2 \mathrm{d}t + \sin x - 1, \quad x \in [0, \pi]; \tag{34}$$

$$\int_0^t \Big(\frac{x}{\sin x}\Big)^2 \mathrm{d}x < 2\mathrm{tg}\Big(\frac{t}{2}\Big) + \frac{2}{3}\mathrm{tg}^3\Big(\frac{t}{2}\Big), \quad t \in \Big(0, \frac{\pi}{2}\Big]; \tag{35}$$

$$\int_0^t \frac{x}{\sin x}\,\mathrm{d}x < 2\mathrm{tg}\Big(\frac{t}{2}\Big), \quad t \in \Big(0, \frac{\pi}{2}\Big]. \tag{36}$$

3.2. Proofs of inequalities for trigonometric functions

Putting $p(x) = 1$, $f(x) = \sin x$, $g(x) = x$, $x \in [a,b] = [0,t]$, $t \in [0, \pi/2]$ in (1), we have

$$\int_0^t \sin x\,\mathrm{d}x \int_0^t x\,\mathrm{d}x \leqslant \int_0^t \mathrm{d}x \int_0^t x\sin x\,\mathrm{d}x$$

By calculating directly we obtain inequality (26).

Assuming $p(x)=x$, $f(x)=x$, $g(x)=\cos x$, $x\in[a,b]=[0,t]$, $t\in[0,\pi/2]$ in (1), we get

$$\int_0^t x^2\,\mathrm{d}x\int_0^t x\cos x\,\mathrm{d}x \geqslant \int_0^t x\,\mathrm{d}x\int_0^t x^2\cos x\,\mathrm{d}x$$

This implies the inequality (27).

By similar arguments, if we set $p(x)=\cos x$, $f(x)=g(x)=x$, then inequality (28) is yielded; if we put $p(x)=\sin x$, $f(x)=g(x)=x$ in (1), the inequality (29) is established.

3.3. Proofs of inequalities for integrals

In this subsection, the following lemma is necessary.

LEMMA. *For $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and $x\neq 0$, we have*

$$\frac{3}{\pi}-\frac{4}{\pi^3}x^2<\frac{\sin x}{x}<1-\frac{4(\pi-2)}{\pi^3}x^2. \tag{37}$$

Proof. Set $f(x)=\sin x-x+4(\pi-2)x^3/\pi^3$, $x\in[0,\pi/2]$, then

$$f'(x)=\cos x-1+\frac{12(\pi-2)}{\pi^3}x^2,\quad f''(x)=\frac{24(\pi-2)}{\pi^3}x-\sin x,$$
$$f'''(x)=\frac{24(\pi-2)}{\pi^3}-\cos x,\qquad f^{(4)}(x)=\sin x\geqslant 0.$$

Thus, $f''(x)$ is convex. Since $f''(0)=0$, $f''(\pi/2)=(12\pi-24-\pi^2)/\pi^2>0$, then $f''(x)$ has only one zero on $(0,\pi/2]$, that is, $f'(x)$ has only one minimum on $(0,\pi/2]$. From $f'(0)=0$, $f'(\pi/2)=2-6/\pi>0$, it follows that $f(x)$ also has only one minimum on the interval $(0,\pi/2)$. Therefore, from $f(0)=f(\pi/2)=0$, we conclude that $f(x)\leqslant 0$, $x\in[0,\pi/2]$. This proves the right hand side of the inequality (37).

By the same procedure we can obtain the left side of (37). The proof of Lemma is completed.

□

Remark 3. Inequality (37) refines the results in references [1, 7, 9] and sharpens Jordan's inequality. Its detailed discussion can be found in [18].

Now we prove some integral inequalities. Squaring (26) and integrating on both sides over $[0,\pi/2]$ and $[0,\pi/4]$, respectively, we could establish the left hand sides of (30) and (31), respectively.

Integrating (27) on both sides over $[0,\pi/2]$, we can get inequality (32).

Integrating (28) and (29) on both sides over $[0,x]$, $x\in[0,\pi/2]$, respectively, then inequalities (33) and (34) are obtained, respectively.

Rewriting inequality (26) as

$$\frac{x}{\sin x}\leqslant\frac{2}{1+\cos x}=\Big(\sec\frac{x}{2}\Big)^2,\quad x\in\Big[0,\frac{\pi}{2}\Big]. \tag{38}$$

Squaring (38) on both sides and integrating over $[0,t]$, then the inequality in (35) is established. Integrating (38) on both sides over $[0,t]$, $t \in [0,\pi/2]$, then the inequality in (36) follows.

Integrating the right inequality of (37) over $[\pi/4,\pi/2]$, we get the first inequality of (32).

Squaring the right side of (37) and integrating over $[0,\pi/2]$ and $[0,\pi/4]$, respectively, then we have the right-hand sides of (30) and (31), respectively.

This completes the proofs of integral inequalities.

Remark 4. Inequality (26) implies

$$\sin x \geqslant \frac{4x}{4+x^2}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{39}$$

The referee pointed out that the constant 4 is not the optimal value.

3.4. More integral inequalities related to sin *x/x*

Choosing $p(x) = \sin x$, $f(x) = (1+x)^{-1}$, $g(x) = 1+x$, $x \in [0,\pi/2]$ in (1) and computing directly arrives at

$$\int_0^x \frac{\sin t}{1+t}\,\mathrm{d}t \geqslant \frac{(1-\cos x)^2}{1-\cos x - x\cos x + \sin x}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{40}$$

Taking $p(x) = 1$, $f(x) = x^\alpha \sin x$, $g(x) = x^{[\alpha]+1-\alpha}$ for $\alpha > 0$, $x \in [0,\pi/2]$ into inequality (1) yields

$$\int_a^b t^\alpha \sin t\,\mathrm{d}t \leqslant \frac{([\alpha]-\alpha+2)(b-a)}{b^{[\alpha]-\alpha+2} - a^{[\alpha]-\alpha+2}} \int_a^b t^{[\alpha]+1} \sin t\,\mathrm{d}t, \tag{41}$$

where $a,b \in [0,\pi/2]$, $b > a$, and $[\alpha]$ denotes the Gauss function.

It is well-known that the function $\sin x/x$ decreases on $[0,\pi]$. If let p, f, g be as follows

$$p(x) = 1, \quad f(x) = x^\alpha, \quad g(x) = \frac{\sin x}{x^\alpha};$$
$$p(x) = 1, \quad f(x) = \sin x, \quad g(x) = \frac{x^\alpha}{\sin x};$$

then we have, respectively

$$\int_a^b \frac{\sin t}{t^\alpha}\,\mathrm{d}t \geqslant \frac{(\alpha+1)(b-a)(\cos a - \cos b)}{b^{\alpha+1} - a^{\alpha+1}}, \; a,b \in [0,\pi],\; b > a,\; \alpha \geqslant 1; \tag{42}$$

$$\int_a^b \frac{t^\alpha}{\sin t}\,\mathrm{d}t \leqslant \frac{(b-a)(b^{\alpha+1} - a^{\alpha+1})}{(\alpha+1)(\cos a - \cos b)}, \; a,b \in \left[0, \frac{\pi}{2}\right],\; b > a,\; \alpha > 1. \tag{43}$$

If $a,b \in [\frac{\pi}{2},\pi]$, the inequality (43) is reversed.

4. Miscellaneous inequalities

4.1. Let $\alpha>0$, $\beta>0$ be real constants, define

$$f_{\alpha,\beta}(x)=\int_0^x \frac{t^\alpha}{(1-t)^\beta}\,\mathrm{d}\,t,\quad g_{\alpha,\beta}(x)=\int_0^x \frac{t^\alpha}{1-t^\beta}\,\mathrm{d}\,t,\quad x\in[0,1). \tag{44}$$

Note that we also can take $\beta=0$ in $f_{\alpha,\beta}(x)$.

Let p, f, and g be as follows

$$\begin{aligned}
&p(t)=1,\quad f(t)=\frac{t^\alpha}{1-t^\beta},\quad g(t)=1-t^\beta;\\
&p(t)=1,\quad f(t)=t^\gamma,\quad g(t)=\frac{t^\alpha}{(1-t)^\beta};\\
&p(t)=1,\quad f(t)=(1-t)^\gamma,\quad g(t)=\frac{t^\alpha}{(1-t)^\beta};\\
&p(t)=1,\quad f(t)=\frac{1}{1-t},\quad g(t)=\frac{t^\alpha}{(1-t)^\beta};\\
&p(t)=1,\quad f(t)=t^\gamma,\quad g(t)=\frac{t^\alpha}{1-t^\beta};
\end{aligned}$$

then we obtain

$$g_{\alpha,\beta}(x)\geqslant\frac{(\beta+1)x^{\alpha+1}}{(\alpha+1)(\beta+1-x^\beta)},\quad x\in[0,1); \tag{45}$$

$$f_{\alpha,\beta}(x)\leqslant\frac{\gamma+1}{x^\gamma}f_{\alpha+\gamma,\beta}(x),\quad \alpha,\beta,\gamma>0,\quad x\in(0,1); \tag{46}$$

$$f_{\alpha,\beta}(x)\geqslant\frac{x(\gamma+1)}{1-(1-x)^{\gamma+1}}f_{\alpha,\beta-\gamma}(x),\quad 0\leqslant\gamma\leqslant\beta,\quad x\in(0,1); \tag{47}$$

$$f_{\alpha,\beta}(x)\leqslant-\frac{x}{\ln(1-x)}f_{\alpha,\beta+1}(x),\quad x\in(0,1); \tag{48}$$

$$g_{\alpha,\beta}(x)\leqslant\frac{\gamma+1}{x^\gamma}g_{\alpha+\gamma,\beta}(x),\quad \gamma>0,\quad x\in(0,1). \tag{49}$$

For $\gamma<0$ and $\gamma\neq-1$, the inequality (45) is reversed.

For $x<0$, we have

$$\frac{(n+2)|x|^{n+1}}{(n+1)[n+2-(n+1)x]}\leqslant(-1)^{n+1}\int_0^x\frac{t^n}{1-t}\,\mathrm{d}\,t\leqslant\frac{2|x|^{n+1}}{(n+1)(2-x)}. \tag{50}$$

Inequality (50) improves the related results in [7, p. 597].

4.2. Let $h_{\alpha,\beta}(x)=\int_0^x t^\alpha/(1+t^\beta)\,\mathrm{d}\,t$, $\alpha>0$, $\beta>0$, $x>0$. For $\alpha>0$, $\beta>0$, $x>0$, let $p(t)=t^\alpha$, $f(t)=\frac{1}{1+t^\beta}$, and $g(t)=1+t^\beta$, we have

$$h_{\alpha,\beta}(x)\geqslant\frac{(\alpha+\beta+1)x^{\alpha+1}}{(\alpha+1)[\alpha+\beta+1+(\alpha+1)x^\beta]}. \tag{51}$$

If let $p(t) = 1$, $f(t) = 1 + t$, and $g(t) = \frac{t^\alpha}{1+t}$, we get

$$h_{\alpha,1}(x) \leqslant \frac{2x^{\alpha+1}}{(\alpha+1)(x+2)} \tag{52}$$

for $x > 0$ and $\alpha > 1$ or $0 < x < \frac{\alpha}{1-\alpha}$ and $0 < \alpha < 1$.

4.3. For $p(x) = 1$, $f(x) = (\sin x)^r$, $g(x) = \sin x$ or $\cos x$ in inequality (1) with $r > 0$, calculating directly produces

$$\int_0^x (\sin t)^r \,\mathrm{d}\,t \geqslant \frac{x}{r+1}(\sin x)^r, \quad x \in \left[0, \frac{\pi}{2}\right]; \tag{53}$$

$$\int_0^x (\sin t)^r \,\mathrm{d}\,t \leqslant \frac{x}{1-\cos x}\int_0^x (\sin t)^{r+1} \,\mathrm{d}\,t, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{54}$$

In particular, taking $x = \pi/2$, $r = n$ being a positive integer leads to inequalities similar to Wallis' inequality below

$$\frac{(2n-1)!!}{(2n)!!} \leqslant \frac{(2n)!!}{(2n+1)!!} \leqslant \frac{\pi^2}{4} \cdot \frac{(2n+1)!!}{(2n+2)!!}. \tag{55}$$

4.4. Define

$$q_\alpha(x) = \int_0^x \operatorname{tg}^\alpha t \,\mathrm{d}\,t, \quad \alpha > 0, \quad x \in \left[0, \frac{\pi}{2}\right). \tag{56}$$

Let $p(x) = 1$, $f(x) = \operatorname{tg}^\alpha x$, $g(x) = \sec^2 x$ or $\operatorname{tg} x$, then, from (1), we establish two inequalities

$$q_\alpha(x) \leqslant \frac{x}{\alpha+1} \operatorname{tg}^\alpha x, \quad x \in \left[0, \frac{\pi}{2}\right); \tag{57}$$

$$q_\alpha(x) \leqslant -\frac{x}{\ln\cos x} q_{\alpha+1}(x), \quad x \in \left[0, \frac{\pi}{2}\right). \tag{58}$$

4.5. Assume

$$\begin{aligned}
&p(x) = 1, \quad f(x) = \frac{\cos x}{\sqrt{1-x^2}}, \quad g(x) = \sqrt{1-x^2};\\
&p(x) = 1, \quad f(x) = \sec x, \quad g(x) = \frac{\cos x}{\sqrt{1-x^2}};\\
&p(x) = 1, \quad f(x) = \cos x, \quad g(x) = \sqrt{1-x^2};\\
&p(x) = 1, \quad f(x) = \cos x, \quad g(x) = \frac{\sqrt{1-x^2}}{\cos x};\\
&p(x) = 1, \quad f(x) = \frac{1}{\sqrt{1-x^2}}, \quad g(x) = \frac{\sqrt{1-x^2}}{\cos x}
\end{aligned}$$

in (1), then

$$\frac{4\sin 1}{\pi} \leqslant \int_0^1 \frac{\cos x}{\sqrt{1-x^2}} \,\mathrm{d}\,x \leqslant \frac{\pi}{2\ln(\sec 1 + \operatorname{tg} 1)}, \tag{59}$$

$$\frac{\pi \sin 1}{4} \leqslant \int_0^1 \sqrt{1-x^2} \cos x \,\mathrm{d}\,x, \tag{60}$$

$$\frac{\pi}{4\sin 1} \geqslant \int_0^1 \frac{\sqrt{1-x^2}}{\cos x} \,\mathrm{d}\,x \geqslant \frac{2\ln(\sec 1 + \operatorname{tg} 1)}{\pi}. \tag{61}$$

These inequalities refine or generalize some related results in [7, p. 598].

4.6. Substitute $p(x)=1$, $f(x)=(1-x^\alpha)^\beta$ or $(1+x^\alpha)^\beta$, $g(x)=x^{\alpha-1}$ into (1), then we get

$$\int_0^x (1-t^\alpha)^\beta \, \mathrm{d}t \geqslant \frac{x^{1-\alpha}[1-(1-x^\alpha)^{\beta+1}]}{\beta+1}, \quad \alpha>1,\ \beta>0,\ x\in[0,1); \quad (62)$$

$$\int_0^x (1+t^\alpha)^\beta \, \mathrm{d}t \leqslant \frac{x^{1-\alpha}[(1+x^\alpha)^{\beta+1}-1]}{\beta+1}, \quad \alpha>1,\ \beta>0,\ x>0. \quad (63)$$

When $\beta<0$ and $\beta\neq-1$, inequalities (62) and (63) are reversed. If $\beta=-1$, then

$$\int_0^x (1+t^\alpha)^{-1} \, \mathrm{d}t \geqslant x^{1-\alpha}\ln(1+x^\alpha), \quad \alpha>1, \quad x>0; \quad (64)$$

$$\int_0^x (1-t^\alpha)^{-1} \, \mathrm{d}t \leqslant -x^{1-\alpha}\ln(1-x^\alpha), \quad \alpha>1, \quad x\in(0,1). \quad (65)$$

4.7. Let $I_n=\int_0^1(1-x^2)^n\,\mathrm{d}x$, $J_n(x)=\int_0^x(1+t^2)^n\,\mathrm{d}t$ for $x>0$, n being a positive integer. Substitution of

$$p(x)=1, \quad f(x)=(1-x^2)^n, \quad g(x)=\frac{1}{1+x};$$
$$p(x)=1, \quad f(x)=\frac{1}{1+x^2}, \quad g(x)=(1+x^2)^n;$$
$$p(x)=1, \quad f(x)=1+x^2, \quad g(x)=(1+x^2)^n;$$

into (1) leads respectively to

$$(\ln 2)I_n \leqslant I_{n-1}-\frac{1}{2n}; \quad (66)$$
$$(\operatorname{arctg} x)J_n(x) \geqslant xJ_{n-1}(x), \quad n\geqslant 2; \quad (67)$$
$$(3+x^2)J_n(x) \leqslant 3J_{n+1}(x), \quad n\geqslant 1. \quad (68)$$

The inequality (68) is sharper than inequality (67). The referee pointed out that the constant $\ln 2$ in (66) could be replaced by $25/32$.

Remark 5. In [6, pp. 442–444] and [7, p. 597], the following inequalities are given

$$\int_0^1 (1-x^2)^n \, \mathrm{d}x \geqslant \frac{2}{3\sqrt{n}}, \qquad \int_{-1}^1 (1-x^2)^n \, \mathrm{d}x > \left(\frac{\pi}{n+1}\right)^{1/2} > \frac{1}{\sqrt{n}}.$$

Remark 6. Using Tchebysheff's integral inequality, more general results could be yielded, for details, see [9, 12, 13, 14, 19, 21, 22]. From Tchebysheff's integral inequality, we also can generalize the concepts of means to the generalized weighted mean values with two parameters, and prove their monotonicities, see [12, 13, 19, 21, 22, 23].

Acknowledgements. The authors express their thanks to the referee for many helpful and valuable comments and suggestions to this paper.

REFERENCES

[1] U. ABEL AND D. CACCIA, *A sharpening of Jordan's inequality*, Amer. Math. Monthly **93** (1986), no. 7, 568.

[2] J. T. CHU, *On bounds for the normal integral*, Biometrika **42** (1955), 263–265.

[3] J. M. CONTE, *Solution of problem 5607*, Revue Math. Spec. **74** (1963/64), 227–230.

[4] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, 1952.

[5] A. A. KRISHNASWAMI AYYANGAR, *Problem 1800*, Math. Student **11** (1943), 63.

[6] G. KLAMBAUER, *Mathematical Analysis*, Chinese Edition, The People's Press of Hunan, Changsha, China, 1981. (Translated by Ben-Wang Sun)

[7] JI-CHANG KUANG, *Applied Inequalities*, 2nd edition, Hunan Education Press, Changsha, China, 1993. (Chinese)

[8] D. S. MITRINOVIĆ, *Problem 5555*, Amer. Math. Monthly **75** (1968), 84 and 1129–1130.

[9] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[10] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[11] A. OSTOWSKI, *Vorlesungen über Differential-und Integralrechnung*, Vol. 3, Basel-Stuttgart, 1954.

[12] FENG QI, *Proofs for a class of integral inequalities and their applications*, In "Collected Papers of the First Youth Academic Annual Meeting of Henan Province", 91–93, Press of Science and Technology of China, Beijing, China, 1995. (Chinese)

[13] FENG QI, *Generalized weighted mean values with two parameters*, Proc. Roy. Soc. London Ser. A **454** (1998), no. 1978, 2723–2732.

[14] FENG QI, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics **29** (1998), no. 2, 155–163.

[15] FENG QI, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, submitted for publication.

[16] FENG QI AND BAI-NI GUO, *Lower bound of the first eigenvalue for the Laplace operator on compact Riemannian manifold*, Chinese Quarterly Journal of Mathematics, **8** (1993), no. 2, 40–49.

[17] FENG QI AND SEN-LIN GUO, *Inequalities for the incomplete gamma and related functions*, Math. Ineq. Appl. **2** (1999), no. 1, 47–53.

[18] FENG QI AND QIN-DAO HAO, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (1998), no. 3, 116–120.

[19] FENG QI AND ZHENG HUANG, *Inequalities of the complete elliptic integrals*, Tamkang Journal of Mathematics **29** (1998), no. 3, 165–169.

[20] FENG QI AND JIA-QIANG MEI, *Some inequalities of the incomplete gamma and related functions*, Zeitschrift für Analysis und ihre Anwendungen **18** (1999), no.3, 793–799.

[21] FENG QI AND SEN-LIN XU, *The function* $(b^x - a^x)/x$ *: Inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359.

[22] FENG QI AND SEN-LIN XU, *Refinements and extensions of an inequality,* II, J. Math. Anal. Appl. **211** (1997), 616–620.

[23] FENG QI AND SHI-QIN ZHANG, *Note on monotonicity of generalized weighted mean values, Proc. Roy. Soc. London Ser. A* **455** *(1999), no. 1989, 3259–3260.*

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com

(Received November 23, 1998)

Feng Qi
Department of Mathematics
Jiaozuo Institute of Technology
Jiaozuo City
Henan 454000
The People's Republic of China
e-mail: qifeng@jzit.edu.cn

Li-Hong Cui
Department of Fundamental Courses
Zhengzhou Grain College
Zhengzhou City
Henan 450052
The People's Republic of China

Sen-Lin Xu
Department of Mathematics
University of Science and Technology of China
Hefei City
Anhui 230026
The People's Republic of China