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ON POWERS OF CLASS A(k) OPERATORS INCLUDING p–HYPONORMAL AND LOG–HYPONORMAL OPERATORS

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Abstract. In [11], we introduced class A(k) as a class of operators including p -hyponormal and log -hyponormal operators. In this paper, we shall show that "if T is an invertible class A(k) operator for $k \in (0,1]$, then T^n is a class A($\frac{k}{n}$) operator for all positive integer n ." Moreover, we shall show a similar result on powers of class AI(s,t) operators which were introduced in [7] as extensions of class A(k) operators, that is, "if T is a class AI(s,t) operator for $s,t \in (0,1]$, then T^n is a class AI($\frac{s}{n}, \frac{t}{n}$) operator for all positive integer n ."

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H . An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$.

An operator T is said to be a p *-hyponormal* operator if $(T^*T)^p \geqslant (TT^*)^p$ for a positive number p , and Tanahashi [19] defined log *-hyponormal* as a class of operators such that T is invertible and $\log T^*T \geqslant \log TT^*$. p -Hyponormal and log -hyponormal operators were defined as extensions of hyponormal operators, i.e., $T^*T \geqslant TT^*$. It is well known that "*every* p *-hyponormal operator is a* q *-hyponormal operator for* $p \geqslant q > 0$" by the celebrated Löwner-Heinz theorem "$A \geqslant B \geqslant 0$ *ensures* $A^\alpha \geqslant B^\alpha$ *for any* $\alpha \in [0,1]$," and every invertible p -hyponormal operator is a log -hyponormal operator since $\log t$ is an operator monotone function. It is also well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator [14, Problem 209].

Recently, these operator classes which were defined by operator inequalities were studied by using the following Theorem F.

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THEOREM F. (Furuta inequality [8])

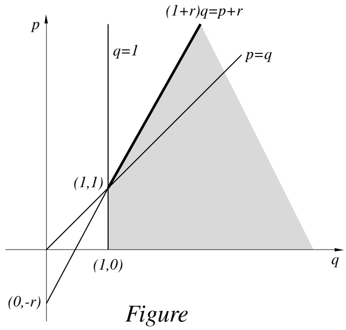
If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



Figure

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [6] and [17] and also an elementary one page proof in [9]. Tanahashi [18] showed that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F.

Very recently, Aluthge and Wang [4] obtained the following theorem on p-hyponormal operators by using Theorem F.

THEOREM A.1. ([4]) *Let* T *be a* p*-hyponormal operator for* $p \in (0,1]$. *Then*

$$(T^{n*}T^n)^{\frac{p}{n}} \geqslant (T^*T)^p \geqslant (TT^*)^p \geqslant (T^nT^{n*})^{\frac{p}{n}}$$

hold for all positive integer n.

Theorem A.1 is a very interesting result because Theorem A.1 asserts that "if T is a p-hyponormal operator for $p \in (0,1]$, then T^n is a $(\frac{p}{n})$-hyponormal operator." Related to Theorem A.1, Furuta and Yanagida [13] obtained Theorem A.2 as follows.

THEOREM A.2. ([13]) *Let* T *be a* p*-hyponormal operator for* $p \in (0,1]$. *Then*

(i) $(T^{n*}T^n)^{\frac{p+1}{n}} \geqslant \cdots \geqslant (T^{2*}T^2)^{\frac{p+1}{2}} \geqslant (T^*T)^{p+1}$

and

(ii) $(TT^*)^{p+1} \geqslant (T^2T^{2*})^{\frac{p+1}{2}} \geqslant \cdots \geqslant (T^nT^{n*})^{\frac{p+1}{n}}$

hold for all positive integer n.

Theorem A.2 is an extension of Theorem A.1. In fact, raising each sides of (i) and (ii) in Theorem A.2 to the power $\frac{p}{p+1} \in [0,1]$ by Löwner-Heinz theorem and using p-hyponormality of T, we have Theorem A.1.

Moreover, Aluthge and Wang [3] obtained the following result on $\log$-hyponormal operators.

THEOREM B.1. ([3]) *If* T *is a* $\log$*-hyponormal operator, then* T^{2^n} *is also a* $\log$*-hyponormal operator for any positive integer* n.

Related to Theorem B.1, we obtained the following Theorem B.2 in [20] as an extension of Theorem B.1.

THEOREM B.2. ([20]) *Let* T *be a* $\log$*-hyponormal operator. Then*

(i) $\quad (T^{n*}T^n)^{\frac{1}{n}} \geqslant \cdots \geqslant (T^{2^*}T^2)^{\frac{1}{2}} \geqslant T^*T$
and
(ii) $\quad TT^* \geqslant (T^2T^{2^*})^{\frac{1}{2}} \geqslant \cdots \geqslant (T^nT^{n*})^{\frac{1}{n}}$
hold for all positive integer n.

COROLLARY B.3. ([20]) *If T is a* log *-hyponormal operator, then T^n is also a* log *-hyponormal operator for any positive integer n.*

We remark that Theorem B.2 for log -hyponormal operators is a parallel result to Theorem A.2 for p -hyponormal operators.

On the other hand, related to p -hyponormal and log -hyponormal operators, we introduced classes of operators in [11] as follows.

DEFINITION 1. ([11])
(1) An operator T belongs to class A if $|T^2| \geqslant |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$.
(2) For each $k > 0$, an operator T belongs to class A(k) if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2. \tag{1.1}$$

We remark that class A(1) coincides with class A.

By using Theorem F, we obtained the following theorem on the inclusion relations among these classes.

THEOREM C.1. ([11])
(i) *Every* log *-hyponormal operator is a class A(k) operator for $k > 0$.*
(ii) *Every class A operator is a paranormal operator,*
i.e., $\|T^2x\| \geqslant \|Tx\|^2$ for every unit vector $x \in H$.
(iii) *Every invertible class A(k) operator is a class A(l) operator for $l \geqslant k > 0$.*

Ito [16] showed that the properties of Theorem B.2 for log -hyponormal operators remain valid for class A operators as follows.

THEOREM C.2. ([16]) *Let T be an invertible class A operator. Then*
(i) $\quad |T^n|^{\frac{2}{n}} \geqslant \cdots \geqslant |T^2| \geqslant |T|^2$
and
(ii) $\quad |T^*|^2 \geqslant |T^{2^*}| \geqslant \cdots \geqslant |T^{n*}|^{\frac{2}{n}}$
hold for all positive integer n.

COROLLARY C.3. ([16]) *If T is an invertible class A operator, then T^n is also a class A operator for any positive integer n.*

Very recently, the following classes of operators were defined in [7] as extensions of class A(k).

DEFINITION 2. ([7])
(1) For each $s > 0$ and $t > 0$, an operator T belongs to *class A(s,t)*
if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geqslant |T^*|^{2t}$.
(2) For each $s > 0$ and $t > 0$, an operator T belongs to *class AI(s,t)*
if T is an invertible and class A(s,t) operator.

Related to class A(s,t) operators, the following theorem was obtained in [7] as a nice application of Theorem F.

THEOREM D. ([7])

(i) *For each* $p > 0$ *and* $r > 0$, *every class A*(p,r) *operator is a class A*(p,t) *operator for* $t \geqslant r > 0$.

(ii) *For each* $p > 0$ *and* $r > 0$, *every class AI*(p,r) *operator is a class AI*(s,t) *operator for* $s \geqslant p > 0$ *and* $t \geqslant r > 0$.

Firstly, we shall show a more precise estimation for powers of invertible class A(k) operators for $k \in (0,1]$ in Theorem 1 than Corollary C.3. Secondly, we shall show a similar result on class AI(s,t) operators in Theorem 4.

2. Powers of class A(k) operators

THEOREM 1. *If* T *is an invertible and class A*(k) *operator for* $k \in (0,1]$, *then* T^n *is a class A*$(\frac{k}{n})$ *operator for all positive integer* n.

COROLLARY 2. *Let* T *be an invertible and class A operator. Then* T^n *is a class A*$(\frac{1}{n})$ *operator for all positive integer* n.

By using (iii) of Theorem C.1, Corollary 2 yields Corollary C.3 since class A$(\frac{1}{n})$ is included in class A, so that Corollary 2 is an extension of Corollary C.3.

To prove Theorem 1, the following Lemma E is important.

LEMMA E. ([10, 11]) *Let* A *and* B *be invertible operators. Then*

$$(BAA^*B^*)^{\lambda} = BA(A^*B^*BA)^{\lambda-1}A^*B^*$$

holds for any real number λ.

Proof of Theorem 1. Suppose that T is an invertible class A(k) operator for $k \in (0,1]$, i.e.,

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2. \tag{1.1}$$

By (iii) of Theorem C.1, T is a class A operator, and also we have the following inequalities by Theorem C.2;

$$|T|^2 \leqslant |T^n|^{\frac{2}{n}}, \tag{2.1}$$

$$|T^*|^2 \geqslant |T^{n*}|^{\frac{2}{n}}. \tag{2.2}$$

Then we have

$$(T^*|T^n|^{\frac{2k}{n}}T)^{\frac{1}{k+1}} \geqslant (T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2 \tag{2.3}$$

by (1.1), (2.1) and Löwner-Heinz theorem. (2.3) implies the following by Lemma E.

$$T^*|T^n|^{\frac{k}{n}}(|T^n|^{\frac{k}{n}}TT^*|T^n|^{\frac{k}{n}})^{\frac{1}{k+1}-1}|T^n|^{\frac{k}{n}}T \geqslant T^*T. \tag{2.4}$$

(2.4) is equivalent to the following.

$$|T^n|^{\frac{2k}{n}} \geqslant (|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}. \tag{2.5}$$

By (2.2), (2.5) and Löwner-Heinz theorem, we have

$$|T^n|^{\frac{2k}{n}} \geqslant (|T^n|^{\frac{k}{n}}|T^*|^2|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}} \geqslant (|T^n|^{\frac{k}{n}}|T^{n*}|^{\frac{2}{n}}|T^n|^{\frac{k}{n}})^{\frac{k}{k+1}}. \tag{2.6}$$

(2.6) implies the following by Lemma E.

$$|T^n|^{\frac{2k}{n}} \geqslant |T^n|^{\frac{k}{n}}|T^{n*}|^{\frac{1}{n}}(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{k}{k+1}-1}|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{k}{n}}.$$

Then we have

$$(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}} \geqslant |T^{n*}|^{\frac{2}{n}}.$$

Put $A = (|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{1}{k+1}}$ and $B = |T^{n*}|^{\frac{2}{n}}$, then $A \geqslant B > 0$. By using (i) of Theorem F, we have

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant B^{1+r} \qquad \text{for } p \geqslant 1 \text{ and } r \geqslant 0. \tag{2.7}$$

Put $p = k+1 \geqslant 1$ and $r = n-1 \geqslant 0$ in (2.7), then we have

$$(B^{\frac{n-1}{2}}A^{k+1}B^{\frac{n-1}{2}})^{\frac{n}{k+n}} \geqslant B^n. \tag{2.8}$$

(2.8) is equivalent to

$$\left\{|T^{n*}|^{\frac{n-1}{n}}(|T^{n*}|^{\frac{1}{n}}|T^n|^{\frac{2k}{n}}|T^{n*}|^{\frac{1}{n}})^{\frac{k+1}{k+1}}|T^{n*}|^{\frac{n-1}{n}}\right\}^{\frac{n}{k+n}} \geqslant |T^{n*}|^2.$$

Then we have

$$(|T^{n*}||T^n|^{\frac{2k}{n}}|T^{n*}|)^{\frac{1}{\frac{k}{n}+1}} \geqslant |T^{n*}|^2. \tag{2.9}$$

We remark the following: Let $T^n = U_n|T^n|$ be the polar decomposition of T^n, then the polar decomposition of T^{n*} can be expressed as follows $T^{n*} = U_n^*|T^{n*}|$. Hence we obtain the following.

$$\begin{aligned}(T^{n*}|T^n|^{2\frac{k}{n}}T^n)^{\frac{1}{\frac{k}{n}+1}} &= (U_n^*|T^{n*}||T^n|^{2\frac{k}{n}}|T^{n*}|U_n)^{\frac{1}{\frac{k}{n}+1}} \\ &= U_n^*(|T^{n*}||T^n|^{2\frac{k}{n}}|T^{n*}|)^{\frac{1}{\frac{k}{n}+1}}U_n \\ &\geqslant U_n^*|T^{n*}|^2U_n \qquad \text{by (2.9)} \\ &= |T^n|^2.\end{aligned}$$

Whence

$$(T^{n*}|T^n|^{2\frac{k}{n}}T^n)^{\frac{1}{\frac{k}{n}+1}} \geqslant |T^n|^2 \qquad \text{holds for all positive integer } n,$$

i.e., T^n is a class A($\frac{k}{n}$) operator.

□

Proof of Corollary 2. Put $k = 1$ in Theorem 1.

□

3. Properties of class AI(s,t) operators

In this section, we shall discuss some properties of class AI(s,t) operators.

Firstly, we show a relation between class A($k,1$) operators and class A(k) operators for $k>0$ as follows.

PROPOSITION 3. *Every class A($k,1$) operator is a class A(k) operator for $k>0$ and vice versa.*

Secondly, we obtain the following theorem as a parallel result to Theorem 1.

THEOREM 4. *If T is a class AI(s,t) operator for $s,t\in(0,1]$, then T^n is a class AI($\frac{s}{n},\frac{t}{n}$) operator for all positive integer n.*

COROLLARY 5. *If T is an invertible and class A(k) operator for $k\in(0,1]$, then T^n is a class AI($\frac{k}{n},\frac{1}{n}$) operator for all positive integer n.*

By Proposition 3 and (i) of Theorem D, Corollary 5 yields Corollary 2 since class A($\frac{k}{n}$) includes class A($\frac{k}{n},\frac{1}{n}$).

Proof of Proposition 3. Let $T=U|T|$ be the polar decomposition of T. We remark that the polar decomposition of T^* is $T^*=U^*|T^*|$. Suppose that T is a class A($k,1$) operator. Then

$$\begin{aligned}(T^*|T|^{2k}T)^{\frac{1}{k+1}} &= (U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}}\\ &= U^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}U\\ &\geqslant U^*|T^*|^2U \qquad \text{since } T \text{ is a class A}(k,1) \text{ operator}\\ &= |T|^2.\end{aligned}$$

Hence T is a class A(k) operator.

Conversely, suppose that T is a class A(k) operator. Then

$$\begin{aligned}(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}} &= UU^*(|T^*||T|^{2k}|T^*|)^{\frac{1}{k+1}}UU^*\\ &= U(U^*|T^*||T|^{2k}|T^*|U)^{\frac{1}{k+1}}U^*\\ &= U(T^*|T|^{2k}T)^{\frac{1}{k+1}}U^*\\ &\geqslant U|T|^2U^* \qquad \text{since } T \text{ is a class A}(k) \text{ operator}\\ &= |T^*|^2.\end{aligned}$$

Hence T is a class A($k,1$) operator.

Whence the proof of Proposition 3 is complete.

□

We omit the proof of Theorem 4 because the proof of Theorem 4 is quite similar to the proof of Theorem 1 by using (ii) of Theorem D instead of (iii) of Theorem C.1 in the proof.

We remark that Aluthge and Wang defined *w-hyponormal* operators in [5] which was related to hyponormal operators as follows. An operator T is said to be a w-hyponormal operator if $|\tilde{T}|\geqslant|T|\geqslant|\tilde{T}^*|$ where $T=U|T|$ is the polar decomposition of T and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation. Aluthge transformation was studied in [1, 2, 12, 15, 19, 21]. Related to w-hyponormal operators, the following results were shown in [5].

THEOREM G.1. ([5]) *An operator T is a w-hyponormal operator if and only if*

$$(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geqslant |T^*| \qquad \textit{and} \qquad |T| \geqslant (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}.$$

THEOREM G.2. ([5]) *If T is an invertible w-hyponormal operator, then T^2 is also a w-hyponormal operator.*

By Theorem G.1, an invertible w-hyponormal operator coincides with a class AI($\frac{1}{2}, \frac{1}{2}$) operator since the first inequality in Theorem G.1 is equivalent to the second inequality in Theorem G.1 in case T is invertible by Lemma E. Then using Theorem 4, we obtain that if T is an invertible w-hyponormal operator then T^2 belongs to class AI($\frac{1}{4}, \frac{1}{4}$) which is a smaller class of operators than w-hyponormal by (ii) of Theorem D. Hence Theorem 4 is a more precise result for w-hyponormal operators than Theorem G.2.

4. Concluding remarks

Firstly, it is interesting to point out the contrast between the following two facts: Theorem A.1 asserts that if T is a p-hyponormal operator for $p \in (0,1]$ then T^n belongs to the class of $(\frac{p}{n})$-hyponormal operators which is a *larger class* of operators than the class of p-hyponormal operators, and contrary to Theorem A.1, Theorem 1 asserts that if T is an invertible class A(k) operator for $k \in (0,1]$ then T^n belongs to class A($\frac{k}{n}$) which is a *smaller class* of operators than class A(k).

Secondly, it was shown in (i) of Theorem C.1 that every $\log$-hyponormal operator is a class A(k) operator for all $k > 0$. Here, we shall discuss a more precise relation between class A(k) and the class of $\log$-hyponormal operators than (i) of Theorem C.1. Assume T is an invertible class A(k) operator, i.e.,

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2. \tag{1.1}$$

Then by using Lemma E, (1.1) is equivalent to (4.1) as follows:

$$T^*|T|^k(|T|^kTT^*|T|^k)^{\frac{-k}{k+1}}|T|^kT \geqslant |T|^2. \tag{4.1}$$

Hence an invertible class A(k) operator satisfies the following inequality by (4.1).

$$|T|^{2k} \geqslant (|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}}.$$

Then

$$\log|T|^{2k} \geqslant \log(|T|^k|T^*|^2|T|^k)^{\frac{k}{k+1}} \tag{4.2}$$

holds since $\log t$ is an operator monotone function. (4.2) is equivalent to

$$\log|T|^{2(k+1)} \geqslant \log(|T|^k|T^*|^2|T|^k). \tag{4.3}$$

Let $k \to 0$ in (4.3), then we have $\log T^*T \geqslant \log TT^*$. Briefly speaking, the class of log-hyponormal operators can be regarded as invertible class $A(0)$. And it is well known that log-hyponormal also can be regarded as 0-hyponormal. It is interesting to point out this contrast.

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