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INEQUALITIES FOR THE INCOMPLETE GAMMA FUNCTION

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(*communicated by A. Laforgia*)

Abstract. We prove some monotonicity results for the incomplete gamma function,

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt,$$

from which some inequalities for $\Gamma(a,x)$ follow.

1. Introduction

Let $\Gamma(a,x)$ denote the incomplete gamma function, i.e.

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt,$$

where both a and t are positive. It is well-known that $\Gamma(a,0) = \Gamma(a)$, and $\lim_{x\to\infty}\Gamma(a,x) = 0$. For positive integer a, say $a = n$, $\Gamma(n,x)$ is an elementary function, since $\Gamma(1,x) = e^{-x}$ and $\Gamma(n,x) = (n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}$ for $n = 2,3,\ldots$. For this reason, we suppose in the sequel that a is not a positive integer.

Also, the complementary error function $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$ is a particular case of the incomplete gamma function, since $\sqrt{\pi}\operatorname{erfc}(x) = \Gamma(\frac{1}{2},x^2)$.

Many authors (see [2],[3],[4]) found inequalities for $\Gamma(a,x)$ and for the related integral $\int_0^x e^{-t^p}dt$. In this paper we prove some inequalities which follow from monotonicity properties.

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2. Main result

THEOREM. *Let a be a positive parameter, and let $q(x)$ be a function, differentiable on $(0,+\infty)$, such that $\lim_{x\to\infty} x^a e^{-x} q(x) = 0$. If we put*

$$T(x) = 1 + (a-x)q(x) + xq'(x), \tag{2.1}$$

then we have the following inequalities:

- *if $T(x) > 0$ for $x > 0$, then $\Gamma(a,x) > x^a e^{-x} q(x)$;*
- *if $T(x) < 0$ for $x > 0$, then $\Gamma(a,x) < x^a e^{-x} q(x)$.*

Proof. Define $j(x)$ as follows:

$$j(x) = \Gamma(a,x) - x^a e^{-x} q(x).$$

By hypotesis, $\lim_{x\to\infty} j(x) = 0$. Taking the derivative, we find

$$\begin{aligned} j'(x) &= -x^{a-1}e^{-x} - ax^{a-1}e^{-x}q(x) + x^a e^{-x} q(x) - x^a e^{-x} q'(x) \\ &= -x^{a-1}e^{-x}[1 + (a-x)q(x) + xq'(x)] = -e^{-x}x^{a-1}T(x). \end{aligned}$$

If $T(x) > 0$ on $\mathbb{R}^+$, then $j(x)$ decreases on $\mathbb{R}^+$, therefore $j(x) > 0$, i.e. $\Gamma(a,x) > x^a e^{-x} q(x)$; while, if $T(x) < 0$ on $\mathbb{R}^+$, then j increases, and we have $\Gamma(a,x) < x^a e^{-x} q(x)$.

REMARK. If $T(x) > 0$ for $x > h$ ($h > 0$), then $j(x)$ decreases on $(h,+\infty)$; in this case we may say that the inequality $\Gamma(a,x) > x^a e^{-x} q(x)$ holds at least for $x > h$; similarly for the case $T(x) < 0$ on $(h,+\infty)$.

3. Some particular cases

In this section we find some inequalities for the incomplete gamma function, using the theorem proved in sec. 2 for particular choices of $q(x)$.

3.1. $q(x) = c$, a positive constant.

In this case we have $T(x) = 1 + ac - cx$, which is negative for $x > a + \frac{1}{c}$. Then we have the inequality

$$\Gamma(a,x) < cx^a e^{-x},$$

which is valid at least for $x > a + \frac{1}{c}$.

3.2. $q(x) = B/x$, with $B > 0$.

Since $T(x) = 1 - B + \frac{B(a-1)}{x}$, we must distinguish three subcases:

3.2.1. $0 < B < 1$. If $a > 1$, then $T(x) > 0$ $\forall x > 0$, while if $0 < a < 1$, then $T(x) > 0$ for $x > \frac{B(1-a)}{1-B}$. So we have

$$\Gamma(a,x) > Bx^{a-1}e^{-x}, \tag{3.1}$$

which holds for every $x > 0$ if $a > 1$, and for $x > \frac{B(1-a)}{1-B}$ if $0 < a < 1$.

3.2.2. $B > 1$. If $a > 1$, then $T(x) < 0$ for $x > \dfrac{B(a-1)}{B-1}$, while if $0 < a < 1$, then $T(x) < 0$ for $x > 0$. So we have

$$\Gamma(a,x) < Bx^{a-1}e^{-x}, \tag{3.2}$$

which holds for every $x > 0$ if $0 < a < 1$, and for $x > \dfrac{B(a-1)}{B-1}$ if $a > 1$. Note that if $a > 1$ formula (3.2) cannot hold for every $x \geqslant 0$, because the right-hand side of (3.2) vanishes for $x = 0$.

3.2.3. $B = 1$. Now we have $T(x) = \dfrac{a-1}{x}$, which is positive or negative in $\mathbb{R}^+$, according to the fact that $a > 1$ or $0 < a < 1$. Then we have:

$$\text{if } a > 1 \Rightarrow \Gamma(a,x) > x^{a-1}e^{-x} \text{ for } x > 0, \tag{3.3}$$

$$\text{if } 0 < a < 1 \Rightarrow \Gamma(a,x) < x^{a-1}e^{-x} \text{ for } x > 0, \tag{3.4}$$

As a consequence of (3.2) and (3.3), we have for $a > 1$

$$x^{a-1}e^{-x} < \Gamma(a,x) < Bx^{a-1}e^{-x}, \tag{3.5}$$

for any $B > 1$; in formula (3.5) lower bound holds for $x > 0$, while upper bound holds only from a certain value on. For example, if we put $B = 11/10$, then we have

$$x^{a-1}e^{-x} < \Gamma(a,x) < \frac{11}{10}x^{a-1}e^{-x},$$

where upper bound is valid for $x > 11(a-1)$.

In a similar way, from (3.1) and (3.4) we have for $0 < a < 1$

$$Bx^{a-1}e^{-x} < \Gamma(a,x) < x^{a-1}e^{-x}, \tag{3.6}$$

for any $B \in (0,1)$; in formula (3.6) upper bound holds for $x > 0$, while lower bound is valid for $x > \dfrac{B(1-a)}{1-B}$. For example, if we put $B = 19/20$, we have

$$\frac{19}{20}x^{a-1}e^{-x} < \Gamma(a,x) < x^{a-1}e^{-x},$$

where lower bound is valid for $x > 19(1-a)$.

3.3. $q(x) = \dfrac{1}{x} + \dfrac{C}{x^2}$**.**

In this case we find $T(x) = \dfrac{(a-C-1)x + (a-2)C}{x^2}$. Instead of considering different subcases, let us note that $T(x)$ becomes very simple when $C = a-1$, precisely $T(x) = \dfrac{(a-1)(a-2)}{x^2}$, whose sign is independent of x. Therefore we have for $x > 0$:

$$\text{for } 0 < a < 1 \text{ or } a > 2 \quad \Rightarrow \quad \Gamma(a,x) > [x^{a-1} + (a-1)x^{a-2}]e^{-x}, \tag{3.7}$$

$$\text{for } 1 < a < 2 \quad \Rightarrow \quad \Gamma(a,x) < [x^{a-1} + (a-1)x^{a-2}]e^{-x}. \tag{3.8}$$

From (3.4) and (3.7) we have in the case $0 < a < 1$

$$[x^{a-1} + (a-1)x^{a-2}]e^{-x} < \Gamma(a,x) < x^{a-1}e^{-x}. \tag{3.9}$$

From (3.3) and (3.8) we have in the case $1 < a < 2$

$$x^{a-1}e^{-x} < \Gamma(a,x) < [x^{a-1} + (a-1)x^{a-2}]e^{-x}. \tag{3.10}$$

Inequalities (3.9) and (3.10) are not useful for x near to zero, but they are stringent for high values of x. Besides, inequality (3.7) improves inequality (3.3) in the case $a > 2$.

3.4. $q(x) = \dfrac{1}{x} + \dfrac{a-1}{x^2} + \dfrac{(a-1)D}{x^3}$.

In this case we find $T(x) = \dfrac{(a-1)(a-2-D)x + (a-1)(a-3)D}{x^3}$. Proceeding as above, we note that with the choice $D=a-2$ we have $T(x)=\dfrac{(a-1)(a-2)(a-3)}{x^3}$. Therefore for $0 < a < 1$ or $2 < a < 3$:

$$\Gamma(a,x) < [x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3}]e^{-x}, \tag{3.11}$$

while for $1 < a < 2$ or $a > 3$:

$$\Gamma(a,x) > [x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3}]e^{-x}. \tag{3.12}$$

Comparing (3.11) and (3.12) with (3.7) and (3.8) respectively, we have for $0 < a < 1$ or $2 < a < 3$:

$$[x^{a-1} + (a-1)x^{a-2}]e^{-x} < \Gamma(a,x) < [x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3}]e^{-x},$$

while for $1 < a < 2$:

$$[x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3}]e^{-x} < \Gamma(a,x) < [x^{a-1} + (a-1)x^{a-2}]e^{-x}.$$

Also, for $a > 3$ inequalities (3.12) improves inequality (3.7).

Dealing with case 3.3 we noted that, if $q(x) = \dfrac{1}{x} + \dfrac{C}{x^2}$, best results are obtained for $C = a-1$, while for other values of C we obtained inequalities valid only from an x_0 on. In a similar way, in case 3.4 we obtained simple results for $D = a-2$. But now we note that $[x^{a-1} + (a-1)x^{a-2} + (a-1)(a-2)x^{a-3}]e^{-x}$ are the first three terms of the asymptotic expansion of $\Gamma(a,x)$ ([1, p. 263, (6.5.32)])

$$\Gamma(a,x) \sim x^{a-1}e^{-x}\left[1 + \frac{(a-1)}{x} + \frac{(a-1)(a-2)}{x^2} + \dots\right]. \tag{3.13}$$

This suggests to compare $\Gamma(a,x)$ with the right-hand side of (3.13), truncated to the term $\dfrac{(a-1)(a-2)\dots(a-n+1)}{x^{n-1}}$.

3.5. $q(x)=\dfrac{1}{x}+\dfrac{a-1}{x^2}+\dfrac{(a-1)(a-2)}{x^3}+\dots+\dfrac{(a-1)\dots(a-n+1)}{x^{n-1}}=\displaystyle\sum_{k=1}^{n}\frac{\Gamma(a)}{\Gamma(a-k+1)x^k}$.

Since $q'(x) = -\sum_{k=1}^{n} \frac{k\Gamma(a)}{\Gamma(a-k+1)x^{k+1}}$, we have:

$$\begin{aligned} T(x) &= 1 + (a-x)\sum_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-k+1)x^k} - \sum_{k=1}^{n} \frac{k\Gamma(a)}{\Gamma(a-k+1)x^k} \\ &= 1 + \sum_{k=1}^{n} \frac{(a-k)\Gamma(a)}{\Gamma(a-k+1)x^k} - \sum_{k=1}^{n} \frac{\Gamma(a)}{\Gamma(a-k+1)x^{k-1}} \\ &= 1 + \sum_{k=1}^{n} \frac{(a-k)\Gamma(a)}{\Gamma(a-k+1)x^k} - \sum_{k=0}^{n-1} \frac{\Gamma(a)}{\Gamma(a-k)x^k} \\ &= \frac{(a-n)\Gamma(a)}{\Gamma(a-n+1)x^n} + \sum_{k=1}^{n-1} \frac{\Gamma(a)}{x^k}\left(\frac{a-k}{\Gamma(a-k+1)} - \frac{1}{\Gamma(a-k)}\right), \end{aligned}$$

that is $T(x)=\frac{(a-n)\Gamma(a)}{\Gamma(a-n+1)x^n}=\frac{(a-1)\dots(a-n)}{x^n}$, since $\frac{a-k}{\Gamma(a-k+1)}-\frac{1}{\Gamma(a-k)}=0$.

Therefore, if n is even we have for every $x > 0$:

- for $0 < a < 1 \vee 2 < a < 3 \vee \dots \vee n-2 < a < n-1 \vee a > n \Rightarrow$

$$\Gamma(a,x) > [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+1)x^{a-n}]e^{-x}; \quad (3.14)$$

- for $1 < a < 2 \vee 3 < a < 4 \vee \dots \vee n-1 < a < n \Rightarrow$

$$\Gamma(a,x) < [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+1)x^{a-n}]e^{-x}; \quad (3.15)$$

Instead, if n is odd we have for every $x > 0$:

- for $1 < a < 2 \vee 3 < a < 4 \vee \dots \vee n-2 < a < n-1 \vee a > n \Rightarrow$

$$\Gamma(a,x) > [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+1)x^{a-n}]e^{-x}; \quad (3.16)$$

- for $0 < a < 1 \vee 2 < a < 3 \vee \dots \vee n-1 < a < n \Rightarrow$

$$\Gamma(a,x) < [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+1)x^{a-n}]e^{-x}. \quad (3.17)$$

In the case n even, (3.16) and (3.17) with $n-1$ in place of n give respectively:

- for $1 < a < 2 \vee 3 < a < 4 \vee \dots \vee n-3 < a < n-2 \vee a > n-1 \Rightarrow$

$$\Gamma(a,x) > [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+2)x^{a-n+1}]e^{-x}; \quad (3.18)$$

- for $0 < a < 1 \vee 2 < a < 3 \vee \dots \vee n-2 < a < n-1 \Rightarrow$

$$\Gamma(a,x) < [x^{a-1}+(a-1)x^{a-2}+\dots+(a-1)\dots(a-n+2)x^{a-n+1}]e^{-x}. \quad (3.19)$$

Inequalities (3.14) and (3.19) together give for $0 < a < 1 \vee 2 < a < 3 \vee \dots \vee n-2 < a < n-1$

$$\begin{aligned} &[x^{a-1} + (a-1)x^{a-2} + \dots + (a-1)\dots(a-n+1)x^{a-n}]e^{-x} < \Gamma(a,x) \\ &\qquad < [x^{a-1} + (a-1)x^{a-2} + \dots + (a-1)\dots(a-n+2)x^{a-n+1}]e^{-x}. \end{aligned}$$

Inequalities (3.15) and (3.18) together give for $1 < a < 2 \vee 3 < a < 4 \vee \ldots \vee n-1 < a < n$

$$[x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+2)x^{a-n+1}]e^{-x} < \Gamma(a,x)$$
$$< [x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+1)x^{a-n}]e^{-x}.$$

Also, for even n inequality (3.16) improves inequality (3.18).

In a similar way, in the case n odd, formulas (3.14) and (3.15) with $n-1$ in place of n give respectively:

- for $0 < a < 1 \vee 2 < a < 3 \vee \ldots \vee n-3 < a < n-2 \vee a > n-1 \Rightarrow$

$$\Gamma(a,x) > [x^{a-1}+(a-1)x^{a-2}+\ldots+(a-1)\ldots(a-n+2)x^{a-n+1}]e^{-x}; \quad (3.20)$$

- for $1 < a < 2 \vee 3 < a < 4 \vee \ldots \vee n-2 < a < n-1 \Rightarrow$

$$\Gamma(a,x) < [x^{a-1}+(a-1)x^{a-2}+\ldots+(a-1)\ldots(a-n+2)x^{a-n+1}]e^{-x}. \quad (3.21)$$

Inequalities (3.17) and (3.20) together give for $0 < a < 1 \vee 2 < a < 3 \vee \ldots \vee n-1 < a < n$

$$[x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+2)x^{a-n+1}]e^{-x} < \Gamma(a,x)$$
$$< [x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+1)x^{a-n}]e^{-x}.$$

Inequalities (3.16) and (3.21) together give for $1 < a < 2 \vee 3 < a < 4 \vee \ldots \vee n-2 < a < n-1$

$$[x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+1)x^{a-n}]e^{-x} < \Gamma(a,x)$$
$$< [x^{a-1} + (a-1)x^{a-2} + \ldots + (a-1)\ldots(a-n+2)x^{a-n+1}]e^{-x}.$$

Finally, for $a > n$ inequality (3.16) improves inequality (3.20).

3.6. $q(x) = \dfrac{1}{x+k}$**,** $k > 0$**.**

We have $T(x) = \dfrac{(k+a-1)x + k(k+a)}{x^3}$. Then we consider the following cases:

3.6.1. if $0 < a < 1$, we already know that (see formula (3.6)):

$$Bx^{a-1}e^{-x} < \Gamma(a,x) < x^{a-1}e^{-x},$$

where $0 < B < 1$, and the lower bound is valid for $x > \dfrac{B(1-a)}{1-B}$. Now let us consider the three subcases $k > 1-a$, $k < 1-a$ and $k = 1-a$.

3.6.1.a) For $k = 1-a$, $T(x)$ becomes $\dfrac{1-a}{(x+1-a)^2}$, which is positive for every $x > 0$. So we have

$$\Gamma(a,x) > \frac{x^a e^{-x}}{x+1-a}. \quad (3.22)$$

Inequality (3.22), which holds for $x > 0$, improves the lower bound in (3.6): in fact it is easy to check that for $x > \dfrac{B(1-a)}{1-B}$ it is $Bx^{a-1} < \dfrac{x^a}{x+1-a}$.

REMARK. (3.22) with $a = 1/2$ and x^2 in place of x becomes $\Gamma\left(\frac{1}{2}, x^2\right) > \frac{xe^{-x}}{x^2+1/2}$, that is

$$\int_x^\infty e^{-t^2}\,dt > \frac{xe^{-x}}{2x^2+1}. \tag{3.23}$$

Inequality (3.23) is better than

$$\int_x^\infty e^{-t^2}\,dt > e^{-x}\left(\frac{1}{2x} - \frac{1}{x^2}\right),$$

(see [2, p. 32, (44)]), since $\frac{x}{2x^2+1} > \frac{1}{2x} - \frac{1}{x^2}$ for every $x > 0$.

3.6.1.b) For $0 < k < 1-a$, $T(x)$ is negative for $x > x_0 = \frac{k(k+a)}{1-k-a}$; therefore we have for $x > x_0$

$$\Gamma(a,x) < \frac{x^a e^{-x}}{x+k}. \tag{3.24}$$

Inequality (3.24) improves the upper bound in (3.6) for $x > x_0$, because $\frac{x^a}{x+k} < x^{a-1}$.

3.6.1.c) The subcase $k > 1-a$ is unuseful, because we have $\Gamma(a,x) > \frac{x^a e^{-x}}{x+k}$, which is worse than (3.22).

3.6.2. If $a > 1$, we have $T(x) = \frac{(k+a-1)x + k(k+a)}{(x+k)^2} > 0 \;\; \forall x > 0$; hence $\Gamma(a,x) > \frac{x^a e^{-x}}{x+k}$, which is worse than (3.3).

3.7. $q(x) = \frac{1}{x} + \frac{a-1}{(x+k)^2}$, $k > 0$.

Here is $T(x) = (a-1)\frac{(2k+a-2)x^2 + (3k^2+ak)x + k^3}{x(x+k)^3}$.

For $k = 1 - \frac{a}{2}$ $(0 < a < 2)$ we have $T(x) = (a-1)(a-2)\frac{2(a-6)x-(a-2)^2}{8x(x+1+a/2)^3}$; the fraction is negative for every $x > 0$. So we have:

$$\text{for } 0 < a < 1 \Rightarrow \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{(x+1-a/2)^2}\right), \tag{3.25}$$

$$\text{for } 1 < a < 2 \Rightarrow \Gamma(a,x) > x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{(x+1-a/2)^2}\right), \tag{3.26}$$

In the case $0 < a < 1$ inequalities (3.25) and (3.7) give

$$x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{x^2}\right) < \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x} + \frac{a-1}{(x+1-a/2)^2}\right);$$

while in the case $1 < a < 2$ inequalities (3.26) and (3.8) give

$$x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+1-a/2)^2}\right) < \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{x^2}\right).$$

For $k > 1-\frac{a}{2}$ no new better inequalities are obtained; indeed, the polynomial $(2k+a-2)x^2+(3k^2+ak)x+k^3$ is positive for every $x>0$, and the sign of $T(x)$ only depends on the sign of $a-1$. Therefore:

- for $0<a<1 \Rightarrow \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right)$, which is worse than (3.25).
- for $1<a<2 \Rightarrow \Gamma(a,x) > x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right)$, which is worse than (3.26).
- for $a>2$ we have $\Gamma(a,x) > x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right)$ again, but now k may be any positive number. This inequality does not improve (3.26).

For $0<k<1-\frac{a}{2}$, which implies $0<a<2$, the polynomial $(2k+a-2)x^2+(3k^2+ak)x+k^3$ is negative for $x > x_0 = k\frac{3k+a+\sqrt{(a+k)^2+8k}}{2-a-2k}$; now let us consider, for $x>x_0$, the following subcases:

$$\text{if } 0<a<1 \quad\Rightarrow\quad \Gamma(a,x) > x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right); \tag{3.27}$$

$$\text{if } 1<a<2 \quad\Rightarrow\quad \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right); \tag{3.28}$$

Inequalities (3.25) and (3.27) give for $0<a<1$:

$$x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right) < \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+1+a/2)^2}\right);$$

where $0<k<1-\frac{a}{2}$ and the lower bound holds for $x>x_0$.

Inequalities (3.26) and (3.28) give for $1<a<2$:

$$x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+1+a/2)^2}\right) < \Gamma(a,x) < x^a e^{-x}\left(\frac{1}{x}+\frac{a-1}{(x+k)^2}\right);$$

where $0<k<1-\frac{a}{2}$ and the upper bound holds for $x>x_0$.

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