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ON THE EQUIVALENCE OF CLASSES OF FOURIER COEFFICIENTS

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(*communicated by J. Pečarić*)

Abstract. It is shown that the classes of Fourier coefficients defined by Fomin, furthermore by C. V. Stanojevič and V. B. Stanojevič are identical.

1. Introduction

Several authors have studied the question of L^1-convergence of the following cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \tag{1.1}$$

Before recalling sample results we define some known classes of Fourier coefficients.

1. A sequence $\mathbf{a} := \{a_n\}$ belongs to S if there exists a monotonically decreasing sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leqslant A_n$ for all n.

2. Let BV denote the class of all null-sequences $\mathbf{a}$ of bounded variation.

3. A null-sequence $\mathbf{a}$ belongs to the class C if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_0^{\delta} \Big| \sum_{k=n}^{\infty} (\Delta a_k) D_k(x) \Big| dx < \varepsilon,$$

for all n, where $D_k(x)$ is the Dirichlet kernel.

4. A null-sequence $\mathbf{a}$ belongs to the class F_p if for some $p > 1$

$$\sum_{n=1}^{\infty} n^{-1/p} \Big(\sum_{k=n}^{\infty} |\Delta a_k|^p \Big)^{1/p} < \infty. \tag{1.2}$$

It is easy to see that the class F_p is wider when p is closer to 1.

The definition **1** is due to Telyakovskiĭ [5], the definitions **3** and **4** were given by Garrett-Stanojevič [2] and Fomin [1], respectively.

Telyakovskiĭ proved that if $\mathbf{a} \in S$ then (1.1) is a Fourier series of some $f \in L^1(0, \pi)$ and that

$$\|s_n - f\| = o(1), \quad n \to \infty, \tag{1.3}$$

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if and only if

$$a_n \log n = o(1), \quad n \to \infty, \tag{1.4}$$

where s_n are the partial sums of (1.1) and $\|\cdot\|$ is the $L^1(0,\pi)$-norm.

Garrett and Stanojevič verified that if $\mathbf{a} \in C \cap BV$ then (1.1) is a Fourier series of some $f \in L^1(0,\pi)$ and $(1.3) \iff (1.4)$.

Fomin proved that if $1 < p \leqslant 2$ and $\mathbf{a} \in F_p$ then (1.1) is also a Foiurier series of some $f \in L^1(0,\pi)$ and $(1.3) \iff (1.4)$.

It is easy to see that the following embedding relations

$$S \subseteq F_p \subseteq BV \tag{1.5}$$

uphold. Hence it follows that Fomin's result is stronger than that of Telyakovskiĭ.

Stanojevič [3] verified that if $1 < p \leqslant 2$ then

$$F_p \subset C \cap BV, \tag{1.6}$$

consequently the result by Garrett and Stanojevič is sharper than that of Fomin.

5. A sequence $\mathbf{a}$ belongs to the class S_p if $a_n \to 0$ and there exists a monotoncially decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1). \tag{1.7}$$

The relation

$$S \subseteq S_p$$

clearly holds.

Very recently Tomovski [6] also proved that if $1 < p \leqslant 2$ then

$$S_p \subseteq C \cap BV. \tag{1.8}$$

Reading the proof of (1.8) we noticed that the assumption

$$\sum_{m=1}^{\infty} 2^{(1-\frac{1}{p})} \Big\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \Big\}^{1/p} < \infty, \tag{1.9}$$

which claims less or at least not more than the two conditions in (1.7) jointly, is also a sufficient condition to the embedding statement (1.8).

This note gave the thought to consider a further class F_p^*.

6. A sequence $\mathbf{a}$ belongs to the class F_p^* if $a_n \to 0$ and for some $p > 1$ the inequality (1.9) holds.

We have also shown that

$$F_p \subseteq S_p \subseteq F_p^*. \tag{1.10}$$

Having (1.10) it was a natural to set the following question: Does the embedding relation

$$F_p^* \subseteq C \cap BV \tag{1.11}$$

maintain if $1 < p \leqslant 2$?

We succeeded to confirm (1.11), too.

Unfortunately, only from this time on, we observed that

$$F_p^* \subseteq F_p \tag{1.12}$$

can also be verified. Thus however, by (1.10) and (1.12), it follows that the classes F_p, S_p and F_p^* are identical.

Consequently the statements (1.11) and also (1.8) are already proved by Stanojevič's theorem.

To prove our last assertion we have only to verify the embedding relations (1.10) and (1.12) for $p > 1$.

Addendum. We are indebted to the referee of this paper who called our attention to the following facts: the class S_p was defined by C. V. Stanojevič and V. B. Stanojevič [4], furthermore already G. A. Fomin [1] proved that the classes F_p and F_p^* are equivalent.

According to the latter information it would be sufficient to verify only the embeding relations (1.10) but since our proof for (1.12) is very short and different from that of Fomin, we present it, too, in the original form of our manuscript.

2. Result

Our only task is to prove the following declaration.

THEOREM. *If* $p > 1$ *then*

$$F_p \subseteq S_p \subseteq F_p^* \subseteq F_p.$$

At the end of this note we shall present an example showing that the embedding

$$S \subset F_p$$

is a strict embedding relation. Maybe this fact is well known, but our proof is very short and utilize our new theorem.

ADDITION. *There exists a null-sequence* $\mathbf{a}$ *such that* $\mathbf{a} \in F_p$ *but* $\mathbf{a} \notin S$.

3. Proofs

Proof of Theorem. First we prove the relation $F_p \subseteq S_p$. If $\mathbf{a} \in F_p$ then setting

$$R_n := n^{-1/p}\left(\sum_{k=n}^{\infty} |\Delta a_k|^p\right)^{1/p},$$

by (1.2), we get that

$$\sum_{n=1}^{\infty} R_n < \infty. \tag{3.1}$$

Next we construct a monotone decreasing sequence $\{A_n\}$ such that for any $n \geqslant 1$

$$R_n \leqslant A_n, \tag{3.2}$$

$$A_n \leqslant KA_{2n}, \tag{3.3}$$

and

$$\sum_{k=1}^{\infty} A_k < \infty \tag{3.4}$$

hold. In (3.3) and later in the sequal K denotes a positive constant, not necessarily the same on any two occurrences.

Let $A_1 := R_1$ and for $m \geqslant 1$ let

$$A_{2^m} := \max(R_{2^m}, \frac{1}{4}A_{2^{m-1}}). \tag{3.5}$$

For $2^m < n < 2^{m+1}$ let

$$A_n := A_{2^m}. \tag{3.6}$$

Since the sequence $\{R_n\}$ is monotone decreasing, thus, by (3.5) and (3.6), the sequence $\{A_n\}$ is also monotone decreasing.

According to the definition of $\{A_n\}$ the inequalities (3.2) are clearly satisfied.

It is also plain that

$$A_{2^m} \leqslant \max(4R_{2^{m+1}}, A_{2^m}) = 4A_{2^{m+1}},$$

whence, by (3.6), the validity of the inequality (3.3) obviously follows.

To verify (3.4) we define a sequence $\{\nu_i\}$ of the natural numbers. Let ν_i denote those natural numbers when

$$A_{2^{\nu_i}} = R_{2^{\nu_i}}. \tag{3.7}$$

It can be occured that the number of ν_i satisfying (3.7) is finite. In this case, if i_0 is the largest with property (3.7), then we define $\nu_{i_0+1} := \infty$. Then

$$\sum_{m=\nu_1}^{\infty} 2^m A_{2^m} = \sum_{i \geqslant 1} \sum_{m=\nu_i}^{\nu_{i+1}-1} 2^m A_{2^m} = \sum_{i \geqslant 1} 2^{\nu_i} A_{2^{\nu_1}} + \sum_{i \geqslant 1} \sum_{m=\nu_i+1}^{\nu_{i+1}-1} 2^m A_{2^m}. \tag{3.8}$$

In the second sum merely those i appear which satisfy the inequality $\nu_{i+1} > \nu_i + 1$. In this case

$$\sum_{m=\nu_i+1}^{\nu_{i+1}-1} 2^m A_{2^m} = \sum_{k=1}^{\nu_{i+1}-\nu_i-1} 2^{\nu_i+k} A_{2^{\nu_i+k}} = 2^{\nu_i} \sum_{k=1}^{\nu_{i+1}-\nu_i-1} 2^k \frac{A_{2^{\nu_i}}}{4^k}$$
$$\leqslant 2^{\nu_i} R_{2^{\nu_i}} \sum_{k=1}^{\infty} 2^{-k} = 2^{\nu_i} R_{2^{\nu_i}}. \tag{3.9}$$

Summing up the assertions (3.1), (3.8) and (3.9) we get that

$$\sum_{m=\nu_1}^{\infty} 2^m A_{2^m} \leqslant 2 \sum_{i \geqslant 1} 2^{\nu_i} R_{2^{\nu_i}} \leqslant 2 \sum_{m=1}^{\infty} 2^m R_{2^m} < \infty.$$

Herewith we have proved (3.4), too.

Now we turn back to the proof of $F_p \subseteq S_p$. As we have verified, if $\mathbf{a} \in F_p$ then there exists a monotone decreasing sequence $\{A_n\}$ with the properties (3.2), (3.3) and (3.4). Using these inequalities we get that if $2^i \leqslant n < 2^{i+1}$ then

$$\begin{aligned}\sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} &= \sum_{k=1}^{n} \frac{kR_k^p - (k+1)R_{k+1}^p}{A_k^p} \\ &\leqslant \sum_{m=0}^{i} \sum_{k=2^m}^{2^{m+1}-1} (kR_k^p - (k+1)R_{k+1}^p) A_k^{-p} \\ &\leqslant \sum_{m=0}^{i} 2^m R_{2^m}^p A_{2^{m+1}}^{-p} \leqslant K \sum_{m=0}^{i} 2^m A_{2^m}^p A_{2^m}^{-p} \leqslant Kn,\end{aligned}$$

and this proves that $\mathbf{a} \in S_p$ holds, that is, $F_p \subseteq S_p$.

Next we prove the relation $S_p \subseteq F_p^*$. In order to show this embedding we estimate the following sum assuming that $\mathbf{a} \in S_p$:

$$\begin{aligned}&\sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} \Big\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \Big\}^{1/p} \\ &\leqslant \sum_{m=1}^{\infty} 2^m A_{2^m} \Big\{ 2^{-m} \sum_{n=2^m+1}^{2^{m+1}} \frac{|\Delta a_n|^p}{A_n^p} \Big\}^{1/p} < \infty.\end{aligned} \tag{3.10}$$

Here the last sum is finite because the sum in the curly bracket is $O(1)$ by $\mathbf{a} \in S_p$, see (1.7), furthermore $\sum 2^m A_{2^m} < \infty$ since $\sum A_n < \infty$.

The estimation (3.10) verifies the embedding $S_p \subseteq F_p^*$.

Finally we confirm the embedding statement $F_p^* \subseteq F_p$.

In the following calculations we shall use the well-known inequality

$$\Big(\sum b_n\Big)^{\alpha} \leqslant \sum b_n^{\alpha}, \quad b_n \geqslant 0, \quad 0 < \alpha \leqslant 1,$$

with $\alpha = \frac{1}{p}$ and an Abel-rearrangement.

$$\begin{aligned}\sum_{n=2}^{\infty} n^{-1/p} \Big(\sum_{k=n}^{\infty} |\Delta a_k|^p \Big)^{1/p} &\leqslant \sum_{m=0}^{\infty} 2^{-m/p} \sum_{n=2^m+1}^{2^{m+1}} \Big(\sum_{k=n}^{\infty} |\Delta a_k|^p \Big)^{1/p} \\ &\leqslant \sum_{m=0}^{\infty} 2^{m(1-\frac{1}{p})} \sum_{i=m}^{\infty} \Big(\sum_{k=2^i+1}^{2^{i+1}} |\Delta a_k|^p \Big)^{1/p} \\ &= \sum_{i=0}^{\infty} \Big(\sum_{k=2^i+1}^{2^{i+1}} |\Delta a_k|^p \Big)^{1/p} \sum_{m=0}^{i} 2^{m(1-\frac{1}{p})} \\ &\leqslant K \sum_{i=0}^{\infty} 2^{i(1-\frac{1}{p})} \Big(\sum_{k=2^i+1}^{2^{i+1}} |\Delta a_k|^p \Big)^{1/p}.\end{aligned} \tag{3.11}$$

If $\mathbf{a} \in F_p^*$ then the last sum in (3.11) is finite, thus the first one is also finite, and this means that $\mathbf{a} \in F_p$. This shows that $F_p^* \subseteq F_p$ holds.

The proof of over Theorem is complete.

Proof of Addition. Let us define a sequence $\mathbf{a}$ as follows: Let

$$a_1 := 1 \quad \text{and} \quad a_n := 2^{-m} \quad \text{if} \quad 2^{m-1} < n \leqslant 2^m, \ \ m \geqslant 1. \tag{3.12}$$

Then

$$\sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} \Big\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_k|^p \Big\}^{1/p} \leqslant \sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} 2^{-m} < \infty,$$

thus, by our Theorem, $\mathbf{a}$ belongs to F_p.

On the other hand, if $\mathbf{a} \in S$, then the suitable sequence $\{A_n\}$ appearing in the definition of the class S, would have the properties:

$$A_n \geqslant A_{n+1}, \quad |\Delta a_n| \leqslant A_n \quad \text{and} \quad \sum_{m=1}^{\infty} 2^m A_{2^m} < \infty. \tag{3.13}$$

But in the case of the sequence $\mathbf{a}$ given in (3.12)

$$|\Delta a_{2^m}| = 2^{-m-1},$$

therefore, for this sequence $\mathbf{a}$, no sequence $\{A_n\}$ satisfying the three conditions of (3.13) jointly, can be given.

This proves that there exists such a sequence $\mathbf{a}$ which belongs to F_p but does not to S, that is, the $S \subset F_p$ is a strict embedding, indeed.

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