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A NOTE ON ALZER'S REFINEMENT OF AN ADDITIVE KY FAN INEQUALITY

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(*communicated by J. Pečarić*)

Abstract. An elementary proof of Alzer's generalization of Ky Fan inequality is given.

1. For $N \geqslant 2$, let $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant 1/2$, and let $y_n = 1 - x_n$. For $w_n > 0$ $(n = 1, 2, \cdots, N)$, with $\sum w_n = 1$, we denote by

$$A = \sum_{n=1}^{N} w_n x_n \,, \text{ and}$$

$$G = \prod_{n=1}^{N} x_n^{w_n} \,,$$

the weighted arithmetic and geometric means of the x_n's, and by A' and G' respectively, the similarly weighted arithmetic and geometric means of the y_n's.

In [1], H. Alzer showed with some effort that if not all of the x_n 's are equal, then

$$\frac{x_1}{1-x_1} < \frac{A'-G'}{A-G} < \frac{x_N}{1-x_N}. \tag{1}$$

This estimate improves $0 < \frac{A'-G'}{A-G} < 1$, which itself implies the well-known Ky Fan Inequality $G/G' \leqslant A/A'$ ([1, 2]).

2. It is the purpose of this short note to point out that (1) follows rather easily from the following result of D. I. Cartwright and M. J. Field ([3]), which holds under the same circumstances:

$$\frac{1}{2x_N} \sum_{n=1}^{N} w_n (x_n - A)^2 < A - G < \frac{1}{2x_1} \sum_{n=1}^{N} w_n (x_n - A)^2.$$

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Indeed, taking the quotient of these estimates, and the corresponding ones for $A' - G'$, yields

$$\frac{\frac{1}{2y_1}\sum_{n=1}^{N} w_n(y_n - A')^2}{\frac{1}{2x_1}\sum_{n=1}^{N} w_n(x_n - A)^2} < \frac{A' - G'}{A - G} < \frac{\frac{1}{2y_N}\sum_{n=1}^{N} w_n(y_n - A')^2}{\frac{1}{2x_N}\sum_{n=1}^{N} w_n(x_n - A)^2}.$$

Now one obtains (1), upon substituting $y_n = 1 - x_n$ and $A' = 1 - A$.

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