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THE EULER FORMULÆ AND CONVEX FUNCTIONS

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Abstract. The Euler midpoint and Euler–Simpson formulæ are used with functions possessing various convexity and concavity properties to derive inequalities pertinent to numerical integration.

1. Introduction

One of the cornerstones of nonlinear analysis is the Hadamard inequality, which states that if $[a,b]$ $(a<b)$ is a real interval and $f:[a,b]\to R$ a convex function, then

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Recently, Dragomir and Agarwal [3] considered the trapezoid formula for numerical integration for functions such that $|f'|^q$ is a convex function for some $q\geqslant 1$. Their approach was based on estimating the difference between the two sides of the right–hand inequality in (1.1). Improvements of their results were obtained in [5]. In particular, the following tool was established.

Suppose $f:I^0\subseteq R\to R$ is differentiable on I^0 and that $|f'|^q$ is convex on $[a,b]$ for some $q\geqslant 1$, where $a,b\in I^0$ $(a<b)$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{1.2}$$

Some generalizations to higher–order convexity and applications of these results are given in [2].

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In this paper we consider further related results. The most natural nexus for these developments would appear to be the well–known Euler formula

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \sum_{k=1}^{n-1}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] - \frac{(b-a)^{n-1}}{n!}\int_a^b f^{(n)}(t)\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n^*\left(\frac{x-a}{b-a}\right)\right]dt \quad (1.3)$$

(see [4, p. 17]), which holds for every $x \in [a,b]$ and every function $f : [a,b] \to \mathbf{R}$ with $n \geqslant 2$ continuous derivatives. Here $B_k(\cdot)$ $k \geqslant 0$ is the kth Bernoulli polynomial and $B_k = B_k(0) = B_k(1)$ ($k \geqslant 0$) the kth Bernoulli number. We denote by $B_k^*(\cdot)$ ($k \geqslant 0$) the function of period one with $B_k^*(x) = B_k(x)$ for $0 \leqslant x \leqslant 1$.

For $x = b$ and $n = 2r$, (1.3) becomes

$$f(b) = \frac{1}{b-a}\int_a^b f(t)dt + [f(b)-f(a)]B_1 + \sum_{k=2}^{2r-1}\frac{(b-a)^{k-1}B_k}{k!}\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] - \frac{(b-a)^{2r-1}}{(2r)!}\int_a^b f^{(2r)}(t)\left[B_{2r}^*\left(\frac{b-t}{b-a}\right)-B_{2r}\right]dt.$$

Since $B_1 = 1/2$ and $B_{2j+1} = 0$ for $j \geqslant 1$, this may be rearranged, after a change of variable to $s = (t-a)/(b-a)$ in the final term, as

$$\frac{f(a)+f(b)}{2} = \frac{1}{b-a}\int_a^b f(t)dt + \sum_{j=1}^{r-1}\frac{(b-a)^{2j-1}B_{2j}}{(2j)!}\left[f^{(2j-1)}(b)-f^{(2j-1)}(a)\right] - \frac{(b-a)^{2r}}{(2r)!}\int_0^1 f^{(2r)}(a+s(b-a))\left[B_{2r}(1-s)-B_{2r}\right]ds.$$

Here as subsequently an empty sum (in this case for $r = 1$) is interpreted as zero.

Further, $B_{2r}(1-s) = B_{2r}(s)$, so we may write this as the Euler trapezoidal formula

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a)+f(b)] - \sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{(2k)!}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right] + (b-a)^{2r+1}\int_0^1 P_{2r}(s)f^{(2r)}(a+s(b-a))ds,$$

where $P_k(s) := [B_k(s)-B_k]/k!$ ($k \geqslant 1$). See [1, p. 274].

This has many applications and was the starting point of the analysis in [2], where it was used to prove some integral inequalities germane to numerical integration. Analysis based on the trapezoidal formula devolves eventually on finding a method for handling the uncompromising–looking final term.

A natural quantity in the analysis in [2] is

$$I_r := (-1)^r \left\{ \int_a^b f(x)dx - \frac{b-a}{2}[f(a)+f(b)] + \sum_{k=1}^{r-1} \frac{(b-a)^{2k} B_{2k}}{(2k)!} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \right\}.$$

The results of [2] include in particular the following.

THEOREM 1. *Suppose $f : [a,b] \to R$ is $(2r+2)$–convex. Then*

$$(b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} f^{(2r)}\left(\frac{a+b}{2}\right) \leqslant I_r \leqslant (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \frac{f^{(2r)}(b)+f^{(2r)}(a)}{2}. \quad (1.4)$$

If f is $(2r+2)$–concave, the inequality is reversed.

THEOREM 2. *Suppose $f : [a,b] \to R$ is $(2r)$–times differentiable. If $\left|f^{(2r)}\right|^q$ is convex for some $q \geqslant 1$, then*

$$|I_r| \leqslant (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left[\frac{\left|f^{(2r)}(a)\right|^q + \left|f^{(2r)}(b)\right|^q}{2} \right]^{1/q}.$$

If $\left|f^{(2r)}\right|$ is concave, then

$$|I_r| \leqslant (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left| f^{(2r)}\left(\frac{a+b}{2}\right) \right|.$$

The displayed inequalities are manifestly higher–order cousins of (1.2)

In the next section we take a different path from (1.3), one leading to the Euler midpoint formula instead of the Euler trapezoidal formula. In place of the function P_{2r} of the trapezoidal formula, it turns out that we shall have recourse to

$$p_{2r}(t) = B^*_{2r}\left(\frac{a+b-2t}{2(b-a)}\right) - B_{2r}\left(\frac{1}{2}\right).$$

We note that this does not change sign on the interval $[a,b]$ and that it is symmetric about $t = (a+b)/2$. Further

$$(-1)^{r-1} p_{2r}(t) \geqslant 0 \quad \text{for } t \in [a,b].$$

In Section 3 we explore briefly a third path, one that is associated with the Euler–Simpson formula.

The reader will have noted an asymmetry between the conditions applying in the convex and concave cases of Theorem 2. The reason is that if $|f^{(2r)}|^q$ is concave for some $q \geqslant 1$, then $|f^{(2r)}|$ must also be concave (see [2]). The omission of the index q in the concave case thus allows a weaker assumption to be made. This motif occurs also in the present paper.

2. The Euler midpoint formula

Put $x = (a+b)/2$ and $n = 2r$ in (1.3). Since $B_{2j+1}(1/2) = 0$ for $j \geqslant 0$, we obtain the Euler midpoint formula

$$f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)dt + \sum_{k=1}^{r-1}\frac{(b-a)^{k-1}}{(2k)!}B_{2k}\left(\frac{1}{2}\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]$$
$$-\frac{(b-a)^{2r-1}}{(2r)!}\int_a^b f^{(2r)}(t)p_{2r}(t)dt. \tag{2.1}$$

Note that $B_{2k}(1/2) = (1-2^{1-2k})B_{2k}$.

For the sequel we shall utilise

$$I_r^*(a,b) :=(-1)^{r-1}\left\{\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\sum_{k=1}^{r-1}\frac{B_{2k}(b-a)^{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]\right\},$$

which serves the role assumed by I_r in [2]. Where a fixed interval is understood, we drop the argument from I_r^*.

THEOREM 3. *Suppose* $f : [a,b] \to R$ *is* $(2r+2)$*–convex. Then*

$$(b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)f^{(2r)}\left(\frac{a+b}{2}\right)$$
$$\leqslant I_r^* \leqslant (b-a)^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\frac{f^{(2r)}(b)+f^{(2r)}(a)}{2}. \tag{2.2}$$

If f *is* $(2r+2)$*–concave, the inequality is reversed.*

Proof. We have from (2.1) that

$$I_r^* = (-1)^{r-1}\frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}(t)p_{2r}(t)dt$$
$$= \frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}(t)\left|p_{2r}(t)\right| dt$$
$$= \frac{(b-a)^{2r}}{(2r)!}\int_a^b f^{(2r)}\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\left|p_{2r}(t)\right| dt. \tag{2.3}$$

Using the discrete Jensen inequality for the convex function $f^{(2r)}$, we have

$$\begin{aligned}
&\int_a^b f^{(2r)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)|p_{2r}(t)|\,dt\\
&\qquad \leqslant f^{(2r)}(a)\int_a^b \frac{b-t}{b-a}|p_{2r}(t)|\,dt+f^{(2r)}(b)\int_a^b \frac{t-a}{b-a}|p_{2r}(t)|\,dt.\\
&\qquad = f^{(2r)}(a)K_1+f^{(2r)}(b)K_2,\quad \textit{say}.
\end{aligned} \tag{2.4}$$

Since $p_{2r}(t)$ is symmetric about $t=(a+b)/2$ and has constant sign on $[a,b]$, we have $K_1=K_2$. On the other hand

$$\begin{aligned}
K_1+K_2 &= \int_a^b |p_{2r}(t)|\,dt\\
&= (-1)^{r-1}\int_a^b p_{2r}(t)dt\\
&= (-1)^{r-1}\left(1-2^{1-2r}\right)(b-a)B_{2r}\\
&= \left(1-2^{1-2r}\right)|B_{2r}|\,(b-a),
\end{aligned}$$

so that

$$K_1=K_2=\frac{1}{2}\left(1-2^{1-2r}\right)|B_{2r}|\,(b-a). \tag{2.5}$$

The second inequality in (2.2) follows at once from (2.3)–(2.5).

By Jensen's integral inequality

$$\begin{aligned}
&\int_a^b f^{(2k)}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)|p_{2r}(t)|\,dt\\
&\qquad \geqslant \left(\int_a^b |p_{2r}(t)|\,dt\right) f^{(2k)}\left(\frac{\int_a^b\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)|p_{2r}(t)|\,dt}{\int_a^b |p_{2r}(t)|\,dt}\right)\\
&\qquad = \left(1-\frac{1}{2^{2r-1}}\right)|B_{2r}|\,(b-a)f^{(2k)}\left(\frac{a+b}{2}\right).
\end{aligned} \tag{2.6}$$

The first inequality in (2.2) now derives from (2.3), (2.5) and (2.6).

□

The proof of the following theorem is similar to that of the theorem above and to that of [2, Theorem 2].

THEOREM 4. *Suppose* $f : [a,b] \to R$ *is* $(2r)$*-times differentiable.*
(a) If $|f^{(2r)}|^q$ *is convex for some* $q \geqslant 1$*, then*

$$|I_r^*| \leqslant (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left(1 - \frac{1}{2^{2r-1}}\right) \left[\frac{\left|f^{(2r)}(a)\right|^q + \left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

(b) If $\left|f^{(2r)}\right|$ *is concave, then*

$$|I_r^*| \leqslant (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left(1 - \frac{1}{2^{2r-1}}\right) \left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|.$$

To obtain appropriate results for numerical integration from the Euler midpoint formula, we apply the results above to each interval of the subdivision

$$[a, a+b], [a+h, a+2h], ..., [a+(n-1)h, a+nh].$$

Let us denote

$$T(f;h) := h\left[\frac{1}{2}f(a) + \sum_{k=1}^{n-1} f(a+kh) + \frac{1}{2}f(a+nh)\right],$$

$$M(f;h) := h\sum_{k=1}^{n} f\left(a+kh-\frac{h}{2}\right)$$

and

$$H_r := (-1)^{r-1}\left\{\int_a^{a+nh} f(x)dx - M(f;h) - \sum_{k=1}^{r-1} \frac{B_{2k}h^{2k}}{(2k)!}\left(1-\frac{1}{2^{2k-1}}\right)\left[f^{(2k-1)}(a+nh) - f^{(2k-1)}(a)\right]\right\}.$$

THEOREM 5. *(a) If* $f : [a, a+nh] \to R$ *is* $(2r+2)$*–convex, then*

$$h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right) M\left(f^{(2r)};h\right) \leqslant H_r \leqslant h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right) T\left(f^{(2r)};h\right).$$

(b) If f *is* $(2r+2)$*–concave, the inequalities are reversed.*

Proof. The result is immediate from Theorem 3, since

$$H_r = \sum_{m=1}^{n} I_r^*(a+(m-1)h, a+mh).$$

□

THEOREM 6. *Suppose* $f : [a, a+nh] \to R$ *is* $(2r)$*–times differentiable.*

(a) If $\left|f^{(2r)}\right|^q$ *is convex for some* $q \geqslant 1$*, then*

$$|H_r| \leqslant nh^{2r+1}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)\max\left\{\left|f^{(2r)}(a)\right|, \left|f^{(2r)}(a+nh)\right|\right\}.$$

(b) If $\left|f^{(2r)}\right|$ *is concave, then*

$$|H_r| \leqslant h^{2r}\frac{|B_{2r}|}{(2r)!}\left(1-\frac{1}{2^{2r-1}}\right)M\left(\left|f^{(2r)}\right|;h\right).$$

Proof.

$$\begin{aligned}
|H_r| &= \left|\sum_{m=1}^{n} I_r^*(a+(m-1)h, a+mh)\right| \\
&\leqslant \sum_{m=1}^{n} \left|I_r^*(a+(m-1)h, a+mh)\right| \\
&\leqslant \sum_{m=1}^{n} h^{2r+1}\frac{|B_{2r}|}{(2r)!}(1-2^{1-2r})\left[\frac{\left|f^{(2r)}(a+mh)\right|^q+\left|f^{(2r)}(a+(m-1)h)\right|^q}{2}\right]^{1/q}
\end{aligned}$$

by Theorem 4 applied to each interval $[a+(m-1)h, a+mh]$. Hence

$$|H_r| \leqslant h^{2r+1}\frac{|B_{2r}|}{(2r)!}(1-2^{1-2r})\sum_{m=1}^{n}\max\left\{\left|f^{(2r)}(a+mh)\right|, \left|f^{(2r)}(a+(m-1)h)\right|\right\}.$$

The result of (a) now follows from the convexity of $\left|f^{(2r)}\right|^q$.

The proof (b) is similar.

□

3. The Euler–Simpson formula

If f is defined on an arbitrary finite segment $[a,b]$ and has $2r$ continuous derivatives there, then the Euler–Simpson formula (see, for example, [4, p. 221]) states that

$$\begin{aligned}
\int_a^b f(x)dx =& \frac{b-a}{6}\left[f(b)+4f\left(\frac{a+b}{2}\right)+f(a)\right] \\
&+\sum_{k=1}^{r-1}\frac{(b-a)^{2k}B_{2k}}{3(2k)!}\left(1-2^{2-2k}\right)\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right] \\
&+\frac{(b-a)^{2r+1}}{3(2r)!}\int_0^1 f^{(2r)}(a+u(b-a))F(u)du,
\end{aligned}$$

where

$$F(u)=y_{2r}(u)+2\left[y_{2r}^{*}\left(\frac{1}{2}-u\right)-y_{2r}\left(\frac{1}{2}\right)\right],$$

$y_{2r}(u)=B_{2r}(u)-B_{2r}$ and $y_{2r}^{*}(u)=B_{2r}^{*}(u)-B_{2r}$.

It was proved in [4, p. 224] that $F(1-u)=F(u)$ and that $(-1)^{r-1}F(u)\geqslant 0$. Also,

$$\begin{aligned}\int_0^1|F(u)|du&=(-1)^{r-1}\int_0^1 F(u)du\\&=(-1)^{r-1}\left\{\int_0^1 y_{2r}(u)du+2\int_0^1 y_{2r}^{*}\left(\frac{1}{2}-u\right)du-2y_{2r}\left(\frac{1}{2}\right)\right\}\\&=(-1)^{r-1}\left\{3\int_0^1 y_{2r}(u)du-2y_{2r}\left(\frac{1}{2}\right)\right\}\\&=(-1)^{r-1}\left\{-3B_{2r}+4\left(1-2^{-2r}\right)B_{2r}\right\}\\&=(-1)^{r-1}\left(1-2^{2-2r}\right)B_{2r}\\&=\left(1-2^{2-2r}\right)|B_{2r}|\end{aligned}$$

and

$$\int_0^1 u|F(u)|du=\int_0^1(1-u)\,|F(u)|\,du=\frac{1}{2}\left(1-2^{2-2r}\right)|B_{2r}|\,.$$

We can parallel the development of the previous section with the following two theorems. Define

$$\begin{aligned}L_r:=(-1)^{r-1}\Bigg\{&\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\&-\sum_{k=2}^{r-1}\frac{(b-a)^{2k}}{3(2k)!}\left(1-2^{2-2k}\right)B_{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]\Bigg\}.\end{aligned}$$

THEOREM 7. *If $f:[a,b]\to R$ is $(2r+2)$–convex, then*

$$\begin{aligned}(b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}f^{(2r)}\left(\frac{a+b}{2}\right)&\leqslant L_r\\&\leqslant(b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}\frac{f^{(2r)}(a)+f^{(2r)}(b)}{2}.\end{aligned}$$

If f is $(2r+2)$–concave, the inequalities are reversed.

THEOREM 8. *Suppose* $f:[a,b]\to R$ *is* $(2r)$*–times differentiable.*
(a) If $\left|f^{(2r)}\right|^q$ *is convex for some* $q\geqslant 1$*, then*

$$|L_r|\leqslant (b-a)^{2r+1}\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}\left[\frac{\left|f^{(2r)}(a)\right|^q+\left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

(b) If $\left|f^{(2r)}\right|$ *is concave, then*

$$|L_r|\leqslant (b-a)^{2r+1}\frac{1-2^{2-2r}}{3(2r)!}\left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|.$$

To obtain appropriate results for integration *via* the Simpson formula we apply the above results to each interval of the subdivision

$$[a,a+2h],[a+2h,a+4h],...,[a+2(n-1)h,a+2nh].$$

First we introduce

$$S(f;h):=\frac{h}{3}\left[f(a)+f(a+2nh)+2\sum_{i=1}^{n-1}f(a+2ih)+4\sum_{i=1}^{n-1}f(a+(2i-1)h)\right],$$

$$X_r:=\ (-1)^{r-1}\left\{\int_a^{a+2nh}f(x)dx-S(f;h)-\sum_{k=2}^{r-1}\frac{(2h)^{2k}}{3(2k)!}\left(1-2^{2-2k}\right)\right.$$
$$\left.\times B_{2k}\left[f^{(2k-1)}(a+2nh)-f^{(2k-1)}(a)\right]\right\}.$$

The following theorems apply.

THEOREM 9. *If* $f:[a,a+2nh]\to R$ *is* $(2r+2)$*–convex, then*

$$(2h)^{2r}|B_{2r}|\frac{\left(1-2^{2-2r}\right)|B_{2r}|}{3(2r)!}M\left(f^{(2r)};2h\right)\leqslant X_r\leqslant (2h)^{2r}\frac{\left(1-2^{2-2r}\right)}{3(2r)!}T(f^{(2r)};2h).$$

If f *is* $(2r+2)$*–concave, the reverse inequalities hold.*

THEOREM 10. *Suppose* $f:[a,a+2nh]\to R$ *is* $(2r)$*–times differentiable.*
(a) If $\left|f^{(2r)}\right|^q$ *is convex for some* $q\geqslant 1$*, then*

$$|X_r|\leqslant (2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}$$
$$\times\sum_{m=1}^{n}\left[\frac{\left|f^{(2r)}(a+2mh)\right|^q+\left|f^{(2r)}(a+2(m-1)h)\right|^q}{2}\right]^{1/q}$$
$$\leqslant n(2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}\max\left\{\left|f^{(2r)}(a)\right|,\left|f^{(2r)}(a+2nh)\right|\right\}.$$

(b) If $\left|f^{(2r)}\right|$ *is concave, then*

$$|X_r| \leqslant (2h)^{2r+1}|B_{2r}|\frac{\left(1-2^{2-2r}\right)}{3(2r)!}M\left(\left|f^{(2r)}\right|;2h\right).$$

The resultant formulæ in Theorems 7–10 when $r=2$ and the sums in L_r and X_r are empty are of special interest and we isolate them as corollaries.

COROLLARY 1. *If* $f:[a,b]\to R$ *is* 6*–convex, then*

$$\frac{(b-a)^5}{2880}f^{(4)}\left(\frac{a+b}{2}\right) \leqslant \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(x)dx \leqslant \frac{(b-a)^5}{2880}\frac{f^{(4)}(a)+f^{(4)}(b)}{2}.$$

If f *is* 6*–concave, the reversed inequalities apply.*

COROLLARY 2. *Suppose* $f:[a,b]\to R$ *is* 4*–times differentiable.*
(a) If $\left|f^{(4)}\right|^q$ *is convex for some* $q\geqslant 1$*, then*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \leqslant \frac{(b-a)^5}{2880}\left[\frac{\left|f^{(4)}(a)\right|^q+\left|f^{(4)}(b)\right|^q}{2}\right]^{1/q}.$$

(b) If $\left|f^{(4)}\right|$ *is concave, then*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \leqslant \frac{(b-a)^5}{2880}\left|f^{(4)}\left(\frac{a+b}{2}\right)\right|.$$

COROLLARY 3. *If* $f:[a,a+2nh]\to R$ *is* 6*–convex, then*

$$\frac{(2h)^4}{2880}M\left(f^{(4)};2h\right) \leqslant S(f,h)-\int_a^{a+2nh} f(x)dx \leqslant \frac{(2h)^4}{2880}T\left(f^{(4)};2h\right).$$

If f *is* 6*–concave, the inequalities are reversed.*

COROLLARY 4. *Suppose* $f:[a,a+2nh]\to R$ *is* 4*–times differentiable.*

(a) If $\left|f^{(4)}\right|^q$ *is convex for some* $q \geqslant 1$*, then*

$$\left|\int_a^{a+2nh} f(x)dx - S(f;h)\right|$$
$$\leqslant \frac{(2h)^5}{2880}\sum_{m=1}^{n}\left[\frac{\left|f^{(4)}(a+2nh)\right|^q + \left|f^{(4)}(a+2(m-1)h)\right|^q}{2}\right]^{1/q}$$
$$\leqslant \frac{n(2h)^5}{2880}\max\left\{\left|f^{(4)}(a)\right|, \left|f^{(4)}(a+2nh)\right|\right\}.$$

(b) If If $\left|f^{(4)}\right|$ *is concave, then*

$$\left|\int_a^{a+2nh} f(x)dx - S(f;h)\right| \leqslant \frac{n(2h)^5}{2880} M\left(\left|f^{(2r)}\right|;2h\right).$$

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