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FURTHER CHARACTERIZATIONS OF CHAOTIC ORDER VIA SPECHT'S RATIO

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Abstract. As a characterization of chaotic order, we showed *"If* $MI \geqslant B \geqslant mI > 0$, *then* $\log A \geqslant \log B$ *is equivalent to*

$$M_h(p)A^p \geqslant B^p$$

for all $p > 0$, *where* $h = \frac{M}{m} > 1$ *and* $M_h(p) = \dfrac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}$" in [11].

In this paper, we shall show the following characterization of chaotic order as a parallel result to the result mentioned above:

"If $MI \geqslant B \geqslant mI > 0$, *then* $\log A \geqslant \log B$ *is equivalent to*

$$A^p + L(m^p, M^p) \log M_h(p) I \geqslant B^p$$

for all $p > 0$, *where* $L(m,M) = \frac{M-m}{\log M - \log m}$." And we shall discuss the relations among this result and some related results.

1. Introduction

We shall consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible.

For positive operators A and B, $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for all $p \in [0,1]$ by well-known Löwner-Heinz theorem. However, it is also well known that $A \geqslant B \geqslant 0$ does not always ensure $A^p \geqslant B^p$ for any $p > 1$. Related to this result, the following result was shown in [7].

THEOREM A. ([7]) *Let* $A \geqslant B \geqslant 0$ *satisfying* $MI \geqslant B \geqslant mI > 0$ *with* $M > m > 0$. *Then* $K_+(m,M,p)A^p \geqslant B^p$ *holds for all* $p > 1$, *where*

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}. \tag{1.1}$$

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For positive invertible operators A and B, an order defined by $\log A \geqslant \log B$ is called "chaotic order." Ando showed that $\log A \geqslant \log B$ is equivalent to $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geqslant B^p$ holds for all $p \geqslant 0$ in [1]. The following Theorem B is an extension of this result.

THEOREM B. ([5, 6, 9]) *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geqslant B^p$ *for all* $p \geqslant 0$,

(*iii*) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r$ *for all* $p \geqslant 0$ *and* $r \geqslant 0$.

In [11], we showed parallel results to Theorem A on chaotic order by using Theorem A and Theorem B as follows:

THEOREM C. ([11]) *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $\dfrac{(m^p+M^p)^2}{4m^pM^p}A^p \geqslant B^p$ *holds for all* $p > 0$.

THEOREM D. ([11]) *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $M_h(p)A^p \geqslant B^p$ *holds for all* $p > 0$, *where* $h = \frac{M}{m} > 1$ *and*

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}}. \tag{1.2}$$

We remark that Theorem D is a more precise estimation than Theorem C by considering the following theorem in [11].

THEOREM E. ([11]) *Let $K_+(m,M,p)$ be defined in (1.1). Then*

$$F(p,r,m,M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an increasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$, and $\lim\limits_{r \to +0} K_+\left(m^r, M^r, \frac{p+r}{r}\right) = M_h(p)$ holds, where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in (1.2). And the following inequalities hold:

$$h^p = \left(\frac{M}{m}\right)^p \geqslant K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geqslant M_h(p) \geqslant 1 \tag{1.3}$$

for any $p > 0$, $r > 0$ and $M > m > 0$.

In fact, Put $r = p > 0$ in (1.3) of Theorem E. Then $K_+(m^p, M^p, 2) = \frac{(m^p+M^p)^2}{4m^pM^p} \geqslant M_h(p)$ holds for $p > 0$ and $M > m > 0$. $M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ is called *"Specht's ratio."* Related to Specht's ratio, Specht [8] showed the following: Let $M \geqslant x_1, x_2, \cdots, x_n \geqslant m > 0$, $h = \frac{M}{m} > 1$, $A = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $G = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$. Then $M_h(1)G \geqslant A \geqslant G$ holds. Specht's ratio has been studied in [3, 4]. A simplified proof of Theorem D was shown in [2].

And in our previous paper [10], we showed the following Theorem F as a parallel result to Theorem A.

THEOREM F. ([10]) *Let $A \geqslant B \geqslant 0$ satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then*

$$A^p + \frac{mM^p - Mm^p}{M-m}\left\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\right\} I \geqslant B^p$$

holds for all $p > 1$, where $K_+(m,M,p)$ is defined in (1.1).

In this paper, as parallel results to Theorem C and Theorem D, we shall show other characterizations of chaotic order by using Theorem B and Theorem F which is a parallel result to Theorem A.

2. Other characterizations of chaotic order

THEOREM 1. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $A^p + \frac{(M^p - m^p)^2}{4m^p} I \geqslant B^p$ *holds for all* $p > 0$.

THEOREM 2. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $A^p + L(m^p, M^p)\log M_h(p) I \geqslant B^p$ *for all* $p > 0$,

where $h = \frac{M}{m} > 1$, $M_h(p)$ is defined in (1.2) and

$$L(m,M) = \frac{M-m}{\log M - \log m}. \tag{2.1}$$

We remark that $L(m,M)$ is called "*logarithmic mean.*" Related to $L(m,M)$, it is well known that $\frac{M+m}{2} \geqslant L(m,M) \geqslant \sqrt{mM}$ hold for $m > 0$ and $M > 0$.

Proof of Theorem 1. (*i*) $\Rightarrow$ (*ii*). $\log A \geqslant \log B$ ensures $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geqslant B^p$ for all $p \geqslant 0$ by Theorem B, and $M^pI \geqslant B^p \geqslant m^pI > 0$ hold for all $p > 0$ by the hypothesis

$MI \geqslant B \geqslant mI > 0$. Put $A_1 = (B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}}$, $B_1 = B^p$, $M_1 = M^p$ and $m_1 = m^p$, then $A_1 \geqslant B_1$ and $M_1 I \geqslant B_1 \geqslant m_1 I > 0$ hold. Applying Theorem F, we have

$$A_1^{p_1} + \frac{m_1 M_1^{p_1} - M_1 m_1^{p_1}}{M_1 - m_1}\left\{K_+(m_1, M_1, p_1)^{\frac{1}{p_1-1}} - 1\right\} I \geqslant B_1^{p_1} \tag{2.2}$$

holds for $p_1 > 1$. Put $p_1 = 2 > 1$ in (2.2), we have

$$B^{\frac{p}{2}}A^pB^{\frac{p}{2}} + \frac{m^pM^{2p} - M^pm^{2p}}{M^p - m^p}\{K_+(m^p, M^p, 2) - 1\} I \geqslant B^{2p}. \tag{2.3}$$

And (2.3) is equivalent to (2.4) as follows:

$$A^p + \frac{m^pM^{2p} - M^pm^{2p}}{M^p - m^p}\{K_+(m^p, M^p, 2) - 1\} B^{-p} \geqslant B^p. \tag{2.4}$$

By using $\frac{1}{m^p}I \geqslant B^{-p} \geqslant \frac{1}{M^p}I > 0$, then we have

$$\begin{aligned}
&A^p + \frac{(M^p - m^p)^2}{4m^p}I \\
&= A^p + \frac{m^pM^{2p} - M^pm^{2p}}{M^p - m^p}\left\{\frac{(m^p + M^p)^2}{4m^pM^p} - 1\right\}\frac{1}{m^p}I \\
&= A^p + \frac{m^pM^{2p} - M^pm^{2p}}{M^p - m^p}\{K_+(m^p, M^p, 2) - 1\}\frac{1}{m^p}I \\
&\geqslant A^p + \frac{m^pM^{2p} - M^pm^{2p}}{M^p - m^p}\{K_+(m^p, M^p, 2) - 1\} B^{-p} \geqslant B^p
\end{aligned}$$

hold since the last inequality holds by (2.4).

$(ii) \Rightarrow (i)$. (ii) ensures the following

$$\frac{A^p - 1}{p} - \frac{B^p - 1}{p} \geqslant \frac{-(M^p - m^p)^2}{4pm^p}I = -\left(\frac{M^p - m^p}{p}\right)^2 \frac{p}{4m^p}I.$$

Let $p \to +0$. Then we have $\log A - \log B \geqslant 0$ because

$$\lim_{p \to +0} \frac{M^p - m^p}{p} = \lim_{p \to +0}\left(\frac{M^p - 1}{p} - \frac{m^p - 1}{p}\right) = \log M - \log m = \log \frac{M}{m},$$

$\frac{p}{4m^p}I \to O$ as $p \to +0$ and we recall that $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for any operator $X > 0$.

□

To prove Theorem 2, we use the following results.

THEOREM G. ([11]) *Let $M_h(p)$ be defined in (1.2). Then for $h > 1$,*

$$\lim_{p \to +0}\{M_h(p)\}^{\frac{1}{p}} = 1.$$

LEMMA 3. *Let $K_+(m,M,p)$ be defined in (1.1), and $f(r)$ be a positive continuous function on $r>0$ satisfying $\lim_{r\to+0} f(r)=0$. Then for $p>0$ and $M>m>0$,*

$$\lim_{r\to+0}\frac{1}{f(r)}\left\{K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{f(r)}-1\right\}=\log M_h(p),$$

where $h=\frac{M}{m}>1$ and $M_h(p)$ is defined in (1.2).

Proof. By considering (1.3) of Theorem E and $\frac{a^r-1}{r}\geqslant \log a$ holds for $r>0$ and $a>0$, we obtain

$$\begin{aligned}\frac{1}{f(r)}\left\{M_h(p)^{-f(r)}-1\right\} &\geqslant \frac{1}{f(r)}\left\{K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{-f(r)}-1\right\}\\ &\geqslant \log K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{-1}.\end{aligned}$$

And

$$\lim_{r\to+0}\frac{1}{f(r)}\left\{M_h(p)^{-f(r)}-1\right\}=-\log M_h(p)$$

holds since $\frac{a^r-1}{r}\to\log a$ as $r\to+0$ for $a>0$, and

$$\lim_{r\to+0}\log K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{-1}=-\log M_h(p)$$

holds by Theorem E. Then we have

$$\lim_{r\to+0}\frac{1}{f(r)}\left\{K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{-f(r)}-1\right\}=-\log M_h(p). \tag{2.5}$$

On the other hand,

$$\left(\frac{M}{m}\right)^{pf(r)}\geqslant K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{f(r)}\geqslant 1$$

hold by (1.3) of Theorem E. Then we have

$$\lim_{r\to+0}K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{f(r)}=1. \tag{2.6}$$

Hence

$$\begin{aligned}&\lim_{r\to+0}\frac{1}{f(r)}\left\{K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{f(r)}-1\right\}\\ &=\lim_{r\to+0}K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{f(r)}\\ &\qquad\times\left[\frac{-1}{f(r)}\left\{K_+\left(m^r,M^r,\frac{p+r}{r}\right)^{-f(r)}-1\right\}\right]\\ &=\log M_h(p)\end{aligned}$$

holds by (2.5) and (2.6).

□

Proof of Theorem 2. $(i) \Rightarrow (ii)$. $\log A \geqslant \log B$ ensures $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geqslant B^r$ for all $p \geqslant 0$ and $r \geqslant 0$ by Theorem B, and $M^rI \geqslant B^r \geqslant m^rI > 0$ hold for all $r > 0$ by the hypothesis $MI \geqslant B \geqslant mI > 0$. Put $A_1 = (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$, $B_1 = B^r$, $M_1 = M^r$ and $m_1 = m^r$, then $A_1 \geqslant B_1$ and $M_1I \geqslant B_1 \geqslant m_1I > 0$ hold. Applying Theorem F, we have

$$A_1^{p_1} + \frac{m_1M_1^{p_1} - M_1m_1^{p_1}}{M_1 - m_1}\left\{K_+(m_1,M_1,p_1)^{\frac{1}{p_1-1}} - 1\right\}I \geqslant B_1^{p_1} \tag{2.7}$$

holds for $p_1 > 1$. Put $p_1 = \frac{p+r}{r} > 1$ in (2.7), we have

$$B^{\frac{r}{2}}A^pB^{\frac{r}{2}} + \frac{m^rM^{p+r} - M^rm^{p+r}}{M^r - m^r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\}I \geqslant B^{p+r}. \tag{2.8}$$

And (2.8) is equivalent to (2.9) as follows:

$$A^p + \frac{m^rM^{p+r} - M^rm^{p+r}}{M^r - m^r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\}B^{-r} \geqslant B^p. \tag{2.9}$$

By using $\frac{1}{m^r}I \geqslant B^{-r} \geqslant \frac{1}{M^r}I > 0$, then we have

$$\begin{aligned} A^p &+ \frac{m^rM^{p+r} - M^rm^{p+r}}{M^r - m^r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\}\frac{1}{m^r}I \\ &\geqslant A^p + \frac{m^rM^{p+r} - M^rm^{p+r}}{M^r - m^r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\}B^{-r} \geqslant B^p \end{aligned}$$

hold since the last inequality holds by (2.9). Moreover

$$\begin{aligned} \lim_{r\to+0} &\frac{m^rM^{p+r} - M^rm^{p+r}}{M^r - m^r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\}\frac{1}{m^r} \\ &= \lim_{r\to+0} \frac{M^r(M^p - m^p)}{\left(\frac{M^r-1}{r} - \frac{m^r-1}{r}\right)p} \cdot \frac{p}{r}\left\{K_+\left(m^r, M^r, \frac{p+r}{r}\right)^{\frac{r}{p}} - 1\right\} \\ &= \frac{M^p - m^p}{p(\log M - \log m)}\log M_h(p) \qquad (2.10) \\ &= L(m^p, M^p)\log M_h(p) \qquad \text{by (2.1)} \end{aligned}$$

holds since $\frac{a^r-1}{r} \to \log a$ as $r \to +0$ for $a > 0$, and by Lemma 3 in case $f(r) = \frac{r}{p}$. Then we obtain (ii).

$(ii) \Rightarrow (i)$. (ii) ensures the following:

$$\begin{aligned}\frac{A^p-1}{p}-\frac{B^p-1}{p} &\geqslant \frac{-L(m^p,M^p)}{p}\log M_h(p)I\\ &= \frac{-(M^p-m^p)}{\log M^p-\log m^p}\log\{M_h(p)\}^{\frac{1}{p}}I\\ &= \frac{-(\frac{M^p-1}{p}-\frac{m^p-1}{p})}{\log M-\log m}\log\{M_h(p)\}^{\frac{1}{p}}I.\end{aligned}$$

Let $p\to+0$, then we have $\log A-\log B\geqslant 0$ by Theorem G and $\frac{X^p-I}{p}\to\log X$ as $p\to+0$ for any operator $X>0$.

□

3. Concluding remark

First of all, we show the following result.

PROPOSITION 4. *Let $M_h(p)$ and $L(m,M)$ be defined in (1.2) and* (2.1)*, respectively. Then*

$$\begin{aligned}\frac{(M^p-m^p)^2}{4m^p} &\geqslant \left(\frac{M^p-m^p}{M^p+m^p}\right)^2 M^p\\ &\geqslant \frac{M_h(p)-1}{M_h(p)}M^p \geqslant L(m^p,M^p)\log M_h(p)\end{aligned}$$

hold for $p>0$ and $M>m>0$, where $h=\frac{M}{m}>1$.

Proof of Proposition 4.

(i) *Proof of* $\frac{(M^p-m^p)^2}{4m^p}\geqslant\left(\frac{M^p-m^p}{M^p+m^p}\right)^2M^p$.

$$\begin{aligned}&\frac{(M^p-m^p)^2}{4m^p}-\left(\frac{M^p-m^p}{M^p+m^p}\right)^2M^p\\ =\ & M^p(M^p-m^p)^2\left(\frac{1}{4m^pM^p}-\frac{1}{(M^p+m^p)^2}\right)\\ =\ & M^p(M^p-m^p)^2\frac{(M^p-m^p)^2}{4m^pM^p(M^p+m^p)^2}\geqslant 0.\end{aligned}$$

(ii) *Proof of* $\left(\frac{M^p-m^p}{M^p+m^p}\right)^2M^p\geqslant\frac{M_h(p)-1}{M_h(p)}M^p$. Let $f(t)=\frac{t-1}{t}=1-\frac{1}{t}$. Then $f(t)$ is increasing for $t\geqslant 1$. Hence we obtain

$$\frac{K_+(m^p,M^p,2)-1}{K_+(m^p,M^p,2)}=f(K_+(m^p,M^p,2))\geqslant f(M_h(p))=\frac{M_h(p)-1}{M_h(p)} \tag{3.1}$$

holds by (1.3) of Theorem E in case $r = p$. By considering (3.1) and

$$\left(\frac{M^p - m^p}{M^p + m^p}\right)^2 = \frac{\frac{(m^p+M^p)^2}{4m^pM^p} - 1}{\frac{(m^p+M^p)^2}{4m^pM^p}} = \frac{K_+(m^p, M^p, 2) - 1}{K_+(m^p, M^p, 2)},$$

we have

$$\left(\frac{M^p - m^p}{M^p + m^p}\right)^2 M^p \geqslant \frac{M_h(p) - 1}{M_h(p)} M^p.$$

(iii) Proof of $\frac{M_h(p)-1}{M_h(p)}M^p \geqslant L(m^p, M^p)\log M_h(p)$. Let $g(t) = \frac{t-1}{t\log t}$. Then $g(t)$ is decreasing for $t > 0$ since

$$g'(t) = \frac{t\log t - (t-1)(\log t + 1)}{(t\log t)^2} = \frac{\log t - (t-1)}{(t\log t)^2} \leqslant 0$$

holds for $t > 0$. Hence we have

$$\frac{M_h(p) - 1}{M_h(p)\log M_h(p)} = g(M_h(p)) \geqslant g(h^p) = \frac{h^p - 1}{h^p \log h^p} \tag{3.2}$$

holds by (1.3) of Theorem E. By considering (3.2), $h = \frac{M}{m} > 1$ and

$$\frac{h^p - 1}{h^p \log h^p} = \frac{M^p - m^p}{M^p(\log M^p - \log m^p)} = \frac{L(m^p, M^p)}{M^p},$$

we obtain

$$\frac{M_h(p) - 1}{M_h(p)} M^p \geqslant L(m^p, M^p)\log M_h(p).$$

Whence the proof of Proposition 4 is complete.

□

By using Theorem C and Theorem D, we can easily obtain the following other characterizations of chaotic order which are the same type as Theorem 1 and Theorem 2 type.

PROPOSITION 5. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $A^p + \left(\frac{M^p - m^p}{M^p + m^p}\right)^2 M^p I \geqslant B^p$ *holds for all* $p > 0$.

PROPOSITION 6. *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(*i*) $\log A \geqslant \log B$,

(*ii*) $A^p + \frac{M_h(p) - 1}{M_h(p)} M^p I \geqslant B^p$ *holds for all* $p > 0$,

where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in (1.2).

We remark that Theorem 2 is a more precise estimation than Proposition 5 and Proposition 6 by considering Proposition 4.

Proof of Proposition 5. $(i) \Longrightarrow (ii)$. By using Theorem C and $M^pI \geqslant B^p \geqslant m^pI > 0$, we obtain

$$\begin{aligned} A^p &\geqslant \frac{4m^pM^p}{(m^p+M^p)^2}B^p \\ &= B^p - \left(\frac{M^p-m^p}{M^p+m^p}\right)^2 B^p \\ &\geqslant B^p - \left(\frac{M^p-m^p}{M^p+m^p}\right)^2 M^pI \qquad \text{by } M^pI \geqslant B^p \geqslant m^pI > 0 \end{aligned}$$

for all $p > 0$.

$(ii) \Longrightarrow (i)$. By using Proposition 4, (ii) ensures (ii) of Theorem 1 as follows:

$$A^p + \frac{(M^p-m^p)^2}{4m^p}I \geqslant A^p + \left(\frac{M^p-m^p}{M^p+m^p}\right)^2 M^pI \geqslant B^p.$$

Hence we obtain $\log A \geqslant \log B$ by Theorem 1.

□

Proof of Proposition 6. $(i) \Longrightarrow (ii)$. By using Theorem D, we obtain

$$\begin{aligned} A^p &\geqslant \frac{1}{M_h(p)}B^p \\ &= B^p - \frac{M_h(p)-1}{M_h(p)}B^p \\ &\geqslant B^p - \frac{M_h(p)-1}{M_h(p)}M^pI \end{aligned}$$

for all $p > 0$, and the last inequality holds by $M^pI \geqslant B^p \geqslant m^pI$ and $M_h(p) \geqslant 1$ which is asserted in (1.3) of Theorem E.

$(ii) \Longrightarrow (i)$. By using Proposition 4, (ii) ensures (ii) of Theorem 1 as follows:

$$A^p + \frac{(M^p-m^p)^2}{4m^p}I \geqslant A^p + \frac{M_h(p)-1}{M_h(p)}M^pI \geqslant B^p.$$

Hence we obtain $\log A \geqslant \log B$ by Theorem 1.

□

We remark that as parallel results to Theorem C and Theorem D, we showed Theorem 1 and Theorem 2 respectively in the previous section. But the proof of Theorem 2 is not easy and requires some lemmas. We remark that Proposition 5 and Proposition 6 are easily obtained by only using Theorem C and Theorem D respectively. However it is interesting to point out the following two facts. Firstly, Theorem 2 is a more precise estimation than Proposition 5 and Proposition 6. Secondly, as a parallel result to Theorem 2, we showed Theorem 1 in the previous section. But Theorem 1 is not a more precise estimation than Proposition 5 and Proposition 6.

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REFERENCES

[1] T. ANDO, *On some operator inequalities*, Math. Ann., **279** (1987), 157–159.

[2] J. I. FUJII, T. FURUTA, T. YAMAZAKI AND M. YANAGIDA, *Simplified proof of characterization of chaotic order via Specht's ratio*, Scientiae Mathematicae, **2** (1999), 63–64.

[3] J. I. FUJII, S. IZUMINO AND Y. SEO, *Determinant for positive operators and Specht's theorem*, Scientiae Mathematicae, **1** (1998), 307–310.

[4] J. I. FUJII AND Y. SEO, *Determinant for positive operators*, Scientiae Mathematicae, **1** (1998), 153–156.

[5] M. FUJII, J. F. JIANG AND E. KAMEI, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.

[6] T. FURUTA, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.

[7] T. FURUTA, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl., **2** (1998), 137–148.

[8] W. SPECHT, *Zur Theorie der elementaren Mittel*, Math. Z., **74** (1960), 91–98.

[9] M. UCHIYAMA, *Some exponential operator inequalities*, Math. Inequal. Appl., **2** (1999), 469–471.

[10] T. YAMAZAKI, *An extension of Specht's theorem via Kantorovich inequality and related results*, to appear in Math. Inequal. Appl.

[11] T. YAMAZAKI AND M. YANAGIDA, *Characterizations of chaotic order associated with Kantorovich inequality*, Scientiae Mathematicae, **2** (1999), 37–50.

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