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SHARP INEQUALITIES FOR SOLUTIONS OF MULTIPOINT BOUNDARY VALUE PROBLEMS

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Abstract. This paper considers the following continuous and discrete multipoint boundary value problems: $x^{(n)}(t) \geqslant 0,\ 0 \leqslant t \leqslant 1,\ x^{(j)}(t_i) = 0$ and $\Delta^n y(k) \geqslant 0,\ k = 0, \cdots, m,\ \Delta^j y(k_i) = 0$, where $j = 0, \cdots, n_i - 1,\ i = 1, \cdots, r,\ \sum_{i=1}^{r} n_i = n,\ 0 = t_1 < t_2 < \cdots < t_r = 1$, and $0 = k_1 < k_1 + n_1 < k_2 < k_2 + n_2 < \cdots < k_r \leqslant k_r + n_r - 1 = m + n$. We offer lower bounds for solutions of these boundary value problems in terms of $\sup_{0 \leqslant t \leqslant 1} |x(t)|$ and $\max_{k \in \{0, \cdots, m+n\}} |y(k)|$. These bounds further lead to inequalities for related Green's functions which are very useful in the study of positive solutions of boundary value problems.

1. Introduction

Let $c, d\ \ (d > c)$ be integers. We shall define the discrete interval $Z[c, d] = \{c, c+1, \cdots, d\}$. For a nonnegative integer ℓ, the factorial expression $k^{(\ell)}$ is defined as $k^{(\ell)} = \prod_{i=0}^{\ell-1}(k - i)$ with $k^{(0)} = 1$. Let $\Delta y(k) = y(k+1) - y(k)$ and for $j \geqslant 2$, $\Delta^j y(k) = \Delta(\Delta^{j-1} y(k))$.

We consider the following continuous and discrete multipoint boundary value problems:

$$x^{(n)}(t) \geqslant 0,\ t \in [0, 1] \tag{1}$$

$$x^{(j)}(t_i) = 0,\ \ j = 0, \cdots, n_i - 1,\ i = 1, \cdots, r \tag{2}$$

and

$$\Delta^n y(k) \geqslant 0,\ k \in Z[0, m] \tag{3}$$

$$\Delta^j y(k_i) = 0,\ \ j = 0, \cdots, n_i - 1,\ i = 1, \cdots, r \tag{4}$$

where $r \geqslant 2,\ n_i \in Z[1, n-1],\ 1 \leqslant i \leqslant r,\ \sum_{i=1}^{r} n_i = n,\ 0 = t_1 < t_2 < \cdots < t_r = 1$, and $k_i,\ 1 \leqslant i \leqslant r$ are integers with $0 = k_1 < k_1 + n_1 < k_2 < k_2 + n_2 < \cdots < k_r \leqslant k_r + n_r - 1 = m + n$. The primary aim of this paper is to derive lower bounds for solutions of (1), (2) and (3), (4) in terms of $\|x\| = \sup_{t \in [0,1]} |x(t)|$ and $\|y\| = \max_{k \in Z[0, m+n]} |y(k)|$ respectively.

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Next, let g be the Green's function of the boundary value problem

$$x^{(n)}(t) = 0, \ t \in [0,1] \tag{5}$$

and (2); and G be the Green's function of the boundary value problem

$$\Delta^n y(k) = 0, \ k \in Z[0,m] \tag{6}$$

and (4). Using the lower bounds obtained earlier, we shall provide analogous inequalities for g and G. We remark that these inequalities are crucial in the study of positive solutions of boundary value problems as well as in eigenvalue problems, e.g., see [4,7,10,12–15,17].

By using a *different* technique, namely, the concept of concavity, Eloe and Henderson [9] have recently obtained the following lower bound for a solution of (1), (2):

$$(-1)^{\alpha_i} x(t) \geqslant \|x\| \left(\frac{a}{4}\right)^{\max\{n-n_1, n-n_r\}}, \ \ t \in \left[\frac{3t_i + t_{i+1}}{4}, \frac{t_i + 3t_{i+1}}{4}\right], \ i = 1, \cdots, r-1 \tag{7}$$

where $a = \min_{i \in Z[1,r-1]}(t_{i+1} - t_i)$ and $\alpha_i = \sum_{j=i+1}^{r} n_j, \ 1 \leqslant i \leqslant r-1$. In addition to extending their results to discrete case, our bound in the continuous case is *sharper* than (7). Our work further generalizes related investigation done on two-point boundary value problems [6,8,16].

2. Preliminaries

For each $i = 1, \cdots, r-1$, we shall denote $\alpha_i = \sum_{j=i+1}^{r} n_j$.

LEMMA 1. [1,2,5,11] *The Green's functions g and G satisfy the following:*

$$(-1)^{\alpha_i} g(t,s) > 0, \ \ (t,s) \in (t_i, t_{i+1}) \times (0,1), \ i = 1, \cdots, r-1 \tag{8}$$

$$g(t,s) \prod_{i=1}^{r} (t - t_i)^{n_i} \geqslant 0, \ (t,s) \in [t_1, t_r] \times [t_1, t_r] \tag{9}$$

$$(-1)^{\alpha_i} G(k,\ell) > 0, \ \ (k,\ell) \in Z[k_i + n_i, k_{i+1} - 1] \times Z[0,m], \ i = 1, \cdots, r-1 \tag{10}$$

$$G(k,\ell) \prod_{i=1}^{r} (k - k_i)^{(n_i)} \geqslant 0, \ (k,\ell) \in Z[0, m+n] \times Z[0,m]. \tag{11}$$

3. Boundary value problem (1), (2)

THEOREM 1. *Suppose that $x(t) \in C^{(n)}[0,1]$ satisfies (1), (2). Then, for $t \in [(3t_i + t_{i+1})/4, (t_i + 3t_{i+1})/4]$, $i = 1, \cdots, r-1$,*

$$(-1)^{\alpha_i} x(t) \geqslant \|x\| \min\left\{ \min\left\{ f\left(\frac{3t_i + t_{i+1}}{4}\right), f\left(\frac{t_i + 3t_{i+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} f(t), \right.$$
$$\left. \min\left\{ h\left(\frac{3t_i + t_{i+1}}{4}\right), h\left(\frac{t_i + 3t_{i+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} h(t) \right\} \tag{12}$$

where $f(t) = \prod_{j=1}^{r-1} |t-t_j|^{n_j}(1-t)^{n_r-1}$ *and* $h(t) = t^{n_1-1}\prod_{j=2}^{r} |t-t_j|^{n_j}$.

Proof. First, we shall prove (12) in the case that $x(t)$ satisfies (2) and

$$x^{(n)}(t) > 0,\ t \in (0,1). \tag{13}$$

Noting that $x(t) = \int_0^1 g(t,s)x^{(n)}(s)ds$, we employ (13) and (8) to get

$$(-1)^{\alpha_i}x(t) > 0,\ \ t \in (t_i, t_{i+1}),\ i = 1,\cdots,r-1. \tag{14}$$

Therefore, there exists $t_* \in (t_\ell, t_{\ell+1})$ for some $\ell \in Z[1, r-1]$ such that $\|x\| = (-1)^{\alpha_\ell}x(t_*)$, or equivalently,

$$x(t_*) = (-1)^{\alpha_\ell}\|x\|. \tag{15}$$

We shall consider four cases.

<u>Case 1</u> $2 \leqslant n_r \leqslant n-1$. Obviously, $x(t)$ fulfills (15) and

$$x^{(j)}(t_i) = 0,\ j = 0,\cdots,n_i-1,\ i = 1,\cdots,r-1, \quad x^{(j)}(1) = 0, j = 0,\cdots,n_r-2. \tag{16}$$

Following [1,3], $x(t)$ has the following expression

$$\begin{aligned} x(t) = {} & \frac{\prod_{j=1}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t)^{n_j}(1-t)^{n_r-1}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t_*)^{n_j}(1-t_*)^{n_r-1}}\,(-1)^{\alpha_\ell}\|x\| \\ & + \int_0^1 g_1(t,s)x^{(n)}(s)ds,\ t \in [0,1] \end{aligned} \tag{17}$$

where $g_1(t,s)$ is the Green's function of the boundary value problem (5), $x(t_*) = 0$ and (16). By (9) we have $\operatorname{sgn} g_1(t,s) = \operatorname{sgn} \prod_{i=1}^{r-1}(t-t_i)^{n_i}(t-1)^{n_r-1}(t-t_*)$.

If $t \in [t_\ell, t_*]$, then $\operatorname{sgn} g_1(t,s) = (-1)^{\alpha_\ell}$. Also, if $t \in [t_i, t_{i+1}]$, $i = 1,\cdots,\ell-1$, then $\operatorname{sgn} g_1(t,s) = (-1)^{\alpha_i}$. Using these together with (13), it follows from (17) that

$$\begin{aligned} (-1)^{\alpha_\ell}x(t) &\geqslant \frac{\prod_{j=1}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t)^{n_j}(1-t)^{n_r-1}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t_*)^{n_j}(1-t_*)^{n_r-1}}\,\|x\| \\ &= \frac{f(t)}{f(t_*)}\,\|x\|,\ t \in [t_\ell, t_*] \end{aligned} \tag{18}$$

and

$$\begin{aligned} (-1)^{\alpha_i}x(t) &\geqslant \frac{\prod_{j=1}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t)^{n_j}(1-t)^{n_r-1}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t_*)^{n_j}(1-t_*)^{n_r-1}}\,(-1)^{\alpha_\ell+\alpha_i}\|x\| \\ &= \frac{f(t)}{f(t_*)}\,\|x\|, \quad t \in [t_i, t_{i+1}],\ i = 1,\cdots,\ell-1. \end{aligned} \tag{19}$$

<u>Case 2</u> $2 \leqslant n_1 \leqslant n-1$. In this case, $x(t)$ satisfies (15) and

$$x^{(j)}(0) = 0, j = 0,\cdots,n_1-2, \quad x^{(j)}(t_i) = 0,\ \ j = 0,\cdots,n_i-1,\ i = 2,\cdots,r. \tag{20}$$

As in [1,3], $x(t)$ can be written as

$$x(t) = \frac{t^{n_1-1}\prod_{j=2}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t)^{n_j}}{t_*^{n_1-1}\prod_{j=2}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}(-1)^{\alpha_\ell}\|x\| + \int_0^1 g_2(t,s)x^{(n)}(s)ds,\ t\in[0,1] \tag{21}$$

where $g_2(t,s)$ is the Green's function of the boundary value problem (5), $x(t_*)=0$ and (20). Once again it follows from (9) that $\mathrm{sgn}\ g_2(t,s) = \mathrm{sgn}\ t^{n_1-1}\prod_{i=2}^{r}(t-t_i)^{n_i}(t-t_*)$.

If $t\in[t_*,t_{\ell+1}]$, then $\mathrm{sgn}\ g_2(t,s) = (-1)^{\alpha_\ell}$. Also, if $t\in[t_i,t_{i+1}]$, $i=\ell+1,\cdots,r-1$, then $\mathrm{sgn}\ g_2(t,s) = (-1)^{\alpha_i}$. Employing all these and (13) in (21), we obtain

$$(-1)^{\alpha_\ell}x(t) \geqslant \frac{t^{n_1-1}\prod_{j=2}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t)^{n_j}}{t_*^{n_1-1}\prod_{j=2}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}\|x\| = \frac{h(t)}{h(t_*)}\|x\|,\ t\in[t_*,t_{\ell+1}] \tag{22}$$

and

$$(-1)^{\alpha_i}x(t) \geqslant \frac{t^{n_1-1}\prod_{j=2}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t)^{n_j}}{t_*^{n_1-1}\prod_{j=2}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}(-1)^{\alpha_\ell+\alpha_i}\|x\| = \frac{h(t)}{h(t_*)}\|x\|,\quad t\in[t_i,t_{i+1}],\ i=\ell+1,\cdots,r-1. \tag{23}$$

Case 3 $n_r = 1$. Clearly, $x(t)$ fulfills (15) and

$$x^{(j)}(t_i) = 0,\ \ j=0,\cdots,n_i-1,\ i=1,\cdots,r-1. \tag{24}$$

It is noted that $(24) = (16)|_{n_r=1}$. We have the following representation

$$x(t) = \frac{\prod_{j=1}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t)^{n_j}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r-1}(t_j-t_*)^{n_j}}(-1)^{\alpha_\ell}\|x\| + \int_0^{\max\{t_*,t_{r-1}\}} g_3(t,s)x^{(n)}(s)ds,\quad t\in[0,\max\{t_*,t_{r-1}\}] \tag{25}$$

where $g_3(t,s)$ is the Green's function of the boundary value problem (5), $x(t_*)=0$ and (24). Further, with $\mathrm{sgn}\ g_3(t,s) = \mathrm{sgn}\ \prod_{i=1}^{r-1}(t-t_i)^{n_i}(t-t_*)$, we observe that $\mathrm{sgn}\ g_3(t,s) = \mathrm{sgn}\ g_1(t,s)$ for $t\in\cup_{i=1}^{\ell-1}[t_i,t_{i+1}]\cup[t_\ell,t_*]$. Using all these together with (13) in (25) readily leads to $(18)|_{n_r=1}$ and $(19)|_{n_r=1}$.

Case 4 $n_1 = 1$. Here, $x(t)$ satisfies (15) and

$$x^{(j)}(t_i) = 0,\ \ j=0,\cdots,n_i-1,\ i=2,\cdots,r. \tag{26}$$

We note that $(26) = (20)|_{n_1=1}$. Clearly, $x(t)$ can be expressed as

$$x(t) = \frac{\prod_{j=2}^{\ell}(t-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t)^{n_j}}{\prod_{j=2}^{\ell}(t_*-t_j)^{n_j}\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}(-1)^{\alpha_\ell}\|x\| + \int_{\min\{t_*,t_2\}}^{1} g_4(t,s)x^{(n)}(s)ds,\quad t\in[\min\{t_*,t_2\},1] \tag{27}$$

where $g_4(t,s)$ is the Green's function of the boundary value problem (5), $x(t_*) = 0$ and (26). We find that $\operatorname{sgn} g_4(t,s) = \operatorname{sgn} \prod_{i=2}^{r}(t-t_i)^{n_i}(t-t_*)$ has the same values as $\operatorname{sgn} g_2(t,s)$ for $t \in \cup_{i=\ell+1}^{r-1}[t_i,t_{i+1}] \ \cup[t_*,t_{\ell+1}]$. Subsequently, we employ these and (13) in (27) to get $(22)|_{n_1=1}$ and $(23)|_{n_1=1}$.

From Cases 1–4, we conclude that inequalities (18), (19), (22) and (23) hold for $1 \leqslant n_1, n_r \leqslant n-1$. Now, we couple (18) and (22), and also (19) and (23) to get

$$(-1)^{\alpha_i}x(t) \geqslant \|x\| \min\left\{\frac{f(t)}{f(t_*)},\ \frac{h(t)}{h(t_*)}\right\},\ \ t \in [t_i,t_{i+1}],\ i = 1,\cdots,r-1. \tag{28}$$

Hence, for $t \in I_i \equiv [(3t_i+t_{i+1})/4, (t_i+3t_{i+1})/4],\ i = 1,\cdots,r-1,$ it follows from (28) that

$$\begin{aligned}(-1)^{\alpha_i}x(t) \quad &\geqslant \quad \|x\| \min\left\{\min_{t,\, t_*}\frac{f(t)}{f(t_*)},\ \min_{t,\, t_*}\frac{h(t)}{h(t_*)}\right\}\\ &= \quad \|x\| \min\left\{\frac{\min_{t\in I_i} f(t)}{\max_{t_*\in[0,1]} f(t_*)},\ \frac{\min_{t\in I_i} h(t)}{\max_{t_*\in[0,1]} h(t_*)}\right\}.\end{aligned}$$

Since $\min_{t\in I_i} f(t)$ and $\min_{t\in I_i} h(t)$ occur at end points, the above inequality is exactly the same as (12). Thus, we have shown that (12) holds if $x(t)$ satisfies (13), (2).

Now, let $x(t)$ be a solution of (1), (2). For $\epsilon > 0$, we define

$$x_\epsilon(t) = x(t) + \epsilon \prod_{j=1}^{r}(t-t_j)^{n_j}.$$

Clearly, $x_\epsilon(t)$ fulfills (13) and (2). Therefore, (12) holds for each $\epsilon > 0$ and by continuity, (12) holds for $\epsilon = 0$. The proof of the theorem is complete.

REMARK 1. We shall deduce the inequality (7) of Eloe and Henderson [9] from the proof of Theorem 1. To begin, since

$$\frac{\prod_{j=\ell+1}^{r-1}(t_j-t)^{n_j}(1-t)^{n_r-1}}{\prod_{j=\ell+1}^{r-1}(t_j-t_*)^{n_j}(1-t_*)^{n_r-1}} \geqslant 1,\ t \in [0,t_*]$$

and

$$\frac{t^{n_1-1}\prod_{j=2}^{\ell}(t-t_j)^{n_j}}{t_*^{n_1-1}\prod_{j=2}^{\ell}(t_*-t_j)^{n_j}} \geqslant 1,\ t \in [t_*,1]$$

it follows from (18), (19), (22) and (23) respectively that

$$(-1)^{\alpha_\ell}x(t) \geqslant \frac{\prod_{j=1}^{\ell}(t-t_j)^{n_j}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}}\ \|x\|,\ t \in [t_\ell,t_*] \tag{29}$$

$$(-1)^{\alpha_i}x(t) \geqslant \frac{\prod_{j=1}^{\ell}|t-t_j|^{n_j}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}}\ \|x\|,\ \ t \in [t_i,t_{i+1}],\ i = 1,\cdots,\ell-1 \tag{30}$$

$$(-1)^{\alpha_\ell}x(t) \geqslant \frac{\prod_{j=\ell+1}^{r}(t_j-t)^{n_j}}{\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}\ \|x\|,\ t\in[t_*,t_{\ell+1}] \tag{31}$$

and

$$(-1)^{\alpha_i}x(t) \geqslant \frac{\prod_{j=\ell+1}^{r}|t_j-t|^{n_j}}{\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}\ \|x\|,\ \ t\in[t_i,t_{i+1}],\ i=\ell+1,\cdots,r-1. \tag{32}$$

Note that (29)–(32) hold for the cases $n_r = 1$ and $n_1 = 1$ as well. Hence, for $t\in I_i \equiv [(3t_i+t_{i+1})/4,(t_i+3t_{i+1})/4],\ i=1,\cdots,r-1,$ we combine (29)–(32) to get

$$\begin{aligned}(-1)^{\alpha_i}x(t) &\geqslant \|x\|\min\left\{\min_{t,\ t_*}\frac{\prod_{j=1}^{\ell}|t-t_j|^{n_j}}{\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}},\ \min_{t,\ t_*}\frac{\prod_{j=\ell+1}^{r}|t_j-t|^{n_j}}{\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}\right\}\\ &= \|x\|\min\left\{\frac{\min_{t\in I_i}\prod_{j=1}^{\ell}|t-t_j|^{n_j}}{\max_{t_*\in[0,1]}\prod_{j=1}^{\ell}(t_*-t_j)^{n_j}},\ \frac{\min_{t\in I_i}\prod_{j=\ell+1}^{r}|t_j-t|^{n_j}}{\max_{t_*\in[0,1]}\prod_{j=\ell+1}^{r}(t_j-t_*)^{n_j}}\right\}\\ &\geqslant \|x\|\min\left\{\left(\frac{a}{4}\right)^{n-n_r},\ \left(\frac{a}{4}\right)^{n-n_1}\right\}\end{aligned}$$

which is exactly the same as (7).

REMARK 2. It is clear from the proof of Theorem 1 and Remark 1 that inequality (12) is *sharper* than (7).

As in [9] Theorem 1 leads to the following corollary.

COROLLARY 1. *For each* $s\in(0,1)$, *let* $\|g(\cdot,s)\| = \sup_{t\in[0,1]}|g(t,s)|$. *Then, for* $(t,s)\in[(3t_i+t_{i+1})/4,(t_i+3t_{i+1})/4]\times(0,1),\ i=1,\cdots,r-1,$

$$(-1)^{\alpha_i}g(t,s) \geqslant \|g(\cdot,s)\|\min\left\{\min\left\{f\left(\frac{3t_i+t_{i+1}}{4}\right),f\left(\frac{t_i+3t_{i+1}}{4}\right)\right\}\Big/\max_{t\in[0,1]}f(t),\right.$$

$$\left.\min\left\{h\left(\frac{3t_i+t_{i+1}}{4}\right),h\left(\frac{t_i+3t_{i+1}}{4}\right)\right\}\Big/\max_{t\in[0,1]}h(t)\right\}$$ *where the functions* f *and* h *are defined in Theorem* 1.

4. Boundary value problem (3), (4)

THEOREM 2. *Suppose that* $y(k)$ *defined on* $Z[0,m+n]$ *satisfies* (3), (4). *Then, for* $k\in Z[k_i+n_i,k_{i+1}-1],\ i=1,\cdots,r-1,$

$$(-1)^{\alpha_i}y(k)\geqslant\|y\|\min\left\{\frac{\min\{p(k_i+n_i),p(k_{i+1}-1)\}}{\max_{k\in Z[0,m+n]}p(k)},\ \frac{\min\{q(k_i+n_i),q(k_{i+1}-1)\}}{\max_{k\in Z[0,m+n]}q(k)}\right\} \tag{33}$$

where $p(k) = \left|\prod_{j=1}^{r-1}(k-k_j)^{(n_j)}\right|(m+n-k)^{(n_r-1)}$ *and* $q(k) = k^{(n_1-1)}\left|\prod_{j=2}^{r}(k-k_j)^{(n_j)}\right|$.

Proof. We shall employ arguments analogous to that used in Theorem 1. To begin, we shall prove (33) when $y(k)$ satisfies (4) and

$$\Delta^n y(k) > 0,\ k\in Z[0,m]. \tag{34}$$

Since $y(k) = \sum_{\ell=0}^{m} G(k,\ell)\Delta^n y(\ell)$, in view of (34) and (10) we have

$$(-1)^{\alpha_i} y(k) > 0, \quad k \in Z[k_i + n_i, k_{i+1} - 1], \; i = 1, \cdots, r-1. \tag{35}$$

Hence, there exists $k_* \in Z[k_\ell + n_\ell, k_{\ell+1} - 1]$ for some $\ell \in Z[1, r-1]$ such that $\|y\| = (-1)^{\alpha_\ell} y(k_*)$, or equivalently,

$$y(k_*) = (-1)^{\alpha_\ell} \|y\|. \tag{36}$$

Case 1 $2 \leqslant n_r \leqslant n-1$. Clearly, $y(k)$ satisfies (36) and

$$\Delta^j y(k_i) = 0, \; j = 0, \cdots, n_i - 1, \; i = 1, \cdots, r-1, \qquad \Delta^j y(k_r) = 0, \; j = 0, \cdots, n_r - 2. \tag{37}$$

Following [2,4], $y(k)$ can be represented as

$$y(k) = \frac{\prod_{j=1}^{\ell}(k-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k)^{(n_j)} (m+n-k)^{(n_r-1)}}{\prod_{j=1}^{\ell}(k_*-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k_*)^{(n_j)} (m+n-k_*)^{(n_r-1)}} (-1)^{\alpha_\ell}\|y\| \\ + \sum_{\ell=0}^{m} G_1(k,\ell)\Delta^n y(\ell), \; k \in Z[0, m+n] \tag{38}$$

where $G_1(k,\ell)$ is the Green's function of the boundary value problem (6), $y(k_*) = 0$ and (37). From (11) we find that $\operatorname{sgn} G_1(k,\ell) = \operatorname{sgn} \prod_{i=1}^{r-1}(k-k_i)^{(n_i)}(k-k_r)^{(n_r-1)}(k-k_*)$.

If $k \in Z[k_\ell + n_\ell, k_*]$, then $\operatorname{sgn} G_1(k,\ell) = (-1)^{\alpha_\ell}$; and if $k \in Z[k_i + n_i, k_{i+1} - 1]$, $i = 1, \cdots, \ell - 1$, then $\operatorname{sgn} G_1(k,\ell) = (-1)^{\alpha_i}$. Applying these and (34) in (38), we get

$$(-1)^{\alpha_\ell} y(k) \geqslant \frac{\prod_{j=1}^{\ell}(k-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k)^{(n_j)} (m+n-k)^{(n_r-1)}}{\prod_{j=1}^{\ell}(k_*-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k_*)^{(n_j)} (m+n-k_*)^{(n_r-1)}} \|y\| \\ = \frac{p(k)}{p(k_*)} \|y\|, \quad k \in Z[k_\ell + n_\ell, k_*] \tag{39}$$

and

$$(-1)^{\alpha_i} y(k) \geqslant \frac{\prod_{j=1}^{\ell}(k-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k)^{(n_j)} (m+n-k)^{(n_r-1)}}{\prod_{j=1}^{\ell}(k_*-k_j)^{(n_j)} \prod_{j=\ell+1}^{r-1}(k_j + n_j - 1 - k_*)^{(n_j)} (m+n-k_*)^{(n_r-1)}} \times \\ \times (-1)^{\alpha_\ell + \alpha_i} \|y\| \\ = \frac{p(k)}{p(k_*)} \|y\|, \quad k \in Z[k_i + n_i, k_{i+1} - 1], \; i = 1, \cdots, \ell - 1. \tag{40}$$

Case 2 $2 \leqslant n_1 \leqslant n-1$. Here, $y(k)$ fulfills (36) and

$$\Delta^j y(0) = 0, \; j = 0, \cdots, n_1 - 2, \qquad \Delta^j y(k_i) = 0, \; j = 0, \cdots, n_i - 1, \; i = 2, \cdots, r \tag{41}$$

and can be written as

$$y(k) = \frac{k^{(n_1-1)} \prod_{j=2}^{\ell}(k-k_j)^{(n_j)} \prod_{j=\ell+1}^{r}(k_j + n_j - 1 - k)^{(n_j)}}{k_*^{(n_1-1)} \prod_{j=2}^{\ell}(k_*-k_j)^{(n_j)} \prod_{j=\ell+1}^{r}(k_j + n_j - 1 - k_*)^{(n_j)}} (-1)^{\alpha_\ell}\|y\| \\ + \sum_{\ell=0}^{m} G_2(k,\ell)\Delta^n y(\ell), \; k \in Z[0, m+n] \tag{42}$$

where $G_2(k,\ell)$ is the Green's function of the boundary value problem (6), $y(k_*)=0$ and (41). By (11) we have $\text{sgn}\ G_2(k,\ell)=\text{sgn}\ k^{(n_1-1)}\prod_{i=2}^{r}(k-k_i)^{(n_i)}(k-k_*)$.

If $k\in Z[k_*,k_{\ell+1}-1]$, then $\text{sgn}\ G_2(k,\ell)=(-1)^{\alpha_\ell}$; and if $k\in Z[k_i+n_i,k_{i+1}-1]$, $i=\ell+1,\cdots,r-1$, then $\text{sgn}\ G_2(k,\ell)=(-1)^{\alpha_i}$. Using these and (34), it follows from (42) that

$$(-1)^{\alpha_\ell}y(k)\geqslant\frac{k^{(n_1-1)}\prod_{j=2}^{\ell}(k-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k)^{(n_j)}}{k_*^{(n_1-1)}\prod_{j=2}^{\ell}(k_*-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k_*)^{(n_j)}}\ \|y\|$$
$$=\frac{q(k)}{q(k_*)}\ \|y\|,\quad k\in Z[k_*,k_{\ell+1}-1] \tag{43}$$

and

$$(-1)^{\alpha_i}y(k)\geqslant\frac{k^{(n_1-1)}\prod_{j=2}^{\ell}(k-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k)^{(n_j)}}{k_*^{(n_1-1)}\prod_{j=2}^{\ell}(k_*-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k_*)^{(n_j)}}\ (-1)^{\alpha_\ell+\alpha_i}\|y\|$$
$$=\frac{q(k)}{q(k_*)}\ \|y\|,\quad k\in Z[k_i+n_i,k_{i+1}-1],\ i=\ell+1,\cdots,r-1. \tag{44}$$

<u>Case 3</u> $n_r=1$. Here, $y(k)$ satisfies (36) and $(37)|_{n_r=1}$, viz.

$$\Delta^j y(k_i)=0,\ \ j=0,\cdots,n_i-1,\ i=1,\cdots,r-1. \tag{45}$$

Further, $y(k)$ can be expressed as

$$y(k)=\frac{\prod_{j=1}^{\ell}(k-k_j)^{(n_j)}\prod_{j=\ell+1}^{r-1}(k_j+n_j-1-k)^{(n_j)}}{\prod_{j=1}^{\ell}(k_*-k_j)^{(n_j)}\prod_{j=\ell+1}^{r-1}(k_j+n_j-1-k_*)^{(n_j)}}\ (-1)^{\alpha_\ell}\|y\|$$
$$+\sum_{\ell=0}^{\max\{k_*,k_{r-1}+n_{r-1}-1\}-n}G_3(k,\ell)\Delta^n y(\ell),\ k\in Z[0,\max\{k_*,k_{r-1}+n_{r-1}-1\}] \tag{46}$$

where $G_3(k,\ell)$ is the Green's function of the boundary value problem (6), $y(k_*)=0$ and (45). We find that $\text{sgn}\ G_3(k,\ell)=\text{sgn}\ \prod_{i=1}^{r-1}(k-k_i)^{(n_i)}(k-k_*)$ has the same values as $\text{sgn}\ G_1(k,\ell)$ for $k\in\cup_{i=1}^{\ell-1}Z[k_i+n_i,k_{i+1}-1]\cup Z[k_\ell+n_\ell,k_*]$. Hence, (46) implies $(39)|_{n_r=1}$ and $(40)|_{n_r=1}$.

<u>Case 4</u> $n_1=1$. Clearly, $y(k)$ fulfills (36) and $(41)|_{n_1=1}$, viz.

$$\Delta^j y(k_i)=0,\ \ j=0,\cdots,n_i-1,\ i=2,\cdots,r \tag{47}$$

and has the following representation

$$y(k)=\frac{\prod_{j=2}^{\ell}(k-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k)^{(n_j)}}{\prod_{j=2}^{\ell}(k_*-k_j)^{(n_j)}\prod_{j=\ell+1}^{r}(k_j+n_j-1-k_*)^{(n_j)}}\ (-1)^{\alpha_\ell}\|y\|$$
$$+\sum_{\ell=\min\{k_*,k_2\}}^{m}G_4(k,\ell)\Delta^n y(\ell),\ k\in Z[\min\{k_*,k_2\},m+n] \tag{48}$$

where $G_4(k,\ell)$ is the Green's function of the boundary value problem (6), $y(k_*)=0$ and (47). It is noted that $\operatorname{sgn} G_4(k,\ell) = \operatorname{sgn} \prod_{i=2}^{r}(k-k_i)^{(n_i)}(k-k_*) = \operatorname{sgn} G_2(k,\ell)$ for $k \in \cup_{i=\ell+1}^{r-1} Z[k_i+n_i, k_{i+1}-1] \cup Z[k_*, k_{\ell+1}-1]$. Consequently, (48) leads to $(43)|_{n_1=1}$ and $(44)|_{n_1=1}$.

It is clear from Cases 1–4 that inequalities (39), (40), (43) and (44) hold for $1 \leqslant n_1, n_r \leqslant n-1$. Subsequently, for $k \in I_i \equiv Z[k_i+n_i, k_{i+1}-1]$, $i=1,\cdots,r-1$, we combine (39), (43), (40) and (44) to get

$$\begin{aligned}(-1)^{\alpha_i} y(k) &\geqslant \|y\| \min\left\{\frac{p(k)}{p(k_*)},\ \frac{q(k)}{q(k_*)}\right\}\\ &\geqslant \|y\| \min\left\{\min_{k,\,k_*}\frac{p(k)}{p(k_*)},\ \min_{k,\,k_*}\frac{q(k)}{q(k_*)}\right\}\\ &= \|y\| \min\left\{\frac{\min_{k\in I_i} p(k)}{\max_{k_*\in Z[0,m+n]} p(k_*)},\ \frac{\min_{k\in I_i} q(k)}{\max_{k_*\in Z[0,m+n]} q(k_*)}\right\}.\end{aligned}$$

Since $\min_{k\in I_i} p(k)$ and $\min_{k\in I_i} q(k)$ occur at end points, the above inequality is exactly the same as (33). So we have verified that (33) holds if $y(k)$ fulfills (34), (4).

Now, suppose that $y(k)$ satisfies (3), (4). For $\epsilon > 0$, we define

$$y_\epsilon(k) = y(k) + \epsilon \prod_{j=1}^{r}(k-k_j)^{(n_j)}$$

which obviously fulfills (34) and (4). Hence, (33) holds for each $\epsilon > 0$ and by continuity, (33) holds for $\epsilon = 0$. This completes the proof of the theorem.

As in [6] Theorem 2 gives rise to the following corollary.

COROLLARY 2. *For each* $\ell \in Z[0,m]$, *let* $\|G(\cdot,\ell)\| = \max_{k\in Z[0,m+n]} |G(k,\ell)|$. *Then, for* $(k,\ell) \in Z[k_i+n_i, k_{i+1}-1] \times Z[0,m]$, $i=1,\cdots,r-1$,

$$(-1)^{\alpha_i} G(k,\ell) \geqslant \|G(\cdot,\ell)\| \min\left\{\frac{\min\{p(k_i+n_i), p(k_{i+1}-1)\}}{\max_{k\in Z[0,m+n]} p(k)},\ \frac{\min\{q(k_i+n_i), q(k_{i+1}-1)\}}{\max_{k\in Z[0,m+n]} q(k)}\right\}$$

where the functions p *and* q *are defined in Theorem* 2.

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