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ON AN OSTROWSKI TYPE INEQUALITY FOR A RANDOM VARIABLE

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(*communicated by B. Mond*)

Abstract. An Ostrowski's type inequality for a random variable is improved and one application is given.

The following theorem was proved by S. S. Dragomir, N. S. Barnett and S. Wang ([1]):

THEOREM A. *Let* X *be a random variable with the probability density function* $f:[a,b]\subset\mathbf{R}\to\mathbf{R}+$ *and with cumulative distributive function* $F(x)=Pr(X\leqslant x)$. *If* $f\in L_p[a,b]$, $p>1$, *then we have the inequality:*

$$|Pr(X\leqslant x)-\frac{b-E(X)}{b-a}|\leqslant\frac{q}{q+1}||f||_p(b-a)^{\frac{1}{q}}\left(\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}}+\left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right)$$
$$\leqslant\frac{q}{q+1}||f||_p(b-a)^{\frac{1}{q}} \tag{1}$$

for all $x\in[a,b]$, *where* $\frac{1}{p}+\frac{1}{q}=1$.

Here we shall show that the following stronger result can be obtained from the Fink's generalization of Ostrowski's inequality ([2]):

Theorem 1. *Let the assumptions of Theorem A be fulfilled. Then,*

$$|Pr(X\leqslant x)-\frac{b-E(X)}{b-a}|\leqslant\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{(q+1)(b-a)^q}\right)^{\frac{1}{q}}\cdot||f||_p \tag{2}$$

for all $x\in[a,b]$.

Proof. The following result is a special case of a more general result obtained by A. M. Fink (see also [3], p.471). Let g be absolutely continuous on $[a,b]$ with $g'\in L_p[a,b]$. Then, for $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$,

$$|g(x)-\frac{1}{b-a}\int_a^b g(y)dy|\leqslant\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{(q+1)(b-a)^q}\right)^{\frac{1}{q}}\cdot||g'||_p \tag{3}$$

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The inequality (3) is the best possible in the strong sense that for any $x \in [a,b]$ there is a function g for which equality holds. If we put $g = F$ in (3), by using $E(X) = b - \int_a^b F(t)dt$, we obtain the desired inequality (2).

Let's show that (2) is the stronger inequality than the first inequality in (1). Firstly, by the well known inequality for power sum we obtain

$$((x-a)^{q+1} + (b-x)^{q+1})^{\frac{1}{q}} \leqslant (x-a)^{\frac{1+q}{q}} + (b-x)^{\frac{1+q}{q}}. \tag{4}$$

On the other hand, by the weighted arithmetic-geometric mean inequality we have

$$(q+1)^{1-\frac{1}{q}} = (q+1)^{1-\frac{1}{q}} \cdot 1^{\frac{1}{q}} \leqslant \left(1 - \frac{1}{q}\right)(q+1) + \frac{1}{q} = q,$$

i.e., we have

$$\frac{1}{(q+1)^{\frac{1}{q}}} \leqslant \frac{q}{q+1}. \tag{5}$$

From (4) and (5) we have

$$\left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \leqslant \frac{q}{q+1}\left((x-a)^{\frac{1+q}{q}} + (b-x)^{\frac{1+q}{q}}\right),$$

i.e.

$$\left(\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)(b-a)^q}\right)^{\frac{1}{q}} \leqslant \frac{q}{q+1}(b-a)^{\frac{1}{q}}\left(\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right),$$

which shows the improvement of the result of Theorem A.

The improvement of other results in [1] can be done by using Theorem 1. Here, we shall establish only one more result. First, let's remind that the Beta random variable with parameters $(s,t) \in \Omega = \{(s,t) : s,t > 0\}$ has the probability density function

$$f(x,s,t) = \frac{x^{s-1}(1-x)^{t-1}}{B(s,t)}, \quad 0 < x < 1,$$

where $B(s,t)$ is Beta function. Also, as it is noted in [1], for $p > 1$

$$||f(.,s,t)||_p = \frac{1}{B(s,t)}\left(B(p(s-1)+1, p(t-1)+1)\right)^{\frac{1}{p}}$$

provided $s > 1 - \frac{1}{p}$ and $t > 1 - \frac{1}{p}$.

So, by using Theorem 1 we can obtain the following result

Corollary 1. *Let $p > 1$ and X be a Beta random variable with parameters (s,t), $s > 1 - \frac{1}{p}$, $t > 1 - \frac{1}{p}$. Then we have the inequality*

$$|Pr(X \leqslant x) - \frac{t}{s+t}| \leqslant \left(\frac{x^{1+q} + (1-x)^{1+q}}{1+q}\right)^{\frac{1}{q}} \frac{(B(p(s-1)+1, p(t-1)+1))^{\frac{1}{p}}}{B(s,t)}$$

for all $x \in [0,1]$.

Particulary, we have

$$|Pr(X \leqslant \frac{1}{2}) - \frac{t}{s+t}| \leqslant \frac{(B(p(s-1)+1, p(t-1)+1))^{\frac{1}{p}}}{2(1+q)^{\frac{1}{q}} B(s,t)}.$$

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