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SOME NORM INEQUALITIES INVOLVING FUNCTIONS OF TWO VARIABLES

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Abstract. We will prove some inequalities for unitarily invariant norms involving functions of two variables. Some of our results generalize known results.

1. Introduction

Let $m \in \mathbf{N}$. By M_m, we will denote the space of $m \times m$ complex matrices. I_m will denote the identity element in M_m. I, J will be intervals in $\mathbf{R}$. By $S_m(I)$, we will denote the set of all hermitian matrices in M_m whose spectrum is contained in I. P_m will denote the set of all positive definite matrices in M_m and S_m will denote the set of all positive semidefinite matrices in M_m. $|||\cdot|||$ will denote a unitarily invariant norm on M_m where as $\|\cdot\|$ will denote the spectral norm. The Schur product or the Hadamard product of two matrices $X, Y \in M_m$ will be denoted as $X \circ Y$. This is the matrix whose (i,j) entry is $x_{ij}y_{ij}$. The halmos $\square$ will denote the end of the proof.

The classical arithmetic-geometric mean inequality for positive numbers a, b

$$(ab)^{\frac{1}{2}} \leqslant \frac{a+b}{2}$$

has been extended to the case of matrices in several ways (see [5], [6], [8]). In section 2, we will consider a more general case of a generalization considered in [5]. In section 3, we will prove some inequalities for norms involving operator monotone functions of two variables studied in [9]. In this section the influence of Ando [1] is easily discernable.

2. Arithmetic-geometric mean inequalities

A real valued function f defined on an interval I is called convex if

$$f(\lambda s + (1-\lambda)t) \leqslant \lambda f(s) + (1-\lambda)f(t)$$

for all $s, t \in I$ and $0 \leqslant \lambda \leqslant 1$.

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This definition of convex functions is extendable to the case of functions of two variables. A real valued function F defined on $I \times J$ is called convex if

$$F(\lambda s_1 + (1-\lambda)s_2, \lambda t_1 + (1-\lambda)t_2) \leqslant \lambda F(s_1,t_1) + (1-\lambda)F(s_2,t_2)$$

for all $s_1, s_2 \in I$, $t_1, t_2 \in J$ and $0 \leqslant \lambda \leqslant 1$. In case $J = I$, the function F is called diagonally convex if the corresponding function f of single variable defined by

$$f(s) = F(s,s); \quad s \in I$$

is convex. It is clear that if $F : I \times I \to \mathbf{R}$ is convex then F is diagonally convex, however the converse need not be true. The function $F(s,t) = st$ is diagonally convex on $(0,\infty)\times(0,\infty)$ but is not convex on $(0,\infty)\times(0,\infty)$. For properties of convex functions the reader may refer to [11].

It is proved in [5] that the function

$$f(s) = |||A^{1+s}XB^{1-s} + A^{1-s}XB^{1+s}|||$$

is convex on $[-1,1]$, where $A, B \in P_n$ and $X \in M_m$. It is equivalent to saying that the function $F(s,t)$ of two variables defined as $F(s,t) = |||A^{1+s}XB^{1-t} + A^{1-s}XB^{1+t}|||$ is diagonally convex on $[-1,1]\times[-1,1]$. Here we will prove that the function F is convex on $[-1,1]\times[-1,1]$. To prove this we need the following lemma, a proof of which can be found in [5].

LEMMA 2.1. *Let $A, B, X \in M_n$. Then*

$$|||A^*XB||| \leqslant \frac{1}{2}|||A^*AX + XBB^*|||.$$

THEOREM 2.2. *Let $A, B \in P_n$ and $X \in M_n$. Then the function*

$$F(s,t) = |||A^{1+s}XB^{1-t} + A^{1-s}XB^{1+t}|||$$

is convex on $[-1,1]\times[-1,1]$ and attains its minimum at $(0,0)$.

Proof. The function F is continuous and $F(s,t) = F(-s,-t)$. Thus both conclusions will follow if we show that

$$F(s_1,t_1) \leqslant \frac{1}{2}[F(s_1+s_2, t_1+t_2) + F(s_1-s_2, t_1-t_2)],$$

whenever $s_1 \pm s_2, t_1 \pm t_2 \in [-1,1]\times[-1\times 1]$.

Let $s,t \in [-1,1]$. Consider the map

$$M_{(s,t)}(Y) = \frac{1}{2}(A^sYB^{-t} + A^{-s}YB^t),$$

$Y \in M_n$. Then we have for each $Y \in M_n$,

$$|||Y||| = |||A^{-s}(A^sYB^t)B^{-t}||| \leqslant \frac{1}{2}|||A^sYB^{-t} + A^{-s}YB^t||| = |||M_{(s,t)}(Y)|||,$$

using Lemma 2.1.

From this inequality it follows that

$$|||M_{(s_1,t_1)}(AXB)||| \leqslant |||M_{(s_2,t_2)}M_{(s_1,t_1)}(AXB)|||$$

for all s_1, s_2 and t_1, t_2 in $[-1,1]$.

But

$$\begin{aligned}M_{(s_2,t_2)}(M_{(s_1,t_1)}(Y)) &= \frac{1}{4}[A^{s_2}(A^{s_1}YB^{-t_1}+A^{-s_1}YB^{t_1})B^{-t_2}+A^{-s_2}(A^{s_1}YB^{-t_1}+A^{-s_1}YB^{t_1})B^{t_2}] \\ &= \frac{1}{4}[A^{s_1+s_2}YB^{-(t_1+t_2)}+A^{-s_1+s_2}YB^{t_1-t_2}+A^{s_1-s_2}YB^{-t_1+t_2}+A^{-(s_1+s_2)}YB^{(t_1+t_2)}]\end{aligned}$$

i.e.

$$M_{(s_2,t_2)}(M_{(s_1,t_1)}) = \frac{1}{2}(M_{(s_1+s_2,t_1+t_2)} + M_{(s_1-s_2,t_1-t_2)}).$$

Therefore, we have

$$|||M_{(s_1,t_1)}(AXB)||| \leqslant \frac{1}{2}\Big(|||M_{(s_1+s_2,t_1+t_2)}(AXB)||| + |||M_{(s_1-s_2,t_1-t_2)}(AXB)|||\Big)$$

i.e.

$$F(s_1,t_1) \leqslant \frac{1}{2}[F(s_1+s_2,t_1+t_2)+F(s_1-s_2,t_1-t_2)]$$

which completes the proof. □

COROLLARY 2.3. *Let $A,B \in P_n$ and $X \in M_n$. Then the function*

$$G(s,t) = |||A^sXB^{1-t} + A^{1-t}XB^t|||$$

is convex on $[0,1]\times[0,1]$.

Proof. Replacing A by $A^{\frac{1}{2}}$, B by $B^{\frac{1}{2}}$ in above theorem and then putting $\frac{1+s}{2} = s_1$, $\frac{1+t}{2} = t_1$, we get the desired result. □

3. Inequalities involving matrix functions of two variables

Let $T \in M_m$ be given. Define the linear map $H_T : M_m \to M_m$ by

$$H_T(X) \to T \circ X,$$

$X \in M_m$. Define $\|H_T\|$, the induced norm of $\|H_T\|$, to be

$$\|H_T\| = \max\{\|T\circ X\| : \|X\| \leqslant 1\}.$$

It is not easy to compute $\|H_T\|$ for a general matrix, but in special case when $T = (t_{ij})$ is positive semidefinite it is known that

$$\|H_T\| = \max\{t_{ii} : i = 1,2,\dots,m\}.$$

The following lemma will be used in the sequel. For a proof of this, the reader may refer to [2].

LEMMA 3.1. *If $T, X \in M_m$, then we have*

$$|||T \circ X||| \leqslant \|H_T\| \, |||X|||,$$

for any unitarily invariant norm $|||\cdot|||$.

Now let $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $B = diag(\mu_1, \mu_2, \ldots, \mu_n)$. Let $f_k(s,t)$ and $g_k(s,t)$ be continuous real valued functions defined on $I \times J$. Let I and J contains the eigenvalues of A and B respectively. Then $f(A,B)$ is an element of $M_m \otimes M_n$ (see [9]). Consider the map

$$\phi(X,Y) = \sum_k \left[g_k(A,B)(I_m \otimes Y)f_k(A,B)(X \otimes I_n) + (X \otimes I_n)f_k(A,B)(I_m \otimes Y)g_k(A,B)\right]$$

from $M_m \times M_n$ to $M_m \otimes M_n$. The map ϕ is realized as the Hadamard multiplier generated by the matrix

$$\left(\sum_k \left[f_k(\lambda_i, \mu_q)g_k(\lambda_i, \mu_p) + f_k(\lambda_j, \mu_p)g_k(\lambda_j, \mu_q)\right]\right)_{i,p;j,q},$$

i.e., via

$$\begin{aligned}(X,Y) &\to \phi(X,Y) \\ &= \left(\sum_k \left[f_k(\lambda_i, \mu_q)g_k(\lambda_i, \mu_p) + f_k(\lambda_j, \mu_p)g_k(\lambda_j, \mu_q)\right]\right) \circ (X \otimes Y).\end{aligned}$$

Thus using Lemma 3.1, we get the following result.

LEMMA 3.2. *Let $T = \left(\sum_k [f_k(\lambda_i, \mu_q)g_k(\lambda_i, \mu_p) + f_k(\lambda_j, \mu_p)g_k(\lambda_j, \mu_q)]\right)_{i,p;j,q}$. If $\|H_T\| \leqslant \gamma$, then*

$$|||\phi(X,Y)||| \leqslant \gamma |||X \otimes Y|||.$$

THEOREM 3.3. *Let $I = J = (0, \infty)$ and $f_k(s,t)$, $g_k(s,t)$ be continuous functions defined on $I \times J$. Let $A, C \in S_m(I)$ and $B, D \in S_n(J)$. If for every choice of $\lambda_i > 0$, $i = 1, 2, \ldots, 2m$ and $\mu_p > 0$, $p = 1, 2, \ldots, 2n$*

$$\|H_T\| \leqslant \gamma \tag{1}$$

where $T = \left(\sum_k \left[\frac{f_k(\lambda_i, \mu_q)g_k(\lambda_i, \mu_p) + f_k(\lambda_j, \mu_p)g_k(\lambda_j, \mu_q)}{(\lambda_i + \lambda_j)(\mu_p + \mu_q)}\right]\right)_{i,p;j,q}$, then

$$\begin{aligned}\left|\left|\left| \sum_k \left[g_k(A,B)(I_m \otimes Y)f_k(A,D)(X \otimes I_n) + (X \otimes I_n)f_k(C,B)(I_m \otimes Y)g_k(C,D)\right] \right|\right|\right| \\ \leqslant \gamma |||(AX + XC) \otimes (BY + YD)|||\end{aligned}$$

for all unitarily invariant norms.

Proof. First we shall prove the theorem when $C = A$, $D = B$. Choose a system of matrix units for M_m and M_n such that

$$A = diag(\lambda_1, \lambda_2, \ldots, \lambda_m)$$

and

$$B = diag(\mu_1, \mu_2, \dots, \mu_n).$$

Let $X = (x_{ij})_{i,j} \in M_m$ and $Y = (y_{p,q})_{p,q} \in M_n$.

Put $X_1 = AX + XA$, $Y_1 = BY + YB$.

Then

$$\sum_k \left[g_k(A,B)(I_m, \otimes Y)f_k(A,B)(X \otimes I_n) + (X \otimes I_n)f_k(A,B)(I_m \otimes Y)g_k(A,B)\right]$$

$$= \left[\frac{\sum_k \left[f_k(\lambda_i, \mu_q)g_k(\lambda_i, \mu_p) + f_k(\lambda_j, \mu_p)g_k(\lambda_j, \mu_q)\right]}{(\lambda_i + \lambda_j)(\mu_p + \mu_q)}\right]_{i,p;j,q} \circ (X_1 \otimes Y_1).$$

Therefore it follows from (1) and Lemma 3.2 that

$$\left|\left|\left| \sum_k \left[g_k(A,B)(I_m \otimes Y)f_k(A,B)(X \otimes I_n) + (X \otimes I_n)f_k(A,B)(I_m \otimes Y)g_k(A,B)\right] \right|\right|\right|$$

$$\leqslant \gamma |||(AX + XA) \otimes (BY + YB)|||. \tag{2}$$

Now on replacing A by $A_1 = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, B by $B_1 = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$, X by $X_1 = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and Y by $Y_1 = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$ in (2), we get the desired result on using that $\left|\left|\left| \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \right|\right|\right| = |||Z|||$. □

The operator monotone functions of two variables has been studied by Koranyi [9]. A function $f : I \times J \to \mathbf{R}$ is called operator monotone if for $A \geqslant C$, $B \geqslant D$, the following

$$f(A,B) - f(C,B) - f(A,D) + f(C,D) \geqslant 0,$$

holds for all $A, C \in S_m(I)$ and $B, D \in S_n(J)$. Throughout, thereafter we shall assume that the operator monotone function $f : I \times J \to \mathbf{R}$, $0 \in I \cap J$ be such that its first partial derivatives and the mixed second partial derivatives exist and are continuous. Also f satisfies the normalization condition $f(s,0) = f(0,t) = 0$ for all $s \in I, t \in J$. Koranyi [9] gave an integral representation for such an operator monotone function on $(-1,1) \times (-1,1)$.

A function $f : (-1,1) \times (-1,1) \to \mathbf{R}$ is operator monotone if and only if f admits the integral representation

$$f(s,t) = \int_{-1}^{1}\int_{-1}^{1} \frac{s}{1 - \lambda s} \frac{t}{1 - \mu t} d\omega(\lambda, \mu),$$

where ω is a positive measure.

We modify this integral representation for operator monotone function $f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ for our purpose. It follows that an operator monotone function on $[0,\infty) \times [0,\infty)$ with $f(s,0) = f(0,t) = 0$ for all $s, t \in [0,\infty)$, is non negative.

Indeed, we have

$$f(s,t) = f(s,t) - f(0,t) - f(s,0) + f(0,0) \geqslant 0.$$

By applaying the transformation $u \to \dfrac{u-1}{u+1}$ from $[0,\infty)$ to $[-1,1]$ for change of variables in the above integral representation we get the following lemma.

LEMMA 3.4. *A function $f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ is operator monotone if and only if f admits the integral representation*

$$f(s,t) = \int_0^\infty \int_0^\infty \frac{s}{\lambda+s} \frac{t}{\mu+t} d\eta(\lambda,\mu),$$

where η is a positive measure.

LEMMA 3.5. *Let $f(s,t)$ be operator monotone on $[0,\infty)\times[0,\infty)$. Then for every choice of $s_i > 0$, $i = 1,2,\ldots,m$ and $t_p > 0$, $p = 1,2,\ldots,n$, the matrix*

$$\left(\frac{f(s_i,t_p)+f(s_i,t_q)+f(s_j,t_p)+f(s_j,t_q)}{(s_i+s_j)(t_p+t_q)}\right)_{i,p;j,q}$$

is positive semidefinite.

Proof. From Lemma 3.4, it follows that the above matrix can be written as

$$\int_0^\infty \int_0^\infty \left(\frac{\frac{s_i}{\lambda+s_i}+\frac{s_j}{\lambda+s_j}}{s+s_j}\right)_{i,j} \otimes \left(\frac{\frac{t_p}{\mu+t_p}+\frac{t_q}{\mu+t_q}}{t_p+t_q}\right)_{p,q} d\eta(\lambda,\mu).$$

Now the matrix $\left(\frac{\frac{s_i}{\lambda+s_i}+\frac{s_j}{\lambda+s_j}}{s+s_j}\right)_{i,j}$ and $\left(\frac{\frac{t_p}{\mu+t_p}+\frac{t_q}{\mu+t_q}}{t_p+t_q}\right)_{p,q}$ are both positive semidefinite.

Indeed,

$$\left(\frac{\frac{s_i}{\lambda+s_i}+\frac{s_j}{\lambda+s_j}}{s+s_j}\right)_{i,j} = 2DXD + \lambda D_1 J_m D_1,$$

where $D = diag\left(\frac{s_1}{\lambda+s_1},\ldots,\frac{s_m}{\lambda+s_m}\right)$, $X = \left(\frac{1}{s_i+s_j}\right)_{i,j}$, $D_1 = diag\left(\frac{1}{\lambda+s_1},\ldots,\frac{1}{\lambda+s_m}\right)$ and J_m is $m\times m$ matrix with each entry equal to 1. The matrix X being the Cauchy matrix is known to be positive semidefinite. Thus the matrix in question is positive semidefinite.

Similarly the matrix $\left(\frac{\frac{t_p}{\mu+t_p}+\frac{t_q}{\mu+t_q}}{t_p+t_q}\right)_{p,q}$ is positive semidefinite.

Hence the result follows since the tensor product of positive semidefinite matrices is positive semidefinite. □

THEOREM 3.6. *Let* $A, C \in P_m$ *and* $B, D \in P_n$ *be such that* $A \geqslant \varepsilon_1 I_m > 0$, $C \geqslant \varepsilon_1 I_m > 0$ *and* $B \geqslant \varepsilon_2 I_n > 0$, $D \geqslant \varepsilon_2 I_n > 0$ *and let* $f(s,t)$ *be operator monotone on* $[0,\infty) \times [0,\infty)$. *Then*

$$|||f(A,B)(X \otimes Y) + (X \otimes I_n) f(C,B)(I_m \otimes Y) + (I_m \otimes Y) f(A,D)(X \otimes I_n) + (X \otimes Y) f(C,D)||| \leqslant \frac{f(\varepsilon_1, \varepsilon_2)}{\varepsilon_1 \varepsilon_2} |||(AX + XC) \otimes (BY + YYD)|||,$$

for all $X \in M_m$ *and* $Y \in M_n$.

Proof. The function $f(s,t)$ is concave coordinatewise and therefore the function $\frac{f(s,t)}{st}$ is decreasing coordinatewise. Hence for every choice of $s_i > 0$, $i = 1, 2, \ldots, m$ and $t_p > 0$, $p = 1, 2, \ldots, n$, we have

$$\|H_T\| = \max_{s_i > 0, t_j > 0} \frac{f(s_i, t_p)}{s_i t_p} = \frac{f(\varepsilon_1, \varepsilon_2)}{\varepsilon_1 \varepsilon_2}$$

since by Lemma 3.5 the matrix $T = \left(\frac{f(s_i,t_p) + f(s_i,t_q) + f(s_j,t_p) + f(s_j,t_q)}{(s_i + s_j)(t_p + t_q)} \right)_{i,p;j,q}$ is positive semidefinite. The result now follows from Theorem 3.3.

LEMMA 3.7. *Let* $f(s,t)$ *be operator monotone on* $[0,\infty) \times [0,\infty)$. *Then for every choice of* $s_i > 0$, $i = 1, 2, \ldots, m$ *and* $t_p > 0$, $p = 1, 2, \ldots, n$, *the matrix*

$$\left(\frac{f(s_i,t_p) - f(s_i,t_q) - f(s_j,t_p) + f(s_j,t_q)}{(s_i - s_j)(t_p - t_q)} \right)_{i,p;j,q}$$

is positive semidefinite. The entries in above matrix are to be defined appropriately in case $s_i = s_j$ *or* $t_p = t_q$.

A proof of the above lemma can be found in [3].

The Lemma 3.7 gives us the following result, a proof of which is exactly similar to the proof of Theorem 3.6.

THEOREM 3.8. *Let* $A, C \in P_m$ *and* $B, D \in P_n$ *be such that* $A \geqslant \varepsilon_1 I_m > 0$, $C \geqslant \varepsilon_1 I_m > 0$ *and* $B \geqslant \varepsilon_2 I_n > 0$, $D \geqslant \varepsilon_2 I_n > 0$ *and let* $f(s,t)$ *be operator monotone on* $[0,\infty) \times [0,\infty)$. *Then*

$$|||f(A,B)(X \otimes Y) + (X \otimes I_n) f(C,B)(I_m \otimes Y) - (I_m \otimes Y) f(A,D)(X \otimes I_n) + (X \otimes Y) f(C,D)||| \leqslant \left. \frac{\partial^2 f}{\partial s \partial t} \right|_{s=\varepsilon_1, t=\varepsilon_2} |||(AX - XC) \otimes (BY - YD)|||,$$

for all $X \in M_m$ *and* $Y \in M_n$. □

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