Mathematical
Inequalities
& Applications
Volume 3, Number 2 (2000), 177–187

ON HADAMARD'S INEQUALITY FOR THE CONVEX MAPPINGS DEFINED ON A BALL IN THE SPACE AND APPLICATIONS

S. S. DRAGOMIR

(*communicated by T. Rassias*)

Abstract. In this paper we point out an inequality of Hadamard's type for triple integrals which works for convex functions defined on a ball from the space. Some mappings naturally connected with this inequality and related results are also pointed out.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f.

In the paper [4] (see also [5] and [9]) is considered the following mapping naturally connected to Hadamard's result:

$$H : [0,1] \to \mathbb{R},\ H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx.$$

The following properties are also proved:

(h) H is convex and monotonic nondecreasing.

(hh) One has the bounds

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right).$$

Mathematics subject classification (1991): 26D07, 26D15.

Key words and phrases: Convex functions of triple variable, Jensen's integral inequality, Hadamard's inequality.

© ELEMENT, Zagreb
Paper MIA-03-21

Another mapping also closely connected to Hadamard's inequality is the following one [5] (see also [9]):

$$F:[0,1]\to\mathbb{R},\, F(t):=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\,(tx+(1-t)\,y)\,dxdy.$$

The properties of this mapping are the following ones:

(f) F is convex and monotonic nonincreasing on $\left[0,\frac{1}{2}\right]$ and nondecreasing on $\left[\frac{1}{2},1\right]$;

(ff) F is symmetrical relating the element $\frac{1}{2}$. That is,

$$F(t)=F(1-t)\text{ for all } t\in[0,1];$$

(fff) One has the bounds

$$\sup_{t\in[0,1]}F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f\,(x)\,dx$$

and

$$\inf_{t\in[0,1]}F(t)=F\left(\frac{1}{2}\right)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy\geqslant f\left(\frac{a+b}{2}\right);$$

$(ffff)$ The following inequality holds:

$$F(t)\geqslant\max\{H(t),H(1-t)\}\text{ for all } t\in[0,1].$$

In this paper we will point out a similar inequality to Hadamard's one that will work for convex mappings defined on a closed ball from the space $\mathbb{R}^3$. We will also consider some mappings similar in a sense to the mappings H and F and establish their main properties.

For recent refinements, counterparts, generalizations and new Hadamard's type inequalities, see the papers [1]-[12] and [14]-[15] and the book [13].

2. Hadamard's inequality

In this section we will point out some inequalities of Hadamard's type for convex functions defined on the ball $\bar{B}(C,R)$, where $C=(a,b,c)\in\mathbb{R}^3$, $R>0$ and

$$\bar{B}(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\middle|\,(x-a)^2+(y-b)^2+(z-c)^2\leqslant R^2\right\}.$$

The following theorem holds:

THEOREM 1. *Let $f:\bar{B}(C,R)\to\mathbb{R}$ be a convex mapping on the ball $\bar{B}(C,R)$. Then we have the inequality:*

$$\begin{aligned} f\,(a,b,c) &\leqslant \frac{1}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)} f\,(x,y,z)\,dxdydz \\ &\leqslant \frac{1}{\sigma(\bar{B}(C,R))}\iint_{S(C,R)} f\,(x,y,z)\,ds \end{aligned} \tag{2.1}$$

where

$$S(C,R) := \left\{ (x,y,z) \in \mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2 \right\}$$

and

$$\nu\left(\bar{B}(C,R)\right) = \frac{4\pi R^3}{3}, \ \sigma\left(\bar{B}(C,R)\right) = 4\pi R^2.$$

Proof. To prove the first inequality in (2.1), let us consider the transformation:

$$T_1 : \mathbb{R}^3 \to \mathbb{R}^3, T_1(u,v,w) = (2a-u, 2b-v, 2c-w).$$

It is easy to see that the Jacobian of T_1 is

$$J(T_1) = \det \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1,$$

and T_1 is a one-to-one mapping which transforms the ball $\bar{B}(C,R)$ in itself. Then we have the change of variable:

$$\begin{aligned} \iiint_{\bar{B}(C,R)} f(x,y,z)\, dxdydz &= \iiint_{\bar{B}(C,R)} f(2a-u, 2b-v, 2c-w) \left|J(T_1)\right| dudvdw \\ &= \iiint_{\bar{B}(C,R)} f(2a-x, 2b-y, 2c-z)\, dxdydz. \end{aligned} \tag{2.2}$$

Now, by the convexity of f on the ball $\bar{B}(C,R)$, we have:

$$\frac{1}{2}\left[f(x,y,z) + f(2a-x, 2b-y, 2c-z)\right] \geqslant f(a,b,c)$$

for all $(x,y,z) \in \bar{B}(C,R)$.
Integrating this inequality on $\bar{B}(C,R)$ and taking into account that the equality (2.2) holds, we get

$$\iiint_{\bar{B}(C,R)} f(x,y,z)\, dxdydz \geqslant f(a,b,c) \iiint_{\bar{B}(C,R)} dxdydz = \nu\left(\bar{B}(C,R)\right) f(a,b,c)$$

That is, the first inequality in (2.1).
To prove the second part of the inequality (2.1), let us consider the transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$T_2(r,\psi,\varphi) := (r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c).$$

It is well known that the Jacobian of T_2 is

$$J(T_2) = r^2\cos\psi$$

and T_2 is a one-to-one mapping defined on the interval of $\mathbb{R}^3$, $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$, with values in the ball $\bar{B}(C,R)$ from $\mathbb{R}^3$. Thus we have the change of variable:

$$\begin{aligned} I &:= \iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\ &= \int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left[f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c)\right. \\ &\qquad \left.\times\, r^2\cos\psi\right] drd\psi d\varphi. \end{aligned}$$

Now, let us observe that for $(r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$ we have

$$\begin{aligned} &f(r\cos\psi\cos\varphi + a, r\cos\psi\sin\varphi + b, r\sin\psi + c) \\ &\quad = f\left[\left(1-\frac{r}{R}\right)(a,b,c) + \frac{r}{R}(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\right]. \end{aligned}$$

Using the convexity of f on the ball $\bar{B}(C,R)$ we can state that

$$\begin{aligned} &f\left[\left(1-\frac{r}{R}\right)(a,b,c) + \frac{r}{R}(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\right] \\ &\quad \leqslant \left(1-\frac{r}{R}\right)f(a,b,c) + \frac{r}{R}f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c) \quad (2.3) \end{aligned}$$

for all $(r,\psi,\varphi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$.
If we multiply this inequality with $r^2\cos\psi \geqslant 0$ for $(r,\psi)\in[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and integrating the obtained inequality on $[0,R]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$ we derive:

$$\begin{aligned} I &\leqslant f(a,b,c)\int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} r^2\cos\psi\left(1-\frac{r}{R}\right)drd\psi d\varphi \\ &\quad + \frac{1}{R}\int_0^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \left[r^3\cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\right] drd\psi d\varphi \\ &= \frac{\pi R^3}{3} f(a,b,c) + J, \qquad (2.4) \end{aligned}$$

where

$$J := \frac{R^3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\psi f(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\,d\psi d\varphi.$$

Now, let us compute the surface integral of the first type

$$K := \iint_{S(C,R)} f(x,y,z)\,dS,$$

where

$$S(C,R) := \left\{(x,y,z)\in\mathbb{R}^3 \middle| (x-a)^2 + (y-b)^2 + (z-c)^2 = R^2\right\}.$$

If we consider the parametrization of $S(C,R)$ given by:

$$S(C,R):\begin{cases} x = R\cos\psi\cos\varphi + a \\ y = R\cos\psi\sin\varphi + b \\ z = R\sin\psi + c \end{cases} ;(\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi]$$

and putting

$$A := \begin{vmatrix} \frac{\partial y}{\partial\psi} & \frac{\partial z}{\partial\psi} \\ \frac{\partial y}{\partial\varphi} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = -R^2\cos^2\psi\cos\varphi,$$

$$B := \begin{vmatrix} \frac{\partial x}{\partial\psi} & \frac{\partial z}{\partial\psi} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = R^2\cos^2\psi\sin\varphi,$$

and

$$C := \begin{vmatrix} \frac{\partial x}{\partial\psi} & \frac{\partial y}{\partial\psi} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \end{vmatrix} = -R^2\sin\psi\cos\psi,$$

then we have that

$$A^2+B^2+C^2 = R^4\cos^2\psi \text{ for all } (\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

Thus,

$$\begin{aligned} K &= \iint_{S(C,R)} f(x,y,z)\,dS \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\Big[f(R\cos\psi\cos\varphi+a, R\cos\psi\sin\varphi+b, R\sin\psi+c) \\ &\qquad\times\sqrt{A^2+B^2+C^2}\Big]\,d\psi d\varphi \\ &= R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f(R\cos\psi\cos\varphi+a, R\cos\psi\sin\varphi+b, R\sin\psi+c)\,d\psi d\varphi. \end{aligned}$$

Consequently, using the above notations, we deduce: $J = \frac{R}{4}K$.
Now, using the inequality (2.4) we get

$$I \leqslant \frac{\pi R^3}{3} f(a,b,c) + \frac{R}{4}\iint_{S(C,R)} f(x,y,z)\,dS. \tag{2.5}$$

If we divide this inequality by $\nu(\bar{B}(C,R)) = \frac{4\pi R^3}{3}$, we get the following inequality which is interesting in itself:

$$\begin{aligned} &\frac{1}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)} f(x,y,z)\,dxdydz \\ &\qquad \leqslant \frac{1}{4}f(a,b,c) + \frac{3}{4}\cdot\frac{1}{\sigma(\bar{B}(C,R))}\iint_{S(C,R)} f(x,y,z)\,dS. \end{aligned} \tag{2.6}$$

Now, taking into account that we proved the inequality

$$f\ (a,b,c) \leqslant \frac{1}{\bar{\nu}\left(\bar{B}\left(C,R\right)\right)} \iint_{\bar{B}(C,R)} f\ (x,y,z)\,dxdydz,$$

then, from (2.6), we derive

$$\frac{3}{4} \cdot \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ (x,y,z)\,dxdydz \leqslant \frac{3}{4} \cdot \frac{1}{\sigma\left(\bar{B}\left(C,R\right)\right)} \iint_{S(C,R)} f\ (x,y,z)\,dS.$$

That is, the second part of the inequality (2.1).
The proof of the theorem is thus completed.

□

3. Some mappings connected to Hadamard's inequality

As above, assume that the mapping $f : \bar{B}\left(C,R\right) \to \mathbb{R}$ is a convex mapping on the ball $\bar{B}\left(C,R\right)$ centered at the point $C = (a,b,c) \in \mathbb{R}^3$ and having the radius $R > 0$. Consider the mapping $H : [0,1] \to \mathbb{R}$ associated with the function f and given by:

$$H(t) := \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ (t\,(x,y,z) + (1-t)\,C)\,dxdydz$$

which is well defined for all $t \in [0,1]$.

The following theorem contains the main properties of this mapping.

THEOREM 2. *With the above assumption, we have:*

(i) *The mapping H is convex on $[0,1]$;*

(ii) *One has the bounds:*

$$\inf_{t\in[0,1]} H\left(t\right) = H\left(0\right) = f\ \left(C\right) \tag{3.1}$$

and

$$\sup_{t\in[0,1]} H\left(t\right) = H\left(1\right) = \frac{1}{\bar{\nu}\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ (x,y,z)\,dxdydz; \tag{3.2}$$

(iii) *The mapping H is monotonic nondecreasing on $[0,1]$.*

Proof. (i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. Then we have:

$$\begin{aligned}
&H\left(\alpha t_1 + \beta t_2\right) \\
&\quad= \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ (\alpha\left[t_1\left(x,y,z\right) + \left(1-t_1\right)C\right] \\
&\qquad\quad + \beta\left[t_2\left(x,y,z\right) + \left(1-t_2\right)C\right])\,dxdydz \\
&\quad\leqslant \alpha \cdot \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ \left(t_1\left(x,y,z\right) + \left(1-t_1\right)C\right)dxdydz \\
&\qquad\quad + \beta \cdot \frac{1}{\nu\left(\bar{B}\left(C,R\right)\right)} \iiint_{\bar{B}(C,R)} f\ \left(t_2\left(x,y,z\right) + \left(1-t_2\right)C\right)dxdydz \\
&\quad= \alpha H\left(t_1\right) + \beta H\left(t_2\right)
\end{aligned}$$

which proves the convexity of f on $[0,1]$.

(ii). We will prove the following identity:

$$H(t) = \frac{1}{t^3 \nu(\bar{B}, C)} \iiint_{\bar{B}(C,tR)} f(x,y,z)\, dxdydz \tag{3.3}$$

for all $t \in [0,1]$.
Fix t in $[0,1]$ and consider the mapping $g = (\psi, \eta, \mu) : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\begin{cases} \psi(x,y,z) = tx + (1-t)\, a \\ \eta(x,y,z) = ty + (1-t)\, b \\ \mu(x,y,z) = tz + (1-t)\, c \end{cases}, (x,y,z) \in \mathbb{R}^3.$$

We have:

$$\left| \frac{D(\psi,\eta,\mu)}{D(x,y,z)} \right| = t^3$$

and $g(\bar{B}(C,R)) = \bar{B}(C,R)$. Indeed

$$(\psi - a)^2 + (\eta - b)^2 + (\mu - c)^2 = t^2 \left[(x-a)^2 + (y-b)^2 + (z-c)^2 \right] \leqslant t^2 R^2$$

which shows that $(\psi,\eta,\mu) \in \bar{B}(C,R)$, and, conversely, for $(\psi,\eta,\mu) \in \bar{B}(C,tR)$ there exists $(x,y,z) \in \bar{B}(C,R)$ such that $g(x,y,z) = (\psi,\eta,\mu)$. So, g is a one-to-one mapping and we have the following change of variable:

$$\begin{aligned} &\iiint_{\bar{B}(C,tR)} f(\psi,\eta,\mu)\, d\psi d\eta d\mu \\ &\quad = \iiint_{\bar{B}(C,R)} f(\psi(x,y,z), \eta(x,y,z), \mu(x,y,z)) \left| \frac{D(\psi,\eta,\mu)}{D(x,y,z)} \right| dxdydz \\ &\quad = \iiint_{\bar{B}(C,R)} f(t(x,y,z) + (1-t)\, C)\, t^3 dxdydz \end{aligned}$$

and the equality (3.3) is proved.
Now, by the first inequality in (2.1) we get:

$$\frac{1}{\nu(\bar{B}(C,tR))} \iiint_{\bar{B}(C,tR)} f(x,y,z)\, dxdydz \geqslant f(C)$$

which gives us $H(t) \geqslant f(C)$ for all $t \in [0,1]$. Since $H(0) = f(C)$, we obtain the bound (3.1).

By the convexity of f on the ball $\bar{B}(C,R)$ we have:

$$\begin{aligned} H(t) &\leqslant \frac{1}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}[tf(x,y,z)+(1-t)f(C)]\,dxdydz \\ &= \frac{t}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}f(x,y,z)\,dxdydz+(1-t)f(C) \\ &\leqslant \frac{t}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}f(x,y,z)\,dxdydz \\ &\qquad +\frac{1-t}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}f(x,y,z)\,dxdydz \\ &= \frac{1}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}f(x,y,z)\,dxdydz. \end{aligned}$$

As we have

$$H(1)=\frac{1}{\nu(\bar{B}(C,R))}\iiint_{\bar{B}(C,R)}f(x,y,z)\,dxdydz,$$

the bound (3.2) holds.

(iii) Let $0\leqslant t_1<t_2\leqslant 1$. Thus, by the convexity of the mapping H we have

$$\frac{H(t_2)-H(t_1)}{t_2-t_1}\geqslant\frac{H(t_1)-H(0)}{t_1}\geqslant 0$$

as we proved that $H(t_1)\geqslant H(0)$ for all $t_1\in[0,1]$; and the monotonicity of H is established.

□

Further on, we shall introduce another mapping connected to Hadamard's inequality:

$$h:[0,1]\to\mathbb{R},\ h(t):=\begin{cases}\frac{1}{\sigma(B(C,tR))}\iint_{S(C,tR)}f(x,y,z)\,dS \text{ if } t\in(0,1]\\ f(C) \text{ if } t=0\end{cases}$$

where $f:\bar{B}(C,R)\to\mathbb{R}$ is a convex mapping on the ball $\bar{B}(C,R)$ centered at the point $C=(a,b,c)$ and having the radius R and $S(C,R)$ is the sphere:

$$S(C,R):=\left\{(x,y,z)\in\mathbb{R}^3\,\middle|\,(x-a)^2+(y-b)^2+(z-c)^2=R^2\right\}.$$

The main properties of this mapping are embodied in the following theorem:

THEOREM 3. *With the above assumptions, one has:*

(i) *The mapping* $h:[0,1]\to\mathbb{R}$ *is convex on* $[0,1]$;

(ii) *One has the bounds:*

$$\inf_{t\in[0,1]}h(t)=h(0)=f(C) \tag{3.4}$$

and

$$\sup_{t\in[0,1]}h(t)=h(1)=\frac{1}{\sigma(\bar{B}(C,R))}\iint_{S(C,R)}f(x,y,z)\,dS; \tag{3.5}$$

(*iii*) *The mapping* h *is monotonic nondecreasing on* $[0,1]$;
(*iv*) *We have the inequality:*

$$H(t) \leqslant h(t) \text{ for all } t \in [0,1].$$

Proof. For a fixed t in $(0,1]$ consider the surface:

$$S(C,tR):\begin{cases} x = tR\cos\psi\cos\varphi + a \\ y = tR\cos\psi\sin\varphi + b \\ z = tR\sin\psi + c \end{cases} ;(\psi,\varphi)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times[0,2\pi].$$

As in the proof of Theorem 1, we get the equality:

$$\begin{aligned} K &= \iint_{S(C,tR)} f(x,y,z)\,dS \\ &= t^2R^2 \\ &\quad\times\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f(tR\cos\psi\cos\varphi + a, tR\cos\psi\sin\varphi + b, tR\sin\psi + c)\,d\psi d\varphi. \end{aligned}$$

Thus,

$$\begin{aligned} h(t) &= \frac{1}{4t^2\pi^2R^2}\iint_{S(C,tR)} f(x,y,z)\,dS \\ &= \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\psi f(t(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C)\,d\psi d\varphi \end{aligned}$$

for all $t \in (0,1]$.
Using this representation of the mapping h we can prove the following statements:

(i) Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. Then, by the convexity of f, we get that:

$$\begin{aligned} & h(\alpha t_1 + \beta t_1) \\ & = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f\left[\alpha(t_1(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C)\right. \\ & \qquad \left. + \beta(t_2(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C)\right]\cos\psi d\psi d\varphi \\ & \leqslant \alpha\cdot\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f\left[t_1(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C\right]\cos\psi d\psi d\varphi \\ & \qquad + \beta\cdot\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} f\left[t_2(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C\right]\cos\psi d\psi d\varphi \\ & = \alpha h(t_1) + \beta h(t_1) \end{aligned}$$

which proves the convexity of h.

(iv) In the above theorem we proved between other, that

$$H(t) = \frac{1}{\nu(\bar{B}(C,tR))} \iiint_{\bar{B}(C,tR)} f(x,y,z)\,dxdydz$$

for all $t \in (0,1]$.
By Hadamard's inequality (2.1) applied for the ball $\bar{B}(C,tR)$ we have:

$$\frac{1}{\nu(\bar{B}(C,tR))} \iiint_{\bar{B}(C,tR)} f(x,y,z)\,dxdydz \leqslant \frac{1}{\sigma(\bar{B}(C,tR))} \iint_{S(C,tR)} f(x,y,z)\,dS$$

from where we get the inequality

$$H(t) \leqslant h(t) \text{ for all } t \in (0,1].$$

As it is easy to see that $H(0) = h(0) = f(C)$, the statement is thus proved.

(ii) The bound (3.4) follows by the above considerations and we shall omit the details.
By the convexity of f on the ball $\bar{B}(C,R)$ we have:

$$\begin{aligned} h(t) = & \frac{1}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f\,(t\,[(R\cos\psi\cos\varphi, R\cos\psi\sin\varphi, R\sin\psi) + C] \\ & \qquad + (1-t)\,C)\cos\psi d\psi d\varphi \\ \leqslant & \frac{t}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f\,(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\cos\psi d\psi d\varphi \\ & \qquad + (1-t) f(C) \frac{1}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos\psi d\psi d\varphi \\ = & \frac{t}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} f\,(R\cos\psi\cos\varphi + a, R\cos\psi\sin\varphi + b, R\sin\psi + c)\cos\psi d\psi d\varphi \\ & \qquad + (1-t) f(C) \\ = & th(1) + (1-t) f(C) \leqslant th(1) + (1-t)\,h(1) = h(1) \end{aligned}$$

as $f(C) \leqslant h(t)$ for all $t \in [0,1]$. Thus, the bound (3.5) is proved.

(iii) Follows as in the proof of Theorem 2, and we omit the details.

□

REFERENCES

[1] S. S. DRAGOMIR, *Two refinements of Hadamard's inequalities*, Coll. of Sci. Pap. of the Fac. of Sci. Kragujevac (Yugoslavia), **11**(1990), 23–26. ZBL No. 729: 26017.
[2] S. S. DRAGOMIR, J. E. PEČARIĆ AND J. SÁNDOR, *A note on the Jensen-Hadamard inequality*, L' Anal. Num. Theor. L'Approx. (Romania), **19**(1990), 21–28. MR 93c: 26016. ZBL No. 743: 26016.
[3] S. S. DRAGOMIR, *Some refinements of Hadamard's inequalities*, Gaz. Mat. Metod. (Romania) **11**(1990), 189–191.
[4] S. S. DRAGOMIR, *A mapping in connection to Hadamard's inequality*, An Ostro. Akad. Wiss. Math. Natur (Wien) **128** (1991), 17–20. MR 93h: 26032. ZBL No. 747: 26015.

[5] S. S. DRAGOMIR, *Two mappings in connection to Hadamard's inequality*, J. Math. Anal, Appl., **167** (1992), 49-56. MR 93m: 26038. ZBL No. 758: 26014.
[6] S. S. DRAGOMIR AND N. M. IONESCU, *Some integral inequalities for differentiable convex functions*, Coll. Pap. of the Fac. of Sci. Kragujevac (Yugoslavia), **13** (1992), 11–16. ZBL No. 770: 26009.
[7] S. S. DRAGOMIR, *Some integral inequalities for differentiable convex functions*, Contributions, Macedonian Acad. of Sci. and Arts (Scopie), **16**(1992), 77–80.
[8] S. S. DRAGOMIR AND N. M. IONESCU, *Some remarks in convex functions*, L'Anal. Num. Theor. L'Approx. (Romania), **21**(1992), 31–36. MR. 93m: 26023. ZBL No. 770: 26008.
[9] S. S. DRAGOMIR, *On Hadamard's inequality for convex functions*, Mat. Balkanica, **6**(1992), 215–222. MR 934:26033.
[10] S. S. DRAGOMIR, *A refinement of Hadamard's inequality for isotonic linear functionals*, Tamkang J. of Math. (Taiwan), **24**(1993), 101–106.
[11] S. S. DRAGOMIR, D. M. MILOŠEVIĆ AND J. SANDOR, *On some refinements of Hadamard's inequalities and applications*, Univ. Beograd, Publ. Elektrotehn. Fak, Ser. Mat. **4**(1993), 3–10.
[12] S. S. DRAGOMIR, D. BARBU AND C. BUSE, *A Probabilistic argument for the convergence of some sequences associated to Hadamard's inequality*, Studia Univ. Babes-Bolgai, Mathematica, **38**(1993), (1), 29–34.
[13] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
[14] J. E. PEČARIĆ AND S. S. DRAGOMIR, *On some integral inequalities for convex functions*, Bull. Mat. Inst. Pol. Iasi, **36**(1-4) (1990), 19–23. MR 93i: 26019. ZBL No. 765: 26008.
[15] J. E. PEČARIĆ AND S. S. DRAGOMIR, *A generalization of Hadamard's inequality for isotonic linear functionals*, Rodovi Mat. (Sarajevo), **7**(1991), 103–107. MR 93k: 26026. ZBL No. 738: 26006.

(Received October 28, 1999)

School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City
MC 8001 Australia
e-mail: sever@matilda.vu.edu.au