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MERIT FUNCTION FOR VARIATIONAL–LIKE INEQUALITIETS

MUHAMMAD ASLAM NOOR

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Abstract. In this paper, we use the auxiliary principle technique to suggest an iterative method and a merit function for variational-like inequalities. Several special cases, which can be obtained from our results, are also discussed.

1. Introduction

Variational inequalities are being used to study a wide class of diverse unrelated problems arising in various branches of pure and applied sciences in a unified framework. Various generalizations and extensions of variational inequalities have been considered in different directions using novel and innovative techniques, see, for example, [1-30] and the references therein. A useful and important generalization of the variational inequalities is called the variational-like inequalities, considered and studied by Parida and Sen [24]. Yao [28] and Tian [26] used the Berge maximum Theorem and KKM maps to study the existence of a solution of variational-like inequalities in convex settings. It is worth mentioning that their methods are not constructive ones. First of all, we give some background information on the formulation of variational-like inequalities. The variational-like inequalities are closely related to the concept of the invex and preinvex functions, which generalize the notion of convexity of functions. The invex functions were introduced by Hanson [11] in 1981. Hanson's result inspired a great deal of subsequent work which has greatly expanded the role of invexity in optimization. Ben-Israel and Mond [2] proved that the differentiable preinvex functions are invex functions, but the converse is not true. We would like to point out that it was implicit that the preinvex functions are defined on the invex set with respect to function $\eta(.,.)$. We emphasize the fact that the function $\eta(.,.)$ plays a significant and crucial part in the definitions of invex, preinvex functions and invex sets. It is understood that the function $\eta(.,.)$ must be preinvex with respect to each argument. Ironically, we note that all the results in variational-like inequalities are being obtained under the assumptions of standard convexity concepts. No attempt has been made to utilize the concept of invexity theory. This is one of the reasons that all the authors so far have considered that that if an element satisfies the variational-like inequality, then it minimizes that (convex) function , but not the converse, se [26-28]. In early 1990's, Yang and Chen [27] and Noor [16-19]

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proved that the minimum of the differentiable invex and preinvex functions on the invex stes in a normed linear space can be characterized by variational-like inequalities. Noor [16-19] and Weir and Mond [29] proved that many results in mathematical programming involving convex functions and convex sets actually hold for invex(preinvex) functions and their generalizations. In passing, we remark that the preinvex functions and invex sets may not be convex functions and convex sets respectively, see, for example, [29, 2, 16-19] and the references therein.

There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. Due to the presence of the function $\eta(.,.)$, projection, Wiener-Hopf equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving variational-like inequalities. This fact motivated to use the auxiliary principle technique to develop the existence theory for variational-like inequalities. This technique is mainly due to Glowinski, Lions and Tremolieres [10]. The main and basic idea in this technique is to consider an auxiliary variational-like inequality problem related to the original problem. This way one defines a mapping connecting the solutions of both these problems. In this case, one has to show that the mapping connecting the solution is a contraction mapping and consequently it has a fixed point, which is the solution of the original problem. We remark that one may consider a number of auxiliary problems for a given problem. It turned out that solution of the auxiliary problem is the minimum of the auxiliary differentiable functional and the converse is also true. Fukushima [7], Noor [21] and Zhu and Marcotte [30] used the auxiliary principle technique to construct the merit(gap) functions for variational inequalities. These merit functions are being used to develop a number of numerical methods for solving variational and complementarity problems, see [7,8,15,21-23,30,31] and the references therein. In this paper, we prove that the minimum of the differentiable auxiliary preinvex functional on the invex set can be characterized by an auxiliary variational-like inequality under some conditions on the function $\eta(.,.)$. This equivalence is used to suggest an iterative method for solving the variational-like inequalities as well as to construct a merit(gap) for functions for variational-like inequalities. In particular, if $\eta(v,u) = v - u$, then our results collapse to the standard results for variational inequalities.

2. Formulation and Basic Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $F : H \rightarrow H$ and $\eta(.,.) : H \times H \rightarrow H$ be functions.

First of all, we recall the following well known results and concepts.

DEFINITIONS 2.1 [2,17,29]. Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(.,.)$, if , for all $u, v \in K, t \in [0,1]$,

$$u + t\eta(v,u) \in K.$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K$. From now onward K is a nonempty closed invex set in H with respect to η, unless otherwise specified.

DEFINITION 2.2 [2,17,29]. The function $F : K \to H$ is said to be preinvex with respect to η, if, for all $u, v \in K, t \in [0, 1]$,

$$F(u + t\eta(v, u)) \leqslant (1 - t)F(u) + tF(v).$$

The function $F : K \to H$ is said to be preconcave if and only if $-F$ is preinvex.

DEFINITION 2.3. The function $F : H \to H$ is said to be prelinear, if, for all $u, v \in H, t \in [0, 1]$,

$$F(u + t\eta(v, u)) = (1 - t)F(u) + tF(v).$$

We note that the prelinear function is both preinvex and preconcave.

DEFINITION 2.4 [11]. The differentiable function $F : K \to H$ is said to be an invex function with respect to η, if, for all $u, v \in K$,

$$F(v) - F(u) \geqslant \langle F'(u), \eta(v, u)\rangle ,$$

where $F'(u)$ is the differential of F at u. It is known [2,17,29] that the differentiable preinvex functions are invex functions, but the converse is not true. The concept of the invex and preinvex functions has played an important role in the development of convex programming. For the recent applications and generalizations of the invex functions, see [2,15,17,27] and the references therein.

DEFINITION 2.5. A function F is said to be strongly preinvex function on K with respect to the function η with modulus μ, if, for all $u, v \in K, t \in [0, 1]$,

$$F(u + t\eta(v, u)) \leqslant (1 - t)F(u) + tF(v) - t(1 - t)\mu||v - u||^2.$$

We remark that the differentiable strongly preinvex function F is a strongly invex functions, that is,

$$F(v) - F(u) \geqslant \langle F'(u), \eta(v, u)\rangle + \mu||v - u||^2,$$

but the converse is not true.

For given functions $F, \varphi : K \to H$, consider the problem of finding the minimum of $I[v]$ on K, where

$$I[v] = F(v) + \varphi(u), \tag{2.1}$$

which is known as the energy (potential) function. It is shown in [16] that the minimum of $I[v]$ on K can be characterized by a class of variational-like inequalities.

THEOREM 2.1 [16]. *Let K be an invex set in H with respect to the function η and $F : K \to H$ be a differentiable preinvex function with respect to η. If $\varphi : K \to H$ is a preinvex function with respect to η, then $u \in K$ minimizes the function $I[v]$, defined by* (2.1)*, if and only if $u \in K$ satisfies the inequality*

$$\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \textit{for all } v \in K, \tag{2.2}$$

where $F'(u)$ is the differentiable of F at u.

REMARK 2.1. Inequalities of the type (2.2) are known as the mixed variational-like inequalities. Note that for $\eta(v,u) = v - u$, Theorem 2.1 is exactly the same as in [13,14], that is, the minimum of $I[v]$ on the convex set K in H can be characterized by the mixed variational inequalities of the type

$$\langle F'(u), v - u\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \text{for all } v \in K. \tag{2.3}$$

For the applications, formulations and numerical results of mixed variational inequalities, see [4,8-10, 14,23].

DEFINITION 2.6. For all $u, v \in H$ and a given function $\eta : H \times H \longrightarrow H$, the operator $T : H \to H$ is said to be :

(i) η**-strongly monotone** if there exists a constant $\alpha_1 > 0$ such that

$$\langle Tu - Tv, \eta(u,v)\rangle \geqslant \alpha_1 ||u - v||^2.$$

(ii) η**- co-strongly monotone**, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \eta(u,v)\rangle \geqslant \alpha ||Tu - Tv||^2.$$

(iii) η**-monontone** if

$$\langle Tu - Tv, \eta(u,v)\rangle \geqslant 0.$$

(iv) η**-Lipschitz continuous** if there exists a constant $\beta > 0$ such that

$$\langle Tu - Tv, \eta(u,v)\rangle \leqslant \beta ||u - v||^2.$$

Note that for $\eta(v,u) = g(v) - g(u)$, where $g : H \to H$, is an operator, Definition 2.6 reduces to the definition of g-strongly monotone, g-co-strongly monotone, g-monotone and g-Lipschitz continuity of the operator T. For $g \equiv I$, the identity operator, that is, $\eta(v,u) = v - u$, Definition 2.6 reduces to the standard definition of the strongly monotone, co-strongly monotone(co- coercive), monotonicity and Lipschitz continuity of the operator. It is remarked that η- co-strongly monotonicity is weaker than the strongly monotonicity of the operator T.

DEFINTION 2.7. For all $u, v \in H$, the operator $\eta : H \times H \longrightarrow H$ is said to be Lipschitz continuous if there exists a constant $\beta_1 > 0$ such that

$$||\eta(v,u)|| \leqslant \beta_1 ||v - u||.$$

We also need the following assumption about the functions $\eta : H \times H \longrightarrow H$, which plays an important part in obtaining our results.

ASSUMPTION 2.1. For all $u, v, z \in H,$ the operator $\eta : H \times H \longrightarrow H$ satisfies the condition

$$\eta(u, v) = \eta(u, z) + \eta(z, v).$$

REMARK 2.2. From Assumption 2.1, we have

(i) $\eta(u, u) = 0$, for all $u \in H$

(ii) $\eta(u, v) = -\eta(v, u)$, for all $u, v \in H.$

These assumptions were used in [24,26,27] to suggest and analyze iterative methods for variational-like inequalities. Clearly (ii) implies (i).

3. Iterative Methods

In this section, we prove the existence of a solution of the generalized variational-like inequality (3.1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [8] as developed and modified by Noor [17-26] and to suggest an iterative algorithm. We consider the convergence analysis for the iterative scheme involving the $\eta-$ co-strongly monotone operators.

For a given operator $T : H \rightarrow H$, we consider the problem of finding $u \in H$ such that

$$\langle Tu, \eta(v, u)\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \text{for all } v \in H, \tag{3.1}$$

where φ is a preinvex function with respect to the function η. Clearly problem (2.2) is a special case of problem (3.1).

It is well known that the projection method and its variant forms, Wiener-Hopf equaions and resolvent equations cannot be extended and modified to to suggest and analyze projection and proximal type methods for solving variational-like inequalities due to the presence of the nonlinear term φ and the function η. These facts motivated Noor [16-20] to apply the auxiliary principle technique of Glowinski, Lions and Tremolieres [9] for solving variational-like inequalities. The main advantage of this technique is that it enables us to prove the existence of a solution of variational-like inequalities as well as provides us with a variational formulation of variational-like inequalities. To be more precise, we now consider the auxiliary variational-like inequality associated with problem (3.1).

For a given $u \in H,$ consider the problem of finding a unique $w \in H$ satisfying the auxiliary variational-like inequality

$$\begin{aligned} \langle E'(w), \eta(v, w)\rangle \geqslant{}& \langle E'(u) - \rho Tu, \eta(v, w)\rangle \\ &+\rho\varphi(w) - \rho\varphi(v), \quad \text{for all } v \in H, \end{aligned} \tag{3.2}$$

where $E'(u)$ is the differential of a strongly preinvex function $E(u)$. Problem (3.2) has a unique solution due to the strongly preinvexity of the function $E(u)$. We now show that the solution of the auxiliary variational-like inequality (3.2) is the minimum of the functional $I[w]$ on H, where

$$I[w] = E(w) - E(u) - \langle E'(u) - \rho Tu, \eta(w, u)\rangle + \rho\varphi(w) - \rho\varphi(u), \tag{3.3}$$

is known as the auxiliary functional.

REMARK 3.1. The function $B(w,u) = E(w) - E(u) - \langle E'(u), \eta(w,u)\rangle$ associated with the preinvex function $E(u)$ is called the generalized Bregman function. We note that if $\eta(w,u) = w-u$, then $B(w,u) = E(w)-E(u)-\langle E'(u), w-u\rangle$ is the well known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [5,6] and the references therein.

THEOREM 3.1. *Let F be a differentiable preinvex function and φ be a nondifferentiable preinvex function. If Assumption* 2.1 *holds and $\eta(.,.)$ is prelinear in the first argument, then the minimum of $I[v]$, defined by* (3.3)*, can be characterized by the auxiliary variational-like inequality* (3.2).

Proof. Let $w \in H$ be the minimum of $I[w]$ on H, then, for all $w,u \in H$, $t \in [0,1]$, $w_t = w + t\eta(v,w) \in H$ and

$$I[w] \leqslant I[w + t\eta(v,w)]. \tag{3.4}$$

Since $\eta(.,.)$ is prelinear in the first argument, from (3.3) and (3.4), we have

$$\begin{aligned} E(w) - E(u) - \langle E'(u) - \rho Tu, \eta(w,u)\rangle + \rho\varphi(w) &\leqslant E(w_t) - E(u) \\ &\quad -\langle E'(u) - \rho Tu, \eta(w_t,u)\rangle + \rho\varphi(w_t) \\ &\leqslant E(w_t) - (1-t)\langle E'(u) - \rho Tu, \eta(w,u)\rangle \\ &\quad -t\langle E'(u), -\rho Tu, \eta(v,u)\rangle + \rho\varphi(w) + t(\varphi(v) - \varphi(w)), \end{aligned}$$

which implies that

$$\begin{aligned} E(w + t\eta(v,w)) - E(w) &\geqslant t\langle E'(u) - \rho Tu, \eta(v,w)\rangle \\ &\quad -t\langle E'(u) - \rho Tu, \eta(w,u)\rangle + t(\varphi(w) - \varphi(v)). \end{aligned} \tag{3.5}$$

Now using Assumption 2.1, we have

$$\langle E'(u), \eta(v,u)\rangle = \langle E'(u), \eta(v,w)\rangle + \langle E'(u), \eta(w,u)\rangle \tag{3.6}$$

$$\langle Tu, \eta(v,u)\rangle = \langle Tu, \eta(v,w)\rangle + \langle Tu, \eta(w,u)\rangle \tag{3.7}$$

From (3.5), (3.6) and (3.7), we obtain

$$E(w + t\eta(v,w)) - E(w) \geqslant t\langle E'(u) - \rho Tu, \eta(v,w)\rangle + t\rho(\varphi(w) - \varphi(v)).$$

Dividing both sides by t and letting $t \to 0$, we have

$$\langle E'(w), \eta(v,w)\rangle \geqslant \langle E'(u) - \rho Tu, \eta(v,w)\rangle + \rho(\varphi(w) - \varphi(v)),$$

the required inequality (3.2).

Conversely, let $u \in H$ be a solution of (3.2). Then

$$\begin{aligned} I[w] - I[v] &= E(w) - E(v) - \langle E'(u) - \rho Tu, \eta(w,u)\rangle \\ &\quad + \langle E'(u) - \rho Tu, \eta(v,u)\rangle + \rho(\varphi(v) - \varphi(u)) \\ &\leqslant -\langle E'(w), \eta(v,w)\rangle + \langle E'(u), \eta(v,u) - \eta(w,u)\rangle \\ &\quad - \rho\langle Tu, \eta(v,u) - \eta(w,u)\rangle + \rho(\varphi(v) - \varphi(u)) \end{aligned}$$

$$\begin{aligned}
&\leqslant \langle E'(u), \eta(v,w)\rangle - \langle E'(w) - \rho Tu, \eta(v,w)\rangle \\
&\qquad + \langle E'(w) + \rho Tu, \eta(v,w)\rangle - \langle E'(u), \eta(v,w)\rangle \\
&\leqslant 0.
\end{aligned}$$

Thus it follows that $I[w] \leqslant I[v]$, showing that $v \in H$ is the minimum of the functional $I[w]$ on H, the required result.

We remark that if $w = u$, then w is a solution of the variational-like inequality (3.1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (3.1) as long as (3.2) or (3.3) is easier to solve than (3.1).

ALGORITHM 3.1.

(a) At $n = 0$, start with the initial value w_0.

(b) At step n, solve the auxiliary problem (3.2) or (3.3) with $u = w_n$. Let w_{n+1} denote the solution of the problem (3.2).

(c) If $||w_{n+1} - w_n|| \leqslant \epsilon$, for given $\epsilon \geqslant 0$, stop. Otherwise repeat (b).

We now study those conditions under which the approximate solution w_n obtained from Algorithm 3.1 converges to the exact solution w of problem (3.1) using the η-co-strongly monotonicity of the operator T and the technique of Zhu and Marcotte [31] for the variational-like inequalities.

THEOREM 3.2. *Let T be $\eta-$ co-strongly monotone with respect to the function η with constant α. Let E be strongly differentiable preinvex function with modulus β and the function $\eta(.,.)$ be Lipschitz continuous with constant μ. If Assumption* 2.1 *holds, then there exists a unique solution of (3.2). If $0 < \rho < 2\alpha\beta/\mu^2$, then the solution $\{u_n\}$ is bounded and converges to a solution of problem* (3.1).

Proof. Since the function E is strongly preinvex function, the solution u_{n+1} of (3.2) is unique. Let u be any fixed solution of the variational-like inequality (3.1). As in Zhu and Marcotte [31], we consider the function

$$\begin{aligned}
D(w) &= E(u) - E(w) - \langle E'(w), \eta(u,w)\rangle \\
&\geqslant (\beta/2)||u-w||^2, \text{using the strongly invexity of } E.
\end{aligned} \tag{3.8}$$

Now using (3.8) and Assumption 2.1, we have

$$\begin{aligned}
D(u_n) - D(u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u,u_n)\rangle \\
&\quad + \langle E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
&= E(u_{n+1}) - E(u-n) - \langle E'(u_n) - E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\
&\quad - \langle E'(u_n), \eta(u_{n+1},u_n)\rangle \\
&\geqslant (\beta/2)||u_{n+1} - u_n||^2 \\
&\quad + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle .
\end{aligned} \tag{3.9}$$

Taking $w = u_{n+1}$, $u = u_n$, $v = u$ in (3.2) and using $\eta(v,w) = -\eta(w,v)$, we have

$$\langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \geqslant \rho\langle Tu_n, \eta(u_{n+1},u)\rangle + \rho(\varphi(u_{n+1}) - \varphi(u)). \tag{3.10}$$

Combining (3.9), (3.10) and using Assumption2.1, we obtain

$$\begin{aligned}
D(u_n)-D(u_{n+1}) &\geqslant (\beta/2)||u_{n+1}-u_n||^2+\rho\langle Tu_n, \eta(u_{n+1},u)\rangle \\
&\quad +\rho(\varphi(u_{n+1})-\varphi(u)) \\
&\geqslant (\beta/2)||u_{n+1}-u_n||^2+\rho\langle Tu_n-Tu, \eta(u_{n+1},u)\rangle\,, \\
&\quad \text{using (3.1) with} v=u_{n+1} \\
&= (\beta/2)||u_{n+1}-u_n||^2+\rho\langle Tu_n-Tu, \eta(u_{n+1},u_n)\rangle \\
&\quad +\rho\langle Tu_n-Tu, \eta(u_n,u)\rangle \\
&\geqslant \frac{\beta}{2}||u_{n+1}-u_n||^2+\alpha\rho||Tu_n-Tu||^2 \\
&\quad +\rho\langle Tu_n-Tu, \eta(u_n,u)\rangle \\
&\geqslant \frac{\beta}{2}||u_{n+1}-u_n||^2+\alpha\rho||Tu_n-Tu||^2 \\
&\quad -\rho||Tu_n-Tu||||\eta(u_{n+1},u_n)|| \\
&\geqslant (\beta/2)||u_{n+1}-u_n||^2-(\rho/4\alpha)||\eta(u_{n+1},u_n)||^2 \\
&= (\beta/2-\rho\mu^2/4\alpha)||u_{n+1}-u_n||^2,
\end{aligned}$$

where we have used the $\eta-$ co-strongly monotonicity of the operator T and the inequality $ab\leqslant \varepsilon/2||a||^2+1/2\varepsilon||b||^2$, for a positive ε.

If $u_{n+1}=u_n$, then clearly u_n is a solution of the variational inequality (3.2). Otherwise, the assumption $\rho<2\alpha\beta/\mu^2$ implies that the sequence $D(u_n)-D(u_{n+1})$ is nonnegative, and we must have

$$\lim_{n\longrightarrow\infty}(||u_{n+1}-u_n||)=0.$$

Now by using the technique of Zhu and Marcotte [31], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the variational-like inequality (3.1).

REMARK 3.2. (i) Our result is an important and significant extension of the result of Zhu and Marcotte [31] for the variational-like inequalities, which are closely related with preinvex functions. It is noted that the preinvex functions are not convex functions, see [29].

(ii) If $\eta(v,u)=g(v)-g(u)$, where $g: H\longrightarrow H$ is a single-valued operator, then problem (3.1) is equivalent to finding $u\in H$ such that

$$\langle Tu, g(v)-g(u)\rangle+\varphi(g(v))-\varphi(g(u))\geqslant 0, \quad \text{for all } v\in H, \tag{3.11}$$

which is known as the general mixed variational inequality problem, see, for example, [18,20]. For a given $u\in H$, we consider the problem of finding a unique $w\in H$ satisfying the auxiliary variational inequality

$$\langle E'(w)-E'(u)+\rho Tu, g(v)-g(w)\rangle+\rho\varphi(g(v))-\rho\varphi(g(w))\geqslant 0, \tag{3.12}$$

where E' is the differential of a strongly convex function E.

Using the technique of Theorem 3.2 one can prove the convergence of the iterative method for solving the general mixed variational inequality (3.11). For suitable and appropriate choice of the operators T, η, φ and the space H, one can use Theorems 3.1 and 3.2 to study the existence results for various classes of variational inequalities as well as to suggest a number of iterative methods.

4. Merit Functions

In this section, we introduce and study a class of merit(gap) functions for the variational-like inequalities using the auxiliary principle technique. In recent years, there is a substantial interest for solving the variational inequalities and complementarity problems via the merit (gap) functions. The main idea is to find the differentiable equivalent optimization problem for variational inequalities and then using this equivalent formulation, one usually suggests the iterative methods for solving the variational inequalities. For the recent state-of-the art, see [4,7,15,30] and the references therein. We remark that there is no such merit(gap) function for the variational-like inequalities due to partly the presence of the function η and partly, the variational-like inequalities are closely related to the preinvex functions, which are not convex functions.

In order to construct a merit function for variational-like inequalities, we recall the following well known concept.

DEFINITION 4.1. A function $\Psi : H \longrightarrow R \cup \{+\infty\}$ is called a merit(gap) function for the variational-like inequality (3.1), if

(i) $\Psi(u) \geqslant 0$, for all $u \in H$

(ii) $\Psi(\overline{u}) = 0$ if and only if $\overline{u}$ solves (3.1)

Using this definition of the merit function, we reformulate the variational-like inequality as an equivalent optimization problem:

$$\text{Minimize} \quad \Psi(u) \quad \text{subject to } u \in H$$

This approach is due to Fukushima [7], Zhu and Marcotte [30], Larsson and Patriksson [15] and Chen, Goh and Yang [4] for variational inequalities and complementarity problems. Giannessi [8] has also derived a very general merit function for quasi variational inequalities using the alternate(separation) theorem. Following Zhu and Marcotte [30] and Chen, Goh and Yang [4], we consider the function $L : H \times H \longrightarrow R$, with

$$\begin{aligned} L(u,w) &= E(u) - E(w) + \varphi(u) - \varphi(w) \\ &\quad + \langle Tu - E'(u), \eta(u,w) \rangle, \quad \text{for all } u,w \in H, \end{aligned} \tag{4.1}$$

where $E : H \longrightarrow R \cup \{+\infty\}$ is a differentiable preinvex function and φ is a preinvex function. We now define the function Ψ as:

$$\Psi(u) = Max_{w \in H}\{L(u,w)\} \tag{4.2}$$

and the optimization problem

$$Inf_{u \in H}\{\Psi(u)\}. \tag{4.3}$$

For sake of simplicity, we rewrite (4.2) in the following form

$$\psi(u) = Max_{w\in H}\{\langle Tu, \eta(u,w)\rangle + \varphi(u) - \varphi(w) - B(u,w)\}, \tag{4.4}$$

where

$$B(w,u) = E(w) - E(u) - \langle E'(u), \eta(w,u)\rangle .$$

Note that

$$B(u,u) = 0, \quad \text{for all } u \in H$$

$$B(w,u) = E(w) - E(u) - \langle E'(u), \eta(w,u)\rangle \geqslant 0, \quad \text{for all } u \in H,$$

since E is differentiable preinvex function.

We now show that the function Ψ defined by (4.4) is a merit function for the variational-like inequalities (3.1) and this is the motivation of our next result.

THEOREM 4.1. *If Assumption* 2.1 *holds and the function* $\eta(.,)$ *is prelinear in the first argument,then the function* Ψ *defined by* (4.4) *is a merit function for the variational-like inequality* (3.1).

Proof. Since $B(u,u) =, 0,$ for each $u \in H,$ it is clear that $\Psi(u) \geqslant 0,$ for all $u \in H$. Now assume that $u \in H$ is a solution of the variational-like inequality (3.1). Then

$$\langle Tu, \eta(w,u)\rangle \geqslant \varphi(u) - \varphi(w), \quad \text{for all } w \in H.$$

As $B(u,w) \geqslant 0,$ for all $w \in H,$

$$\langle Tu, \eta(w,u)\rangle \geqslant \varphi(u) - \varphi(w) - B(u,w), \quad \text{for all } w \in H.$$

This implies , by using the assumption $\eta(w,u) = -\eta(u,w),$ that

$$\Psi(u) \leqslant 0.$$

Thus we conclude that

$$\Psi(u) = 0.$$

Conversely, let $\Psi(u) = 0,$ for all $u \in H.$ Then by Theorem 3.1, we have

$$\langle E'(u), \eta(w,u)\rangle \geqslant \langle E'(u) - Tu, \eta(w,u)\rangle + \varphi(u) - \varphi(w),$$

that is

$$\langle Tu, \eta(w,u)\rangle + \varphi(w) - \varphi(u) \geqslant 0,$$

which implies that $u \in H$ satisfies the variational-like inequality (3.1), the required result.

It is also possible to establish lower bound for the merit function. The following result is an extention of a result of Zhu and Marcotte [30] and Chen, Goh and Yang [4].

THEOREM 4.2. *Let T be η-strongly monotone with constant $\alpha_1 > 0$ and E' be η-Lipschitz continuous with constant $\beta_1 > 0$. Let u be the unique solution of* (3.1). *If $\beta_1 < \alpha_1$, then*

$$||v-u||^2 \leqslant \Psi(v)/(\alpha_1-\beta_1), \quad \textit{for all } v \in H. \tag{4.5}$$

Proof. Let $u \in H$ be the unique solution of (3.1). Then

$$\langle Tu, \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \text{for all } v \in H,$$

which implies that

$$\begin{aligned} \langle Tv, \eta(v,u)\rangle &\geqslant \varphi(u) - \varphi(v) + \langle Tv - Tu, \eta(v,u)\rangle \\ &\geqslant \varphi(u) - \varphi(v) + \alpha_1||v-u||^2, \end{aligned} \tag{4.6}$$

since T is η-strongly monotone with constant $\alpha_1 > 0$.

Now using (4.6), invexity of E and η-Lipschitz continuity of T, we have

$$\begin{aligned} \Psi(v) &= E(v) - E(u) - \langle E'(v) - Tv, \eta(v,u)\rangle + \varphi(v) - \varphi(u) \\ &\geqslant E(v) - E(u) - \langle E'(v), \eta(v,u)\rangle + \alpha_1||v-u||^2 \\ &\geqslant \langle E'(u) - E'(v), \eta(v,u)\rangle + \alpha_1||v-u||^2 \\ &\geqslant (\alpha_1-\beta_1)||v-u||^2, \end{aligned}$$

from which the required result (4.5) follows.

REMARK 4.1. Following the techniques and ideas of Larsson and Patriksson [15] and Zhu and Marcotte [30], one can easily discuss continuity properties of the merit function including the characterization of the solution of the variational-like inequalities (3.1) as stationary points of the problem (3.2). In a similar way, using this merit function Ψ, we can develop the descent framework for solving the variational-like inequalities. The development and implementations of such algorithms is the subject of future research efforts.

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Muhammad Aslam Noor
Department of Mathematics and Statistics
Dalhousie University
Halifax
Nova Scotia
Canada B3H 3J5
e-mail: noor@mscs.dal.ca