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ON A WEIGHTED GENERALIZATION OF IYENGAR TYPE INEQUALITIES INVOLVING BOUNDED FIRST DERIVATIVE

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(*communicated by R. P. Agarwal*)

Abstract. Inequalities are obtained for weighted integrals in terms of bounds involving the first derivative of the function. Quadrature rules are obtained and the classical Iyengar inequality for the trapezoidal rule is recaptured as a special case when the weight function $w(x) \equiv 1$. Applications to numerical integration are demonstrated.

1. Introduction

In 1938, Iyengar proved the following theorem obtaining bounds for a trapezoidal quadrature rule for functions whose derivative $|f'(x)| < M$ for $x \in (a,b)$ (see for example [3, p. 471]).

THEOREM 1. *Let f be a differentiable function on (a,b) and assume that there is a constant $M > 0$ such that $|f'(x)| \leqslant M$, $\forall x \in (a,b)$. Then we have*

$$\left|\int_b^a f(x)\,dx - (b-a)\frac{f(a)+f(b)}{2}\right| \leqslant \frac{M(b-a)^2}{4} - \frac{1}{4M}(f(b)-f(a))^2. \quad (1.1)$$

Using a classical inequality due to Hayashi (see, [2, pp. 311-312]), Agarwal and Dragomir proved in [1] the following generalization of Theorem 1.

THEOREM 2. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on $\mathring{I}$, the interior of I, and let $a,b \in \mathring{I}$ with $a < b$. Let $M = \sup_{x\in[a,b]} f'(x) < \infty$ and $m = \inf_{x\in[a,b]} f'(x) > -\infty$. If $m < M$ and f' is integrable on $[a,b]$, then we have*

$$\left|\int_a^b f(t)\,dt - (b-a)\frac{f(a)+f(b)}{2}\right| \leqslant \frac{[f(b)-f(a)-m(b-a)][M(b-a)-f(b)+f(a)]}{2(M-m)}. \quad (1.2)$$

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Thus, substituting $m = -M$ in (1.2) reproduces Iyengar's result (1.1).

Cerone and Dragomir [6] produced a number of generalizations of the above results and in particular, they obtained a non-symmetric inequality for a generalized trapezoidal quadrature rule. The following theorem is proved by Cerone and Dragomir [6].

THEOREM 3. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be a differentiable mapping on* $\mathring{I}$ *(the interior of* I*) and* $[a,b] \subset \mathring{I}$ *with* $M = \sup_{x\in[a,b]} f'(x) < \infty,\ m = \inf_{x\in[a,b]} f'(x) > -\infty$ *and* $M > m$. *If* f' *is integrable on* $[a,b]$, *then the following inequality holds:*

$$\beta_L \leqslant \int_a^b f(x)\,dx - [(\theta - a)f(a) + (b-\theta)f(b)] \leqslant \beta_U \tag{1.3}$$

where

$$\beta_U = \frac{(b-a)^2}{2(M-m)}\left[S(2\gamma_U - S) - mM\right],$$
$$\beta_L = \frac{(b-a)^2}{2(M-m)}\left[S(S - 2\gamma_L) + mM\right]$$

with

$$\gamma_U = \left(\frac{\theta - a}{b-a}\right)M + \left(\frac{b-\theta}{b-a}\right)m,\ \gamma_L = M + m - \gamma_U,$$

and

$$S = \frac{f(b) - f(a)}{b-a}.$$

Taking $\theta = \frac{a+b}{2}$ in (1.3) will reproduce the result of Agarwal and Dragomir [1], (1.2). In addition, if $m = -M$ then (1.1) of Iyengar is obtained.

In the current article a trapezoidal type rule is obtained for the weighted integral $\int_a^b w(x)f(x)\,dx$ and thus may be looked upon as a generalization of the Agarwal and Dragomir [1] result. Placing $m = -M$ gives a generalization of the Iyengar result (1.1) for weighted integrals.

2. Preliminaries

Some definitions are required to simplify the subsequent work.

DEFINITION 1. Let $w(x)$ be a positive integrable function on $[a,b]$. Let ν be its zeroth moment about zero so that

$$\nu = \int_a^b xw(x)\,dx < \infty. \tag{2.1}$$

DEFINITION 2. P and Q will be used to denote the zeroth and first moments of $w(x)$ over a subinterval of $[a,b]$. In particular, for $\lambda > 0$ the subscript a or b will be used to indicate the intervals $[a, a+\lambda]$ and $[b-\lambda, b]$ respectively. Thus, for example,

$$P_a = \int_a^{a+\lambda} w(x)\,dx$$

and

$$Q_b = \int_{b-\lambda}^{b} xw(x)\,dx.$$

The following theorem is due to Hayashi (see for example [2, pp. 311-312]).

THEOREM 4. *Let* $h : [a,b] \to \mathbf{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $g : [a,b] \to \mathbf{R}$ *an integrable mapping on* $[a,b]$ *with*

$$0 \leqslant g(x) \leqslant A, \text{ for all } x \in [a,b].$$

Then

$$A\int_{b-\lambda}^{b} h(x)\,dx \leqslant \int_{a}^{b} h(x)\,g(x)\,dx \leqslant A\int_{a}^{a+\lambda} h(x)\,dx \tag{2.2}$$

where

$$\lambda = \frac{1}{A}\int_{a}^{b} g(x)\,dx.$$

Hayashi's inequality (2.2) will now be used to obtain inequalities for weighted integrals to give trapezoidal type quadrature rules.

3. Trapezoidal inequality for weighted integrals

An inequality for weighted integrals will now be developed for a trapezoidal rule, but firstly, two lemmas will need to be proved by the use of the Hayashi inequality.

LEMMA 1. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be a differentiable mapping on* $\mathring{I}$ *(the interior of* I*) and* $[a,b] \subset \mathring{I}$ *with* $M = \sup_{x\in[a,b]} f'(x) < \infty$, $m = \inf_{x\in[a,b]} f'(x) > -\infty$ *and* $M > m$. *Let* $w(x) \geqslant 0$ *for all* $x \in [a,b]$ *and* $\nu = \int_a^b xw(x)\,dx < \infty$ *be the first moment of* $w(\cdot)$ *on* $[a,b]$. *If* f' *is integrable on* $[a,b]$ *then the following inequality holds:*

$$\begin{aligned}(M-m)&\left[Q_b-(b-\lambda)P_b\right]\\ &\leqslant \int_a^b w(x)f(x)\,dx - \nu\left(f(a)-ma\right) - \frac{m}{2}\left(b^2-a^2\right)\\ &\leqslant (M-m)\left[Q_a-(\lambda+a)P_a+\lambda\nu\right]\end{aligned} \tag{3.1}$$

where P, Q *are as described in Definition 2 and* $\lambda = \frac{b-a}{M-m}(S-m)$, $S = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $h_b(x) = \int_x^b w(u)\,du$ and $g(x) = f'(x) - m$. Then from Hayashi's inequality (2.2)

$$L_b \leqslant I_b \leqslant U_b \tag{3.2}$$

where

$$I_b = \int_a^b h_b(x)\left(f'(x)-m\right)dx,$$

$$\lambda = \frac{1}{M-m}\int_a^b \left(f'(x)-m\right)dx,$$

and

$$L_b = (M-m)\int_{b-\lambda}^{b} h_b(x)\,dx,$$
$$U_b = (M-m)\int_{a}^{a+\lambda} h_b(x)\,dx.$$

Now, an integration by parts gives

$$I_b = -\nu\left(f(a) - ma\right) - \frac{m}{2}\left(b^2 - a^2\right) + \int_a^b w(x) f(x)\,dx. \tag{3.3}$$

Also,

$$\lambda = \frac{b-a}{M-m}(S-m) \tag{3.4}$$

where

$$S = \frac{f(b) - f(a)}{b-a},$$

the slope of the secant over $[a,b]$.

It should be noted that $0 < \lambda \leqslant b-a$ since $S \leqslant M$.

For the lower bound L_b a change of order of integration gives

$$\begin{aligned} \frac{L_b}{M-m} &= \int_{b-\lambda}^{b} w(x) \int_{b-\lambda}^{u} dx du \\ &= (\lambda - b) P_b + Q_b \end{aligned} \tag{3.5}$$

where P_b and Q_b are as described in Definition 2.

Similarly, the upper bound U_b may be obtained through a change of order of integration to give

$$\begin{aligned} \frac{U_b}{M-m} &= \int_a^{a+\lambda} w(u) \int_a^u dx du + \int_{a+\lambda}^{b} w(u) \int_a^{a+\lambda} dx du \\ &= \int_a^{a+\lambda} (u-a) w(u)\,du + \lambda \int_{a+\lambda}^{b} w(u)\,du \\ &= Q_a - (\lambda + a) P_a + \lambda \nu \end{aligned} \tag{3.6}$$

where P_a and Q_a are as described in Definition 2 and ν is the zeroth moment of $w(x)$ on $[a,b]$.

Using $(3.2)-(3.6)$ the lemma is thus proved. □

LEMMA 2. *Let the conditions be as in Lemma 1 then the following inequality holds:*

$$\begin{aligned} &(M-m)\left[Q_b - (\lambda - b) P_b - \lambda \nu\right] \\ &\quad \leqslant \int_a^b w(x) f(x)\,dx - \nu\left(f(b) - mb\right) - \frac{m}{2}\left(b^2 - a^2\right) \\ &\quad \leqslant (M-m)\left[Q_a - (\lambda + a) P_a\right]. \end{aligned} \tag{3.7}$$

Proof. The proof follows along similar lines to that of Lemma 1.

Let $h_a(x) = -\int_a^x w(u)\,du$ and $g(x) = f'(x) - m$. Then using Hayashi's inequality (2.2) gives:

$$L_a \leqslant I_a \leqslant U_a \tag{3.8}$$

where

$$I_a = \int_a^b h_a(x)\left(f'(x) - m\right)dx$$

and

$$L_a = (M-m)\int_{b-\lambda}^{b} h_a(x)\,dx,$$
$$U_a = (M-m)\int_a^{a+\lambda} h_a(x)\,dx.$$

Now, a straight forward integration by parts yields

$$I_a = -\nu\,(f(b) - mb) - \frac{m}{2}\left(b^2 - a^2\right) + \int_a^b w(x) f(x)\,dx. \tag{3.9}$$

Further, an interchange of the order of integration and simplification of results yields

$$\frac{L_a}{M-m} = Q_b + (\lambda - b)\,P_b - \lambda\nu \tag{3.10}$$

and

$$\frac{U_a}{M-m} = Q_a - (\lambda + a)\,P_a. \tag{3.11}$$

Hence, using (3.8) – (3.11) the lemma is proved. □

THEOREM 5. *Let the conditions of Lemmas 1 and 2 be maintained. Then the following inequality holds:*

$$\begin{aligned}(M-m)&\left[Q_b - (b-\lambda)\,P_b - \frac{\lambda}{2}\nu\right]\\ &\leqslant \int_a^b w(x) f(x)\,dx - \frac{\nu}{2}\left[f(a) + f(b)\right] - m\left(\frac{a+b}{2}\right)(b-a-\nu)\\ &\leqslant (M-m)\left[Q_a - (\lambda+a)\,P_a + \frac{\lambda}{2}\nu\right]\end{aligned} \tag{3.12}$$

where the P's and Q's are as defined in Definition 2.

Proof. Addition of (3.1) and (3.7) produces (3.12) upon division by 2. □

COROLLARY 1. *Let the conditions be as in the previous Lemmas and theorems of this section. Then,*

$$\left|\int_a^b w(x) f(x)\,dx - \frac{\nu}{2}[f(a)+f(b)] - m\left(\frac{a+b}{2}\right)[b-a-\nu]\right| \leqslant \frac{\nu}{2}(b-a)(S-m) \leqslant \frac{M-m}{2}\nu(b-a) \tag{3.13}$$

where S is the slope of the secant on $[a,b]$.

Proof. The corollary follows readily from (3.12) on noting that

$$Q_b = \int_{b-\lambda}^b xw(x)\,dx \geqslant (b-\lambda)\int_{b-\lambda}^b w(x)\,dx,$$
$$Q_a = \int_a^{a+\lambda} xw(x)\,dx \leqslant (\lambda+a)\int_a^{a+\lambda} w(x)\,dx$$

and substituting $(M-m)\lambda = (b-a)(S-m)$. □

REMARK 1. Allowing $w(x)\equiv 1$ in (3.12) gives from Definitions 1 and 2

$$\nu = b-a, P_a = P_b = \lambda, Q_a = \frac{\lambda}{2}(\lambda+2a) \text{ and } Q_b = \frac{\lambda}{2}(2b-\lambda).$$

This reveals the lower bound to be negative the upper bound and the result of Cerone and Dragomir [6] for the unweighted trapezoidal rule is recovered after some algebra.

REMARK 2. If $w(x)\equiv 1$ in (3.13) then the coarser upper bound is obtained for the unweighted trapezoidal rule (1.2) since $\frac{M-S}{M-m} < 1$.

REMARK 3. The bounds in (3.12) are not symmetric in general since for this to be so they must sum to zero. Let L_1 be the lower bound and U_1 be the upper bound. Then

$$U_1 + L_1 = (M-m)\left[(Q_b - (b-\lambda)P_b) - ((\lambda+a)P_a - Q_a)\right].$$

We know from the proof of Corollary 1 that $Q_b \geqslant (b-\lambda)P_b$ and $Q_a \leqslant (\lambda+a)P_a$, so $U_1+L_1 = 0$ when $Q_b-(b-\lambda)P_b = (\lambda+a)P_a - Q_a$.

LEMMA 3. *Let the conditions of Theorem 3 and Lemmas 1 and 2 hold. Then, for $w(x)$ symmetric about the mid-point $\frac{a+b}{2}$, the bounds in (3.12) are symmetric. Hence,*

$$\left|\int_a^b w(x) f(x)\,dx - \frac{\nu}{2}[f(a)+f(b)] - m\left(\frac{a+b}{2}\right)[b-a-\nu]\right| \leqslant (M-m)\left[\frac{\lambda}{2}\nu - \int_0^\lambda uw(\lambda+a-u)\,du\right]. \tag{3.14}$$

Proof. From Remark 3 and Definition 2, the sum of the upper and lower bounds in (3.12), U_1 and L_1 respectively is:

$$\begin{aligned} U_1 + L_1 &= (M-m)\left[\int_{b-\lambda}^{b} [x-(b-\lambda)]\, w(x)\, dx - \int_a^{a+\lambda} (\lambda+a-x)\, w(x)\, dx\right] \\ &= (M-m)\left[\int_0^{\lambda} uw(b-\lambda+u)\, du - \int_0^{\lambda} uw(\lambda+a-u)\, du\right]. \end{aligned}$$

Now,

$$U_1 + L_1 = (M-m)\int_0^{\lambda} u\left[w\left(\frac{a+b}{2}+z\right) - w\left(\frac{a+b}{2}-z\right)\right] du$$

where $z = \frac{b-a}{2} - \lambda + u$.

Thus,

$$\begin{aligned} & U_1 + L_1 \\ & = (M-m)\int_{\frac{b-a}{2}-\lambda}^{\frac{b-a}{2}} \left(z+\lambda-\frac{b-a}{2}\right)\left[w\left(\frac{a+b}{2}+z\right) - w\left(\frac{a+b}{2}-z\right)\right] dz \\ & = 0 \end{aligned}$$

for $w(\cdot)$ symmetric about $\frac{a+b}{2}$. Hence, the bounds in (3.12) are symmetric.

Now, from the upper bound in (3.12), U_1 is such that:

$$\begin{aligned} \frac{U_1}{M-m} &= \frac{\lambda}{2}\nu - [(\lambda+a)P_a - Q_a] \\ &= \frac{\lambda}{2}\nu - \int_a^{a+\lambda} (\lambda+a-x)\, w(x)\, dx \\ &= \frac{\lambda}{2}\nu - \int_0^{\lambda} uw(\lambda+a-u)\, du. \end{aligned}$$

Thus, the lemma is proved. □

It should be noted that the expression for U_1 obtained above may be written as

$$\begin{aligned} \frac{U_1}{M-m} &= \frac{\lambda}{2}\nu - \int_0^{\lambda} uw\left(\frac{a+b}{2}-z\right) dz \\ &= \frac{\lambda}{2}\nu - \int_0^{\lambda} uw\left(z-\frac{a+b}{2}\right) dz \end{aligned}$$

where $z = u + \frac{b-a}{2} - \lambda$. Here, we are using the fact that the weight function $w(\cdot)$ is symmetric about the mid-point.

COROLLARY 2. *Let the conditions be as in the previous lemmas and theorems of this section. Then*

$$\begin{aligned}(M-m)\left[Q_b-(b-\lambda)P_b\right]-\nu\left[\frac{b-a}{2}S+am\right] \\ \leqslant \int_a^b w(x)f(x)\,dx-\frac{\nu}{2}\left[f(a)+f(b)\right] \\ \leqslant (M-m)\left[Q_a-(\lambda+a)P_a\right]+\nu\left[\left(\frac{b-a}{2}\right)S-bm\right] \\ +\frac{m}{2}\left(b^2-a^2\right). \qquad (3.15)\end{aligned}$$

Proof. A simple rearrangement of the terms in (3.12), collecting the coefficient of ν and using the fact that $(M-m)\lambda=(b-a)(S-m)$ produces the result. □

REMARK 4. Using similar approximations as those in Corollary 1, simpler bounds may be obtained viz.,

$$\begin{aligned}\frac{m}{2}\left(b^2-a^2\right)-\nu\left[\left(\frac{b-a}{2}\right)S+am\right] \\ \leqslant \int_a^b w(x)f(x)\,dx-\frac{\nu}{2}\left[f(a)+f(b)\right] \\ \leqslant \frac{m}{2}\left(b^2-a^2\right)+\nu\left[\left(\frac{b-a}{2}\right)S-bm\right]. \qquad (3.16)\end{aligned}$$

REMARK 5. There is no real advantage in using rule (3.15) in preference to (3.12) in practice since an adjustment factor involving only the end points for the $\frac{m}{2}\left(b^2-a^2\right)$ term and a simple sum reflecting the effect of the $\frac{m}{2}(b+a)\nu$ term. This will be discussed further in Section 4.

4. Application in numerical integration

In this section we will demonstrate how the results obtained in Section 3 may be utilized to obtain quadrature rules for weighted functions.

THEOREM 6. *Let $f:[a,b]\to\mathbf{R}$ be a differentiable mapping on (a,b) with $M=\sup_{x\in[a,b]}f'(x)<\infty$, $m=\inf_{x\in[a,b]}f'(x)>-\infty$ and $M>m$. Let I_n be a partition of $[a,b]$ such that $I_n: a=x_0<x_1<\ldots<x_{n-1}<x_n=b$. Further, let $w(x)\geqslant 0$ for all $x\in[a,b]$ and $\nu=\int_a^b w(x)\,dx<\infty$ be the first moment of $w(\cdot)$ on $[a,b]$. Then, the following weighted quadrature rule holds*

$$\int_a^b w(x)f(x)\,dx=A(w,f,I_n)+R(w,f,I_n)$$

where $A(w,f,I_n)$ *is an approximation to the weighted integral. Namely,*

$$\begin{aligned} A(w,f,I_n) &= \frac{1}{2}\sum_{i=0}^{n-1} \nu_i \left[f(x_i)+f(x_{i+1}) - m(x_i+x_{i+1})\right] + \frac{m}{2}\left(b^2-a^2\right) \\ &= \frac{1}{2}\left[\nu_0 g_0 + \nu_{n-1} g_n + \sum_{i=1}^{n-1}(\nu_{i-1}+\nu_i) g_i\right] + \frac{m}{2}\left(b^2-a^2\right) \end{aligned}$$

with $g_i = f(x_i) - m x_i$.

In addition, the remainder term $R(w,f,I_n)$ *satisfies*

$$\begin{aligned} |R(w,f,I_n)| &\leqslant \frac{1}{2}\sum_{i=0}^{n-1} \nu_i \left[f(x_{i+1}) - f(x_i) - m(x_{i+1}-x_i)\right] \\ &= \frac{1}{2}\left[\nu_{n-1} g_n - \nu_0 g_0 + \sum_{i=1}^{n-1}(\nu_{i-1}-\nu_i) g_i\right] \\ &\leqslant \frac{M-m}{2}\sum_{i=0}^{n-1}\nu_i h_i, \end{aligned}$$

where $h_i = x_{i+1} - x_i$.

Proof. Applying inequality (3.13) of Corollary 1 on the interval $[x_i, x_{i+1}]$ for $i = 0,1,\ldots,n-1$ we have

$$\begin{aligned} \Bigg| \int_{x_i}^{x_{i+1}} w(x) f(x)\,dx - \frac{\nu_i}{2}\left[f(x_i)+f(x_{i+1})\right] \\ - \frac{m}{2}\left[x_{i+1}^2 - x_i^2 - (x_{i+1}+x_i)\nu_i\right] \Bigg| \\ \leqslant \frac{\nu_i}{2}\left[f(x_{i+1}) - f(x_i) - m(x_{i+1}-x_i)\right]. \end{aligned}$$

Summing over i for $i = 0,1,\ldots,n-1$ gives the quadrature rule

$$\begin{aligned} A(w,f,I_n) &= \frac{1}{2}\sum_{i=0}^{n-1}\nu_i\left[f(x_i)+f(x_{i+1}) - m(x_i+x_{i+1}) + m\left(x_{i+1}^2 - x_i^2\right)\right] \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\nu_i\left[f(x_i)+f(x_{i+1}) - m(x_i+x_{i+1})\right] + \frac{m}{2}\left(b^2-a^2\right) \\ &= \frac{1}{2}\sum_{i=0}^{n-1}\nu_i\frac{1}{2}(g_i+g_{i+1}) + \frac{m}{2}\left(b^2-a^2\right) \end{aligned}$$

where $g_i = f(x_i) - m x_i$.

Hence

$$A(w,f,I_n) = \frac{1}{2}\left[\nu_0 g_0 + \nu_{n-1} g_n + \sum_{i=1}^{n-1}(\nu_{i-1}+\nu_i) g_i\right] + \frac{m}{2}\left(b^2-a^2\right).$$

The remainder term $R(w,f,I_n)$ is such that

$$\begin{aligned}|R(w,f,I_n)| &\leqslant \frac{1}{2}\sum_{i=0}^{n-1}\nu_i\left[f\left(x_{i+1}\right)-f\left(x_i\right)-m\left(x_{i+1}-x_i\right)\right]\\ &= \frac{1}{2}\sum_{i=0}^{n-1}\nu_i\left[g_{i+1}-g_i\right]\\ &= \frac{1}{2}\left[\nu_{n-1}g_n-\nu_0 g_0+\sum_{i=1}^{n-1}\left(\nu_{i-1}-\nu_i\right)g_i\right].\end{aligned}$$

Using the second inequality in Corollary 1 gives

$$|R(w,f,I_n)| \leqslant \frac{M-m}{2}\sum_{i=0}^{n-1}\nu_i h_i.$$

Hence, the theorem is proved. □

If a uniform grid is taken so that $x_i = x_0 + ih, i = 0,1,...,n,$ then

$$|R(w,f,I_n)| \leqslant \frac{M-m}{2}\cdot h\cdot \nu.$$

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