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INTERLACING OF EIGENVALUES AND INVARIANT FACTORS

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Abstract. In this note, we explain the analogy between interlacing properties of invariant factors of matrices over a principal ideal domain R and of eigenvalues of complex Hermitian matrices. This is done by looking at the interlacing theorems as dealing with very special cases of the problems of describing the invariant factors of a product of R-matrices and the eigenvalues of a sum of Hermitian matrices.

1. Introduction

Let R be a principal ideal domain and let $n \geqslant 2$ be a natural number. Given elements $c_n|\dots|c_1$ and $a_{n-1}|\dots|a_1$ in R, there exists an $n \times n$ matrix over R with the c_i as invariant factors, containing an $(n-1)\times n$ submatrix with the a_i as invariant factors, if and only if ([11, 16])

$$c_{i+1}|a_i|c_i\,, \quad i = 1,\dots,n-1.$$

It is also well known that given real numbers $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ and $\alpha_1 \geqslant \cdots \geqslant \alpha_{n-1}$, there exists an $n \times n$ Hermitian matrix with the γ_i as eigenvalues, containing an $(n-1)\times(n-1)$ principal submatrix with the α_i as eigenvalues if and only if (*e.g.* [3])

$$\gamma_{i+1} \leqslant \alpha_i \leqslant \gamma_i\,, \quad i = 1,\dots,n-1.$$

The observation of the analogy between these two “interlacing" results has led several authors to look for explanations for it, as well as for other analogies between invariant factors of R-matrices and eigenvalues of Hermitian matrices, (see *e.g.* [1], [2], [10]).

In the present note we explain the interlacing analogy by looking at the above results as dealing with very special cases of two important problems in matrix theory: describing the invariant factors of products of matrices over R and the eigenvalues of sums of Hermitian matrices.

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2. Interlacing of invariant factors

THEOREM 1. *Let* $c_n | \dots | c_1$ *and* $a_{n-1} | \dots | a_1$ *be elements of* R. *Then the following three conditions are equivalent:*

(a) There exists an $n \times n$ *matrix over* R *with the* c_i *as invariant factors, containing an* $(n-1) \times n$ *submatrix with the* a_i *as invariant factors.*

(b) There exist $n \times n$ R*-matrices* A *with invariant factors* $a_1, \dots, a_{n-1}, 1$, *and* X *with* $n-1$ *invariant factors equal to* 1 *such that* AX *has invariant factors* c_i.

(c) $c_n | a_{n-1} | c_{n-1} | \dots | c_2 | a_1 | c_1$.

Proof. The assertion (a) $\Leftrightarrow$ (c) is the interlacing result mentioned in the introduction.

We base the proof of (a) $\Leftrightarrow$ (b) on the following matrix factorization suggested in [4]:

$$\begin{bmatrix} A_1 & 0 \\ x & \xi \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \xi \end{bmatrix}, \tag{1}$$

where A_1 is $(n-1) \times (n-1)$, x is $1 \times (n-1)$ and ξ is 1×1. Note that the middle factor is unimodular.

Let us start with an $n \times n$ matrix, call it C, and a submatrix as described in (a). Of course we may assume that C already has the lower triangular block decomposition of the lefthand side of (1), where A_1 has invariant factors a_i. So (1) gives us the factorization we need in (b).

Conversely, assume we have a product AX satisfying all conditions in (b). Now AX is obviously equivalent to a matrix product much like the righthand side of (1), namely

$$\begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix} V \begin{bmatrix} I & 0 \\ 0 & \xi \end{bmatrix}, \tag{2}$$

where V is unimodular and A_1 is $(n-1)$-square with invariant factors $a_{n-1} | \dots | a_1$. But here we cannot guarantee that V has the block-lower-triangular form of the middle factor in the first factorization. We remove this obstacle in the following way. First, we remark that the problem is localizable: For each fixed prime $p \in R$, we can restrict ourselves to matrices over the local domain R_p, *i.e.*, work only with powers of p. Then we use, with minor changes, Lemma 5.1 of [13] and the argument under that lemma to show that V may in fact be transformed into a matrix like the middle unimodular factor of (1). So we started with a product AX as in (b), and we end up with a matrix (1), equivalent to AX and having A_1 as a submatrix. This finishes our proof that (a) and (b) are equivalent. $\square$

We remark that, in the matrix product situation, we can multiply throughout by an element of R, and obtain the following:

COROLLARY. *Given elements* $c_n | \dots | c_1$ *and* $a_n | \dots | a_1$ *in* R, *there exist* $n \times n$ R*-matrices* A *with invariant factors* $a_n | \dots | a_1$ *and* X *with* $n-1$ *invariant factors equal to* 1 *such that* AX *has invariant factors* $c_n | \dots | c_1$ *if and only if*

$$a_n | c_n | a_{n-1} | c_{n-1} | \dots | c_2 | a_1 | c_1 .$$

REMARK. Unfortunately, in the second part of the above proof we used a non-elementary localization technique, combined with a *triangular* decomposition of unimodular matrices over principal local domains. This can hardly be considered an elementary argument. In fact, it seems that we cannot pass without some tricky maneuver to transform AX into a matrix—e.g., like the one on the left side of (1)—exhibiting the desired matrix-submatrix pattern, as we see with the following 2×2 example: Given a, p, q, r, s, ξ in R, such that $ps - qr = 1$, consider the factorization

$$\begin{bmatrix} ap & aq\xi \\ r & s\xi \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix}.$$

If either $p, r,$ or s is a unit (as it holds in the local case) then it is easy to find upper-triangular row and column elementary transformations that reduce the middle unimodular factor to a lower triangular unimodular matrix. But in the general case we cannot expect things to be so easy: As a matter of fact, the existence of LUL- or ULU-factorizations for 2×2 $[n \times n]$ unimodular matrices characterizes local principal ideal domains [13, Lemma 5.1].

On the other hand, using localization, we have shown that our 2×2 matrix is equivalent to one like

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix}.$$

We wonder if a simple proof of this fact can be found without the localization step. More generally, we have the following:

OPEN PROBLEM. *Find an explicit equivalence, without localization, which transforms* (2) *into* (1).

3. Interlacing of eigenvalues of Hermitian matrices

THEOREM 2. *Let $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ and $\alpha_1 \geqslant \cdots \geqslant \alpha_{n-1}$ be nonnegative real numbers. Then the following three conditions are equivalent:*

(a) There exists an $n \times n$ Hermitian matrix with the γ_i as eigenvalues, containing an $(n-1) \times (n-1)$ principal submatrix with the α_i as eigenvalues.

(b) There exist $n \times n$ Hermitian matrices A with eigenvalues $\alpha_1, \ldots, \alpha_{n-1}, 0$ and $X \geqslant 0$ with $n-1$ zero eigenvalues such that $A + X$ has eigenvalues γ_i.

(c) $\gamma_1 \geqslant \alpha_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_{n-1} \geqslant \alpha_{n-1} \geqslant \gamma_n$.

Proof. The assertion (a) $\Leftrightarrow$ (c) is the interlacing result mentioned in the introduction.

The proof of (a) $\Leftrightarrow$ (b) is essentially due to R. C. Thompson [15] and is based on the following observation of Wielandt [17]: If we let N be $n \times (n-1)$, x be $n \times 1$, and denote by M the matrix $[N \,|\, x]$, then the $n \times n$ positive semidefinite matrices

$$MM^* = NN^* + xx^* \text{ and } M^*M = \begin{bmatrix} N^*N & N^*x \\ x^*N & x^*x \end{bmatrix}$$

have the same eigenvalues. □

We remark that in the Hermitian sum situation, we can add a real scalar throughout, and obtain the following:

COROLLARY. *Given real numbers* $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ *and* $\alpha_1 \geqslant \cdots \geqslant \alpha_n$*, there exist* $n \times n$ *Hermitian matrices* A *with eigenvalues* $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ *and* $X \geqslant 0$ *with* $n-1$ *zero eigenvalues such that* $A + X$ *has eigenvalues* $\gamma_1 \geqslant \cdots \geqslant \gamma_n$ *if and only if*

$$\gamma_1 \geqslant \alpha_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_{n-1} \geqslant \alpha_{n-1} \geqslant \gamma_n \geqslant \alpha_n .$$

4. Relating the two situations

We have seen in section 2 that in the invariant factor interlacing situation, we are dealing with a very particular case of the problem of describing the invariant factors of a product of two square R-matrices, given their invariant factors. In this problem, it is enough to consider the non-singular case [12].

NOTATION. We deal with the local version of the problem. Denote by IF_n the set of triples (α, β, γ) of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonincreasing nonnegative integers for which there exist $n \times n$ non-singular R_p-matrices A, B and C with invariant factors $p^{\alpha_1}, \ldots, p^{\alpha_n}$, $p^{\beta_1}, \ldots, p^{\beta_n}$ and $p^{\gamma_1}, \ldots, p^{\gamma_n}$ such that $C = AB$.

Analogously, we have seen in section 3 that in the Hermitian matrix interlacing situation we are dealing with a very particular case of the problem of describing the eigenvalues of a sum of two Hermitian matrices given their eigenvalues.

NOTATION. Denote by E_n the set of triples (α, β, γ) of n-tuples of nonincreasing real numbers for which there exist $n \times n$ Hermitian matrices A, B and C with eigenvalues $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ and $\gamma_1, \ldots, \gamma_n$ such that $C = A + B$. There is no loss of generality in working with nonnegative numbers only, and we include this in the definition of E_n.

THEOREM 3. *One has*

$$\mathrm{IF}_n = \mathrm{E}_n \cap \mathbb{Z}^{3n}. \tag{3}$$

Proof. From [6], we know that $\mathrm{IF}_n = \mathrm{LR}_n$, where LR_n is the set of triples (α, β, γ) of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ of nonincreasing nonnegative integers such that γ can be obtained from α and β, according to the combinatorial Littlewood-Richardson rule [9].

In [7], Klyachko has shown that for integer n-tuples α, β and γ, one has

$$(\alpha, \beta, \gamma) \in \mathrm{E}_n \Longleftrightarrow \exists_{N \in \mathbb{N}} (N\alpha, N\beta, N\gamma) \in \mathrm{LR}_n .$$

And very recently ([8]) Knutson and Tao proved that

$$(N\alpha, N\beta, N\gamma) \in \mathrm{LR}_n \Longrightarrow (\alpha, \beta, \gamma) \in \mathrm{LR}_n .$$

Putting these three facts together gives the stated result. $\square$

This relation between invariant factors and eigenvalues (which, as seen in the proof, follows readily from the deep results in [6], [7], [8]) explains, in light of Theorems 1 and 2, the interlacing analogy for submatrices.

REMARKS. 1. The equality (3) was conjectured for several years, on the strength of analogies between the two sets. In [13] it was shown that the inclusion $\mathrm{LR}_n \subseteq \mathrm{E}_n$ follows from a theorem of Heckman [5] on representations and coadjoint orbits of compact connected Lie groups.

2. The set LR_n, in the case where the second n-tuple is forced to have at most one nonzero component, is described explicitly by the so-called Pieri rule. This, not unexpectedly, is precisely the family of interlacing inequalities

$$\gamma_1 \geqslant \alpha_1 \geqslant \gamma_2 \geqslant \alpha_2 \geqslant \cdots \geqslant \gamma_n \geqslant \alpha_n$$

plus the obvious condition

$$\Sigma\,\gamma_i = \Sigma\,\alpha_i + \beta_1$$

(which is the only restriction involving β_1).

3. By Theorem 3, the integral vectors of E_n are the elements of LR_n. It follows from a result in [7] that E_n is a polyhedral cone in $\mathbb{R}^{3n}$ defined by a finite set of rational homogeneous linear inequalities in α, β and γ. Such a cone is generated by its integral vectors (see *e.g.* [14]), and therefore each inequality satisfied by the integral generators holds for all vectors in the cone. So the study of the case of integral spectra is enough to understand the situation concerning eigenvalues of Hermitian matrices.

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