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# DIFFERENTIAL AND INTEGRAL $f$–MEANS AND APPLICATIONS TO DIGAMMA FUNCTION

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*Abstract.* Some basic comparaison theorems between two elementary means — differential and integral $f$-mean are obtained. This theorems are applied to digamma function.

## 1. Introduction

Let $I \subset \mathbf{R}$ be an open interval, $f : I \to \mathbf{R}$ convex (or concave) differentiable function, $s, t \in I$. Then there exists the unique $\alpha \in [s, t]$ for which

$$f'(\alpha) = \frac{f(t) - f(s)}{t - s}.$$

$\alpha$ is called *differential $f$-mean* of $s$ and $t$, and denoted by

$$D_f = D_f(s,t) := (f')^{-1}\left(\frac{f(t) - f(s)}{t - s}\right). \tag{1}$$

If $f : I \to \mathbf{R}$ is (strictly) monotone, then there exists the unique $\beta \in [s, t]$ for which

$$\frac{1}{t-s}\int_s^t f(u)du = f(\beta).$$

$\beta$ is called *integral $f$-mean* of $s$ and $t$, and denoted by

$$I_f = I_f(s,t) = f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)du\right). \tag{2}$$

Obviously, for convex (or concave) differentiable function $f$, it holds

$$I_{f'} = D_f. \tag{3}$$

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EXAMPLE 1. For the convenience of the reader let us give some example of differential and integral $f$ -means.

$$f(x) = x, \qquad I_x(s,t) = \frac{s+t}{2} = A(s,t),$$

$$f(x) = \frac{1}{x}, \qquad D_{1/x}(s,t) = \sqrt{st} = G(s,t),$$

$$f(x) = \frac{1}{x}, \qquad I_{1/x}(s,t) = \frac{s-t}{\log s - \log t} = L(s,t),$$

$$f(x) = \log x, \qquad I_{\log}(s,t) = \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}} = I(s,t),$$

$$f(x) = x^{r-1}, \quad r \neq 1 \qquad D_{x^{r-1}}(s,t) = \left(\frac{t^r - s^r}{r(t-s)}\right)^{\frac{1}{r-1}} = L_r(s,t),$$

where $A$, $G$, $L$, $I$ and $L_r$ are arithmetic, geometric, logarithmic, identric and generalized logarithmic mean.

From the first example it is clear that differential and integral $f$ -means are quite elementary. But, it seems that their properties are not studied in details, and some of the known results are obtained rather as special cases of more general results. The purpose of this elementary note is to give a general view to the connection between differential and integral means.

Integral $f$ -mean (in our notation) is a special case of $M$-quasi arithmetic mean defined in [3] by

$$M^{-1}\left(\int M(f(u))d\mu(u)\right),$$

where function $M$ being specified, to be (say) $M(x) = 1/x$, $r$-th power, logarithmic function etc. We shall instead regard this mean for general function $M$ and $f(x) = x$.

We shall derive some monotonicity properties and elementary comparaison theorem between $D$ and $I$. General theorems will be applied to the case of digamma function. The apeareance of this paper is motivated by theorems given in [5] in the same journal.

## 2. Comparaison theorems

We start with the following theorem which is well known (see e.g. [4]), and its proof is an easy consequence of Jensen inequality.

THEOREM A. *Let* $f, g : I \to \mathbf{R}$ *be strictly monotonic. If*

$(i)$ $f$ *is increasing, and*

$(ii)$ $f \circ g^{-1}$ *is convex, then*

$$I_g \leqslant I_f. \tag{5}$$

REMARK 1. If $(ii)$ is satisfied, then one says that $f$ is convex with respect to $g$ [4]. If one of the conditions $(i)$ or $(ii)$ changes to the opposite property, then the inequality sign in (5) also changes. Therefore, if $(ii)$ is satisfied and $f$ is decreasing, then $I_g \geqslant I_f$ . Also, (5) holds for decreasing function $f$ for which $f \circ g^{-1}$ is concave, etc.

H. Alzer [2] proved that for strictly increasing function $g$ such that $1/g^{-1}$ is convex, it holds

$$g(L(a,b)) \leqslant \frac{1}{b-a}\int_a^b g(x)dx.$$

It is obvious that this result follows from Lemma A by choosing $f(x) = 1/x$, since for this function we have

$$L_f(a,b) = \left[\frac{1}{b-a}\int_a^b \frac{1}{x}dx\right]^{-1} = L(a,b).$$

In [6] generalization of this result is proved for identric and generalized logarithmic mean: if $x \mapsto f(x^{1/(r-1)})$ is convex, then

$$f(L_r(s,t)) \leqslant \frac{1}{t-s}\int_s^t f(u)du,$$

and if $x \mapsto f(e^x)$ is convex, then

$$f(I(s,t)) \leqslant \frac{1}{t-s}\int_s^t f(u)du.$$

Both inequalities are straighforward applications of Theorem A.

Obviously, one can derive a scale of analogous results for various means by choosing some simple elementary functions. For example,

$$\begin{array}{lll}
f(x) = x, & \dfrac{a+b}{2} \geqslant I_g(a,b), & \text{if } g^{-1} \text{ is convex,} \\
f(x) = 1/x, & \dfrac{a-b}{\log a - \log b} \leqslant I_g(a,b), & \text{if } 1/g^{-1} \text{ is convex,} \\
f(x) = \dfrac{1}{\sqrt{x}}, & \left(\dfrac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leqslant I_g(a,b), & \text{if } (g^{-1})^{-1/2} \text{ is convex,} \\
f(x) = \log x, & \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} \geqslant I_g(a,b), & \text{if } \log(g^{-1}) \text{ is convex,} \\
f(x) = x^{r-1}\ (r>1), & \left(\dfrac{b^r-a^r}{b-a}\right)^{1/(r-1)} \geqslant I_g(a,b), & \text{if } (g^{-1})^{r-1} \text{ is convex,}
\end{array}$$

etc. Here and elsewhere in this article $g^{-1}$ denotes the inverse to $g$.

Another scale of analogous results follows if we specify function $g$. Let us note some of them. Suppose $f$ is increasing (the opposite inequalities holds if $f$ is decreasing):

| | | |
|---|---|---|
| $g(x) = x,$ | $\dfrac{a+b}{2} \leqslant I_f(a,b),$ | if $f$ is convex, |
| $g(x) = 1/x,$ | $L(a,b) \leqslant I_f(a,b),$ | if $f(1/x)$ is convex, |
| $g(x) = \log x,$ | $\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)} \leqslant I_f(a,b),$ | if $f(e^x)$ is convex, |
| $g(x) = x^{r-1}\ (r>1),$ | $\left(\dfrac{b^r - a^r}{r(b-a)}\right)^{1/(r-1)} \leqslant I_f(a,b),$ | if $f(x^{1/(r-1)})$ is convex. |

It is interesting to derive an elementary criterion for convexity of $f \circ g^{-1}$. The following theorem is a slight generalization of Theorem 1. [4]:

THEOREM 1. *Let $f, g \in C^{(2)}(I)$. Suppose:*
$(i)$ *$g$ is decreasing,*
$(ii)$ *$g'/f'$ is increasing.*
*Then $f$ is convex with respect to $g$.*

*Proof.* For the function $h = f \circ g^{-1}$ we have

$$h''(x) = \frac{1}{g'(g^{-1}(x))}\left(\frac{f'(g^{-1}(x))}{g'(g^{-1}(x))}\right)'$$

and theorem follows.

Adequate statement holds if some of the properties from $(i)$–$(ii)$ changes to the opposite one.

In [4] it is additionally supposed that $f$ is increasing function.

Using Theorem A, we can compare integral means:

THEOREM 2. *Let $f, g \in C^{(2)}(I)$, $f$ and $g$ be monotone functions. Suppose:*
$(i)$ *$f$ is increasing,*
$(ii)$ *$g$ is decreasing,*
$(iii)$ *$g'/f'$ is increasing.*
*Then*

$$I_g \leqslant I_f. \tag{6}$$

The choice $g = f'$ leads to the comparaison theorem between differential and integral $f$-mean.

THEOREM 3. *Let $f \in C^{(2)}(I)$ satisfies*
$(i)$ *$f$ is increasing,*
$(ii)$ *$f$ is concave,*
$(iii)$ *$f''/f'$ is increasing.*
*Then*

$$D_f \leqslant I_f. \tag{7}$$

EXAMPLE. It holds $D_{\cos} \leqslant I_{\cos}$, $D_{\sin} \geqslant I_{\sin}$, $D_{\log} \leqslant I_{\log}$.

## 3. Monotonicity results

We shall give a brief motivation. Let us take $f(x) = 1/x$, $x > 0$ . Then

$$\frac{f(t) - f(s)}{t - s} = f'(D_f) \implies D_f = \sqrt{st} = G(s,t),$$

and

$$\frac{1}{t-s}\int_s^t f(u)du = f(I_f) \implies I_f = \frac{t-s}{\log t - \log s} = L(s,t).$$

Since $f$ is decreasing and convex, and $f''(x)/f'(x) = -2/x$ is increasing on $(0,\infty)$, by Theorem 3 it follows $D_f \leqslant I_f$ . But, it is known that

$$G(x+s, x+t) \uparrow A(x+s, x+t) \qquad \text{as} \quad x \to \infty,$$

hence

$$G(x+s,x+t) - x \leqslant L(x+s,x+t) - x \uparrow A(s,t).$$

Therefore one may ask when this property holds in general?

THEOREM 4. *Let $f \in C^{(2)}(I)$ and*

$(i)$ $f$ *is increasing,*
$(ii)$ $f'$ *is decreasing,*
$(iii)$ $f''/f'$ *is increasing.*

*Then $x \mapsto I_f(x+s, x+t) - x$ is increasing.*

*Proof.* Let us denote

$$h(x) = f^{-1}\left(\frac{1}{t-s}\int_{x+s}^{x+t} f(u)du\right) - x.$$

Then

$$h'(x) = \left[f'\left(f^{-1}\left(\frac{1}{t-s}\int_{x+s}^{x+t} f(u)du\right)\right)\right]^{-1}\frac{1}{t-s}\int_s^t f'(x+u)du - 1$$

Since $f'$ is positive and decreasing, $h'(x) > 0$ is equivalent to

$$(f')^{-1}\left[\frac{f(x+t) - f(x+s)}{t-s}\right] \leqslant f^{-1}\left[\frac{1}{t-s}\int_{x+s}^{x+t} f(u)du\right]$$

i.e. $D_f \leqslant I_f$ , which is true by Theorem 3.

THEOREM 5. *Suppose $f : I \to \mathbf{R}$ satisfies*

$(i)$ $f$ *is increasing,*
$(ii)$ $f'$ *is decreasing,*
$(iii)$ $f''$ *is increasing,*
$(iv)$ $f''/f'$, $f'''/f''$ *are increasing.*

*Then* $x \mapsto I_f(x+s, x+t)$ *is concave.*

*Proof.* Let us denote

$$I(g) = \frac{1}{t-s}\int_s^t g(x+u)du$$

and

$$h(x) = I_f(x+s, x+t) = f^{-1}(I(f)).$$

Then

$$\begin{aligned} h' &= \frac{1}{f'(h)} \cdot I(f'), \\ h'' &= -\frac{1}{f'(h)^2} \cdot f''(h) \cdot \frac{1}{f'(h)} \cdot I(f') + \frac{1}{f'(h)} \cdot I(f'') \\ &= \frac{1}{f'(h)^3}\left[f'(h)^2 I(f'') - f''(h) I(f')^2\right] \end{aligned}$$

Since $f$ is increasing, $h'' \leqslant 0$ is equivalent to

$$\frac{I(f'')}{I(f')^2} \leqslant \frac{f''(h)}{f'(h)^2}$$

By Theorem 3, we have $D_{f'} \leqslant I_{f'} = D_f$ and $D_f \leqslant I_f$. Since $f''$, $f''/f'$ and $1/f'$ are increasing,

$$\frac{f''(I_{f''})}{f'(I_f')^2} = \frac{f''(D_{f'})}{f'(D_f)^2} \leqslant \frac{f''(D_f)}{f'(D_f)^2} \leqslant \frac{f''(I_f)}{f'(I_f)^2}$$

which has to be proved.

COROLLARY 1. *Let* $f : I \to \mathbf{R}$ *be increasing function such that* $f'$ *is completely monotonic. Then* $x \mapsto I_f(x+s, x+t) - x$ *is increasing and concave.*

*Proof.* By assumption, properties $(i)-(iii)$ of Theorems 4 and 5 are satisfied. If $\varphi$ is completely monotonic on $I$, then

$$\varphi^{(k+1)}(x)\varphi^{(k-1)}(x) \geqslant \varphi^{(k)}(x)^2, \qquad \forall x \in I, \quad k = 1, 2, \dots$$

and $(iv)$ of Theorem 5 follows.

## 4. Applications to digamma function

Let us apply preceeding theorems to the digamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is known that $\psi$ is increasing concave function. It has the following representation [1], p. 259:

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left[\frac{1}{n} - \frac{1}{n+x-1}\right], \quad x \neq 0, -1, -2, \dots$$

and it follows

$$(-1)^{(k+1)}\psi^{(k)}(x) \geqslant 0, \qquad k = 1, 2, \dots$$

therefore, $\psi'$ is completely monotonic. Hence, $\psi''/\psi'$ and $\psi'''/\psi''$ are increasing.

By Theorem 1, it follows that $\psi \circ (\psi')^{-1}$ is convex, hence, by Theorem 2,

$$D_\psi \leqslant I_\psi.$$

LEMMA 1. *We have, for all* $s, t > 0$

$$\psi\left(\frac{t-s}{\log t - \log s}\right) \leqslant \frac{1}{t-s}\int_s^t \psi(u)du.$$

*Proof.* It holds $L(s,t) = I_{1/x}(s,t)$. Using Theorem A, it is sufficient to prove that $h(x) = \psi(1/x)$ is convex. We have, for $u > 0$

$$\frac{1}{u^3}h''\left(\frac{1}{u}\right) = u\psi''(u) + 2\psi'(u) = -2\sum_{n=0}^{\infty}\frac{u}{(x+u)^3} + 2\sum_{n=0}^{\infty}\frac{1}{(x+u)^2} > 0$$

and Lemma follows.

Hence, we can conclude the following properties of digamma function:

THEOREM 6. *For* $x > 0$ *digamma function* $\psi$ *has the properties:*

$(i)$ $D_\psi \leqslant I_\psi$,

$(ii)$ $x \mapsto I_\psi(x+s, x+t) - x$ *is increasing concave function, and*

$$I_\psi(x+s, x+t) - x \uparrow A(s,t) \qquad \text{as } x \to \infty.$$

*Proof.* We have proved that

$$G(s,t) \leqslant L(s,t) \leqslant I_\psi(s,t).$$

The statements follows by Corollary 1, and the fact that $G(x+s, x+t) \uparrow A(x+s, x+t) = x + A(s,t)$ as $x \to \infty$.

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