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ON SOME GENERALIZATIONS OF OSTROWSKI INEQUALITY FOR LIPSCHITZ FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

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Abstract. Some generalizations of Ostrowski inequality for Lipschitz functions and functions of bounded variation are given.

1. Introduction

In 1938, Ostrowski [1] (see also [2, p. 468]) proved the following integral inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M, \quad \forall x\in[a,b] \tag{1.1}$$

where $f:[a,b]\to\mathbf{R}$ is differentiable function such that $|f'(x)|\leqslant M$ for all $x\in[a,b]$.

G. V. Milovanović and J. Pečarić [3] and A. M. Fink [4] (see also [2, p. 470]) have considered generalizations of (1.1) in the form

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant K(n,p,x)\left\|f^{(n)}\right\|_p, \tag{1.2}$$

where $F_k(x)$ is defined by

$$F_k(x)=\frac{n-k}{k!(b-a)}\left[f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k\right]. \tag{1.3}$$

For $n=1$ the sum above is defined to be zero.

In fact, G. V. Milovanović and J. Pečarić have proved that ([2, p.469]):

$$K(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+1)!n(b-a)}, \tag{1.4}$$

while A. M. Fink proved that

$$K(n,p,x)=\frac{[(x-a)^{np'+1}+(b-x)^{np'+1}]^{1/p'}}{n!(b-a)}B((n-1)p'+1,p'+1)^{1/p'}, \tag{1.5}$$

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where $1 < p \leqslant \infty$, $1/p + 1/p' = 1$, B is the beta function, and

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n!n^n(b-a)} \max\left\{(x-a)^n, (b-x)^n\right\}. \tag{1.6}$$

Further generalization of this result is given in [10].

For $n = 1$ relation (1.6) becomes

$$K(1,1,x) = \frac{1}{b-a} \max\left\{x-a, b-x\right\}. \tag{1.7}$$

This result was recently obtained by S. S. Dragomir and S. Wang [5] in equivalent form

$$K(1,1,x) = \frac{1}{2} + \frac{1}{b-a}\left|x - \frac{a+b}{2}\right|. \tag{1.8}$$

To simplify some expressions, in the rest of this paper we shall use the notation

$$\Delta(x) := \max\left\{x-a, b-x\right\} = \frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|, \quad x \in [a,b].$$

Of course, since $\max\left\{(x-a)^n, (b-x)^n\right\} = \Delta^n(x)$, one can write (1.6) in equivalent form

$$K(n,1,x) = \frac{(n-1)^{n-1}}{n!n^n(b-a)} \Delta^n(x). \tag{1.9}$$

Dragomir and Wang [6] have also obtained (1.5) for $n = 1$ that is

$$K(1,p,x) = \frac{\left[(x-a)^{p'+1} + (b-x)^{p'+1}\right]^{1/p'}}{(b-a)(p'+1)^{1/p'}} \tag{1.10}$$

and gave various applications of this result.

In this paper we shall give some generalizations of previous results and prove some similar inequalities for Lipschitz functions and functions of bounded variation. Such results are generalization of results from [7], [8] and [9].

2. Some identities

Let (P_n) be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geqslant 1$, and $P_0 = 1$. Furthermore, let $I \subset \mathbf{R}$ be a segment and $f : I \to \mathbf{R}$ such that $f^{(n-1)}$ is Lipschitz function or is a continuous function of bounded variation on I, for some $n \geqslant 1$. Consider

$$\begin{aligned}
&(-1)^{n-1}\int_y^x P_{n-1}(t)\,df^{(n-1)}(t) \\
&\quad = (-1)^{n-1}P_{n-1}(t)f^{(n-1)}(t)\Big|_y^x + (-1)^{n-2}\int_y^x P_{n-2}(t)f^{(n-1)}(t)\,dt \\
&\quad = (-1)^{n-1}\left[P_{n-1}(x)f^{(n-1)}(x) - P_{n-1}(y)f^{(n-1)}(y)\right] \\
&\qquad + (-1)^{n-2}\int_y^x P_{n-2}(t)f^{(n-1)}(t)\,dt
\end{aligned}$$

for $x, y \in I$. By applying the same procedure to the last integral we successively get the relation

$$(-1)^{n-1} \int_y^x P_{n-1}(t) df^{(n-1)}(t)$$
$$= \sum_{k=1}^{n-1} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y) \right] + f(x) - f(y) \tag{2.1}$$

for $x, y \in I$. If we set $x = a,\ y = b,\ n = m+1$ and replace $f(t)$ by $\int_a^t f(u)du$ in (2.1) we get

$$\int_a^b f(t)dt = \sum_{k=1}^{m} (-1)^k \left[P_k(a) f^{(k-1)}(a) - P_k(b) f^{(k-1)}(b) \right]$$
$$+ (-1)^m \int_a^b P_m(t) f^{(m)}(t)dt. \tag{2.2}$$

By integration with respect to y, (2.1) becomes

$$\int_a^b f(y)dy = (b-a) \left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) \right]$$
$$- \sum_{k=1}^{n-1} (-1)^k \int_a^b P_k(y) f^{(k)}(y)dy + (-1)^n \int_a^b \int_y^x P_{n-1}(t) df^{(n-1)}(t)dy. \tag{2.3}$$

Using (2.2) we have

$$\int_a^b f(y)dy = (b-a) \left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) \right]$$
$$- \sum_{k=1}^{n-1} \left[\sum_{j=1}^{k} (-1)^j \left[P_j(b) f^{(j-1)}(b) - P_j(a) f^{(j-1)}(a) \right] + \int_a^b f(t)dt \right]$$
$$+ (-1)^n \int_a^b \int_y^x P_{n-1}(t) df^{(n-1)}(t)dy$$

that is

$$n \int_a^b f(y)dy = (b-a) \left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) \right]$$
$$- \sum_{k=1}^{n-1} (-1)^k (n-k) \left[P_k(b) f^{(k-1)}(b) - P_k(a) f^{(k-1)}(a) \right]$$
$$+ (-1)^n \int_a^b \int_y^x P_{n-1}(t) df^{(n-1)}(t)dy. \tag{2.4}$$

Using notation

$$\widetilde{F}_k := \frac{(-1)^k (n-k)}{b-a}\left[P_k(a) f^{(k-1)}(a) - P_k(b) f^{(k-1)}(b)\right]$$

and

$$k(t,x) := \begin{cases} t-a, & t \in [a,x] \\ t-b, & t \in (x,b] \end{cases},$$

relation (2.4) becomes

$$\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1} \widetilde{F}_k\right] - \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t) k(t,x) df^{(n-1)}(t). \tag{2.5}$$

The sums above are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials

$$P_k(t) = \frac{(t-x)^k}{k!}, k \geqslant 0$$

relation (2.5) becomes

$$\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1} F_k(x)\right] - \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1} k(t,x) df^{(n-1)}(t), \tag{2.6}$$

where $F_k(x)$ is defined by (1.3).

For the harmonic sequence of polynomials

$$P_k(t) = \frac{1}{k!}\left(t - \frac{a+b}{2}\right)^k, k \geqslant 0$$

relation (2.5) becomes

$$\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}\left(x - \frac{a+b}{2}\right)^k f^{(k)}(x)\right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}(n-k)}{k! 2^k}\left[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b)\right]\right] - \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{1}{n!(b-a)}\int_a^b \left(\frac{a+b}{2} - t\right)^{n-1} k(t,x) df^{(n-1)}(t). \tag{2.7}$$

For $y \in [a,b]$ and the harmonic sequence of polynomials

$$P_k(t) = \frac{1}{k!}(t-y)^k, k \geqslant 0$$

relation (2.5) gives

$$\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{(y-x)^k}{k!}f^{(k)}(x)\right.$$
$$\left.+\sum_{k=1}^{n-1}\frac{(n-k)}{k!(b-a)}\left[(y-a)^k f^{(k-1)}(a)-(y-b)^k f^{(k-1)}(b)\right]\right]-\frac{1}{b-a}\int_a^b f(t)dt$$
$$=\frac{1}{n!(b-a)}\int_a^b (y-t)^{n-1}k(t,x)df^{(n-1)}(t). \tag{2.8}$$

These identities are important in the context of the present paper and will be used to prove many inequalities.

3. Inequalities for Lipschitz functions

THEOREM 1. *Let (P_k) be a harmonic sequence of polynomials and $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is L-Lipschitz function for some $n\geqslant 1$ i.e.*

$$|f^{(n-1)}(x)-f^{(n-1)}(y)|\leqslant L\cdot|x-y|,\quad x,y\in[a,b]$$

where $L\geqslant 0$ is a constant. Then

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F}_k\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant\frac{L}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,dt$$

where $\widetilde{F}_k$ and $k(t,x)$ are from relation (2.5).

Proof. For integrable function $F:[a,b]\to\mathbf{R}$ we have

$$\left|\int_a^b F(t)df^{(n-1)}(t)\right|\leqslant L\int_a^b |F(t)|\,dt$$

since $f^{(n-1)}$ is L-Lipschitz function.

Let us apply this estimation to the relation (2.5). We get

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F}_k\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$=\left|\frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t)k(t,x)df^{(n-1)}(t)\right|$$
$$\leqslant\frac{L}{n(b-a)}\int_a^b |P_{n-1}(t)k(t,x)|\,dt$$

which proves our assertion. □

COROLLARY 1. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant L\frac{(x-a)^{n+1}+(b-x)^{n+1}}{(n+1)!n(b-a)}$$

where $F_k(x)$ is given by (1.3).

Proof. Put $P_k(t)=\frac{1}{k!}(t-x)^k,\ k\geqslant 0,$ in the theorem above. □

REMARK 1. For $n=1$ the inequality of the corollary above becomes

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant L\frac{(x-a)^2+(b-x)^2}{2(b-a)}$$

It was proved by J. Pečarić and B. Savić in [8, Teorema 8, p. 190] and rediscovered recently by S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang in [9]. This inequality is a special case of a result proved in [8] for Hölder functions.

COROLLARY 2. *Let f be defined as in the theorem above and $n\geqslant 1$. Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{(y-x)^k}{k!}f^{(k)}(x)\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n-1}\frac{(n-k)}{k!(b-a)}\left[(y-a)^kf^{(k-1)}(a)-(y-b)^kf^{(k-1)}(b)\right]\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant\frac{L}{n!(b-a)}I_n(x,y)$$

for every $x,y\in[a,b]$, where

$$I_n(x,y)=\frac{(y-a)^{n+1}+(b-y)^{n+1}}{n(n+1)}+\frac{2}{n+1}|x-y|^{n+1}$$
$$+\frac{2}{n}\left(\frac{a+b}{2}-y\right)(y-x)|x-y|^{n-1}.$$

Proof. Put $P_k(t)=\frac{1}{k!}(t-y)^k,\ k\geqslant 0,$ in the theorem above and use the formula

$$\int_a^b|t-y|^{n-1}|k(t,x)|\,dt=I_n(x,y),\quad n\geqslant 1.$$

□

COROLLARY 3. *Let f be defined as in the theorem above and $n\geqslant 1$. Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\left(x-\frac{a+b}{2}\right)^kf^{(k)}(x)\right.\right.$$
$$\left.\left.+\sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(a)-(-1)^kf^{(k-1)}(b)\right]\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant\frac{L}{(n+1)!}\left[\frac{2}{b-a}\left|x-\frac{a+b}{2}\right|^{n+1}+\frac{(b-a)^n}{n2^n}\right].$$

Proof. Put $y = \frac{a+b}{2}$ in Corollary 2. □

COROLLARY 4. *Let f' be an L-Lipschitz on $[a,b]$. Then*

$$\left|\frac{1}{2}\left[f(x)+(y-x)f'(x)+\frac{(y-a)f(a)-(y-b)f(b)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{L}{2(b-a)}I_2(x,y)$$

for every $x,y \in [a,b]$.

Proof. Put $n=2$ in Corollary 2. □

COROLLARY 5. *Let f be defined as in the theorem above and $n \geqslant 1$. Then*

$$\left|\frac{1}{n}\left[f\left(\frac{a+b}{2}\right)+\sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)\right]\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant L\frac{(b-a)^n}{(n+1)!n2^n}.$$

Proof. Put $x = \frac{a+b}{2}$ in Corollary 3. □

COROLLARY 6. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[f(a)+\sum_{k=1}^{n-1}\frac{(-1)^{k-1}(n-k)}{k!}(b-a)^{k-1}f^{(k-1)}(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant L\frac{(b-a)^n}{(n+1)!n}$$

and

$$\left|\frac{1}{n}\left[f(b)+\sum_{k=1}^{n-1}\frac{(n-k)}{k!}(b-a)^{k-1}f^{(k-1)}(a)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant L\frac{(b-a)^n}{(n+1)!n}.$$

Proof. Put $x=a$ and $x=b$ in Corollary 1. □

COROLLARY 7. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1}\frac{(n-k)}{2\cdot k!}(b-a)^{k-1}\left[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)\right]\right]\right.$$
$$\left.-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant L\frac{(b-a)^n}{(n+1)!n}.$$

Proof. Add the inequalities of Corollary 6 using the triangle inequality. □

COROLLARY 8. *Let f' be L-Lipschitz function on $[a,b]$. Then*

$$\left|\frac{1}{2}\left[f(x)+\frac{f(a)(x-a)-f(b)(x-b)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant \frac{L}{12(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

for every $x\in[a,b]$. Further

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant L\frac{(b-a)^2}{48}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant L\frac{(b-a)^2}{12}.$$

Proof. Put $n=2$ in Corollary 1 to get the first relation and $x=\frac{a+b}{2}$ in the first relation to get the second. Further, put $n=2$ in corollary 7 to get the last relation. □

4. Inequalities for functions of bounded variation

THEOREM 2. *Let (P_k) be a harmonic sequence of polynomials and $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geqslant 1$. Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F}_k\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant \frac{1}{n(b-a)}\max_{a\leqslant t\leqslant b}|P_{n-1}(t)k(t,x)|\cdot V_a^b(f^{(n-1)})$$

where $\widetilde{F}_k$ and $k(t,x)$ are from relation (2.5), and $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. If $F:[a,b]\to\mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral

$$\int_a^b F(t)df^{(n-1)}(t)$$

exists, then

$$\left|\int_a^b F(t)df^{(n-1)}(t)\right| \leqslant \sup_{a\leqslant t\leqslant b}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Let us apply this estimation to the relation (2.5). We have

$$\begin{aligned}
&\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x)f^{(k)}(x)+\sum_{k=1}^{n-1}\widetilde{F}_k\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\quad=\left|\frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t)k(t,x)df^{(n-1)}(t)\right| \\
&\quad\leqslant \frac{1}{n(b-a)}\max_{a\leqslant t\leqslant b}|P_{n-1}(t)k(t,x)|\cdot V_a^b(f^{(n-1)})
\end{aligned}$$

which proves our assertion. □

COROLLARY 9. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}F_k(x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant K(n,1,x)V_a^b(f^{(n-1)})$$

where $F_k(x)$ is defined by (1.3) and $K(n,1,x)$ by (1.9).

Proof. Put $P_k(t)=\frac{1}{k!}(t-x)^k,\ k\geqslant 0,$ in the theorem above. □

REMARK 2. For $n=1$ the inequality of the corollary above becomes

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{2}+\frac{1}{b-a}\left|x-\frac{a+b}{2}\right|\right]\cdot V_a^b(f).$$

This inequality was proved recently by S. S. Dragomir [7].

COROLLARY 10. *Let f be defined as in the theorem above. Then for $n\geqslant 2$ we have*

$$\begin{aligned}
&\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}\frac{(y-x)^k}{k!}f^{(k)}(x)\right.\right. \\
&\qquad\left.\left.+\sum_{k=1}^{n-1}\frac{(n-k)}{k!(b-a)}\left[(y-a)^k f^{(k-1)}(a)-(y-b)^k f^{(k-1)}(b)\right]\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
&\leqslant\frac{1}{n!(b-a)}\max\left\{\frac{(n-1)^{n-1}}{n^n}\Delta^n(y),|x-y|^{n-1}\Delta(x)\right\}\cdot V_a^b(f^{(n-1)})
\end{aligned}$$

for every $x,y\in[a,b]$.

Proof. Put $P_k(t) = \frac{1}{k!}(t-y)^k,\ k \geqslant 0,$ in the theorem above and use the formula

$$\max_{a\leqslant t\leqslant b} |t-y|^{n-1}\,|k(t,x)| = \max\left\{\frac{(n-1)^{n-1}}{n^n}(y-a)^n,\right.$$
$$\left.\frac{(n-1)^{n-1}}{n^n}(b-y)^n, |x-y|^{n-1}(x-a), |x-y|^{n-1}(b-x)\right\}$$
$$= \max\left\{\frac{(n-1)^{n-1}}{n^n}\Delta^n(y), |x-y|^{n-1}\Delta(x)\right\}$$

which can be easily proved. □

COROLLARY 11. *Let f be defined as in the theorem above. Then for $n \geqslant 2$ we have*

$$\left|\frac{1}{n}\left[f(x) + \sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\left(x-\frac{a+b}{2}\right)^k f^{(k)}(x)\right.\right.$$
$$\left.\left. + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b)\right]\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant \frac{1}{n!(b-a)}\max\left\{\frac{(n-1)^{n-1}}{n^n}\left(\frac{b-a}{2}\right)^n, \left|x-\frac{a+b}{2}\right|^{n-1}\Delta(x)\right\}\cdot V_a^b(f^{(n-1)}).$$

Proof. Put $y = \frac{a+b}{2}$, in Corollary 10. □

COROLLARY 12. *Let f be defined as in the theorem above. Then for $n \geqslant 2$ we have*

$$\left|\frac{1}{n}\left[f\left(\frac{a+b}{2}\right) + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}(n-k)}{k!2^k}\left[f^{(k-1)}(a) - (-1)^k f^{(k-1)}(b)\right]\right]\right.$$
$$\left. - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant \frac{(n-1)^{n-1}}{n!n^n 2^n}(b-a)^{n-1}\cdot V_a^b(f^{(n-1)}).$$

Proof. Put $x = \frac{a+b}{2}$ in Corollary 11. □

COROLLARY 13. *If f' has bounded variation on $[a,b]$ then*

$$\left|\frac{1}{2}\left[f(x) + (y-x)f'(x) + \frac{(y-a)f(a)-(y-b)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leqslant \frac{1}{2(b-a)}\max\left\{\frac{1}{4}\Delta^2(y), |x-y|\,\Delta(x)\right\}\cdot V_a^b(f')$$

for every $x, y \in [a,b]$.

Proof. Put $n = 2$ in Corollary 10. □

COROLLARY 14. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[f(a)+\sum_{k=1}^{n-1}\frac{(n-k)}{k!}(b-a)^{k-1}(-1)^{k-1}f^{(k-1)}(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{(n-1)^{n-1}}{n!n^n}(b-a)^{n-1}\cdot V_a^b(f^{(n-1)})$$

and

$$\left|\frac{1}{n}\left[f(b)+\sum_{k=1}^{n-1}\frac{(n-k)}{k!}(b-a)^{k-1}f^{(k-1)}(a)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{(n-1)^{n-1}}{n!n^n}(b-a)^{n-1}\cdot V_a^b(f^{(n-1)}).$$

Proof. Put $x = a$ and $x = b$ in Corollary 9. □

COROLLARY 15. *Let f be defined as in the theorem above. Then*

$$\left|\frac{1}{n}\left[\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1}\frac{(n-k)}{2\cdot k!}(b-a)^{k-1}\left[f^{(k-1)}(a)-(-1)^k f^{(k-1)}(b)\right]\right] -\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{(n-1)^{n-1}}{n!n^n}(b-a)^{n-1}\cdot V_a^b(f^{(n-1)})$$

Proof. Add the inequalities of Corollary 14 using the triangle inequality. □

COROLLARY 16. *Let f' has bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{2}\left[f(x)+\frac{f(a)(x-a)-f(b)(x-b)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{8(b-a)}\Delta^2(x)\cdot V_a^b(f')$$

for every $x\in[a,b]$. Further

$$\left|\frac{1}{2}\left[f(\frac{a+b}{2})+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{32}\cdot V_a^b(f')$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{b-a}{8}\cdot V_a^b(f').$$

Proof. Put $n = 2$ in Corollary 9 to get the first relation and $x = \frac{a+b}{2}$ in the first relation to get the second. To get the last relation put $n = 2$ in corollary 15. □

5. Convergence rate for generalized Taylor formula

Now we shall give some further results about generalized Taylor formula. For some related results see [11].

THEOREM 3. *Let* (P_k) *be a harmonic sequence of polynomials and* $f : [a,b] \to \mathbf{R}$ *such that* $f^{(n-1)}$ *is* L*-Lipschitz function for some* $n \geqslant 2$ *i.e.*

$$|f^{(n-1)}(x) - f^{(n-1)}(y)| \leqslant L \cdot |x-y|, \quad x,y \in [a,b]$$

where $L \geqslant 0$ *is a constant. Then*

$$\left| \sum_{k=1}^{n-1} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y) \right] + f(x) - f(y) \right| \leqslant L \int_y^x |P_{n-1}(t)|\, dt$$

for every $x,y \in [a,b]$, $y \leqslant x$.

Proof. For integrable function $F : [a,b] \to \mathbf{R}$ we have

$$\left| \int_a^b F(t) df^{(n-1)}(t) \right| \leqslant L \int_a^b |F(t)|\, dt$$

since $f^{(n-1)}$ is L-Lipschitz function.

Let us apply this estimation to the relation (2.1). We get

$$\begin{aligned} &\left| \sum_{k=1}^{n-1} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y) \right] + f(x) - f(y) \right| \\ &\qquad = \left| (-1)^{n-1} \int_y^x P_{n-1}(t) df^{(n-1)}(t) \right| \\ &\qquad \leqslant L \int_y^x |P_{n-1}(t)|\, dt \end{aligned}$$

which proves our assertion. □

COROLLARY 17. *Let* f *be defined as in the theorem above. Then*

$$\left| \sum_{k=1}^{n-1} \frac{(x-y)^k}{k!} f^{(k)}(y) - f(x) + f(y) \right| \leqslant L \frac{(x-y)^n}{n!}$$

for every $x,y \in [a,b]$, $y \leqslant x$.

Proof. Put $P_k(t) = \frac{1}{k!}(t-x)^k$, $k \geqslant 0$ in the theorem above. □

COROLLARY 18. *Let* f *be defined as in the theorem above. Then*

$$\begin{aligned} &\left| \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left[(x-a)^k f^{(k)}(x) - (y-a)^k f^{(k)}(y) \right] + f(x) - f(y) \right| \\ &\qquad \leqslant L \frac{(x-a)^n - (y-a)^n}{n!} \end{aligned}$$

for every $x,y \in [a,b]$, $y \leqslant x$.

Proof. Put $P_k(t) = \frac{1}{k!}(t-a)^k,\ k \geqslant 0$ in the theorem above. □

THEOREM 4. *Let (P_k) be a harmonic sequence of polynomials and $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is a function of bounded variation for some $n \geqslant 2$. Then*

$$\left| \sum_{k=1}^{n-1} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y) \right] + f(x) - f(y) \right| \leqslant \max_{y \leqslant t \leqslant x} |P_{n-1}(t)| \cdot V_y^x(f^{(n-1)})$$

for every $x, y \in [a,b],\ y \leqslant x$.

Proof. If $F : [a,b] \to \mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral

$$\int_a^b F(t)\, df^{(n-1)}(t)$$

exists, then

$$\left| \int_a^b F(t)\, df^{(n-1)}(t) \right| \leqslant \sup_{a \leqslant t \leqslant b} |F(t)| \cdot V_a^b(f^{(n-1)}).$$

Let us apply this estimation to the relation (2.1). We have

$$\begin{aligned} &\left| \sum_{k=1}^{n-1} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(y) f^{(k)}(y) \right] + f(x) - f(y) \right| \\ &= \left| (-1)^{n-1} \int_y^x P_{n-1}(t)\, df^{(n-1)}(t) \right| \\ &\leqslant \max_{y \leqslant t \leqslant x} |P_{n-1}(t)| \cdot V_y^x(f^{(n-1)}) \end{aligned}$$

which proves our assertion. □

COROLLARY 19. *Let f be defined as in the theorem above. Then*

$$\left| \sum_{k=1}^{n-1} \frac{(x-y)^k}{k!} f^{(k)}(y) - f(x) + f(y) \right| \leqslant \frac{(x-y)^{n-1}}{(n-1)!} \cdot V_y^x(f^{(n-1)})$$

for every $x, y \in [a,b],\ y \leqslant x$.

Proof. Put $P_k(t) = \frac{1}{k!}(t-x)^k,\ k \geqslant 0$ in the theorem above. □

COROLLARY 20. *Let f be defined as in the theorem above. Then*

$$\left| \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left[(x-a)^k f^{(k)}(x) - (y-a)^k f^{(k)}(y) \right] + f(x) - f(y) \right| \leqslant \frac{(x-a)^{n-1}}{(n-1)!} \cdot V_y^x(f^{(n-1)})$$

for every $x, y \in [a,b],\ y \leqslant x$.

Proof. Put $P_k(t) = \frac{1}{k!}(t-a)^k, \ k \geqslant 0$ in the theorem above. □

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