Mathematical
Inequalities
& Applications
Volume 3, Number 3 (2000), 319–326

L^p INEQUALITIES FOR POLAR DERIVATIVES OF POLYNOMIALS

N. K. GOVIL AND G. NYUYDINKONG

(*communicated by R. N. Mohapatra*)

Abstract. Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n and for $\alpha \in \mathscr{C}$, let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to α. It is well known that the polar derivative generalizes the ordinary derivative. In this paper, we obtain L^p inequalities for polar derivatives of polynomials satisfying $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$ and for polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$. Our results generalize several results in this direction.

1. Introduction and statement of results

If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree at most n, then according to a famous result known as Bernstein's inequality,

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{1}$$

Here, equality holds if and only if only if $p(z)$ has all its zeros at the origin.

In case $p(z)$ does not vanish in $|z| < 1$, it was conjectured by Erdös and proved by Lax [12] that (1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

The above inequality is sharp and equality holds for polynomials of the form $p(z) = \alpha + \beta z^n,\ |\alpha| = |\beta|$.

The L^p analogue of (1) was proved by Zygmund [14] and that of (2) by de Bruijn [4], who proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for $\delta \geqslant 1$,

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \leqslant nC_\delta \left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{3}$$

where $C_\delta = \left(\frac{2\pi}{\int_0^{2\pi} |1+e^{i\theta}|^\delta d\theta}\right)^{\frac{1}{\delta}} = \left(\frac{\sqrt{\pi}\,\Gamma(\frac{\delta}{2}+1)}{2^\delta\,\Gamma(\frac{\delta}{2}+\frac{1}{2})}\right)^{\frac{1}{\delta}}$. This inequality is also sharp and becomes equality for $p(z) = \alpha + \beta z^n,\ |\alpha| = |\beta|$.

Mathematics subject classification (1991): Primary 26D05, 30D15. Secondary 41A17.

Key words and phrases: Inequalities, polar derivatives, self-reciprocal polynomials, self-inversive polynomials.

The inequality (3) was extended for $\delta \geqslant 0$ by Rahman and Schmeisser [13]. Dewan and Govil [5] observed that the inequality (3) also holds for polynomials satisfying $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$ for all $z \in \mathscr{C}$. Such polynomials are said to be self-inversive.

Let $\alpha \in \mathscr{C}$. If $D_\alpha p(z)$ denotes the polar derivative of the polynomial $p(z)$ with respect to the point α, then

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{4}$$

As is well known, the polar derivative generalizes the ordinary derivative. In fact, if we divide both sides of (4) by α and let $\alpha \to \infty$ we get

$$\lim_{\alpha\to\infty} \left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z). \tag{5}$$

Furthermore, if $p(z)$ is a polynomial of degree at most n, then $D_\alpha p(z)$ is a polynomial of degree at most $(n-1)$.

A polynomial $p(z)$ of degree n is said to be self-reciprocal if $p(z) \equiv z^n p(\frac{1}{z})$ for all $z \in \mathscr{C}$. It was suggested by Professor Q. I. Rahman to study this class of polynomials and obtain inequalities analogous to (2). This has been done in many papers (see for example [6, 7, 10, 11]).

In this paper, we prove the following:

THEOREM 1. *If $p(z)$ is a polynomial of degree n satisfying $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$, then for each $\delta \geqslant 0$ and all α with $|\alpha| \geqslant 1$,*

$$\begin{aligned}\frac{n(|\alpha|-1)}{2}\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} &\leqslant \left(\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}\\ &\leqslant n(|\alpha|+1)C_\delta\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \end{aligned}\tag{6}$$

and for all α with $|\alpha| \leqslant 1$,

$$\begin{aligned}\frac{n(1-|\alpha|)}{2}\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} &\leqslant \left(\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}\\ &\leqslant n(|\alpha|+1)C_\delta\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \end{aligned}\tag{7}$$

where $C_\delta = \left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^\delta d\theta}\right)^{\frac{1}{\delta}}$.

If in (6) we divide throughout by $|\alpha|$ and make $|\alpha| \to \infty$, we get

COROLLARY 1. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n\overline{p(\frac{1}{\bar{z}})}$, *then for each* $\delta \geqslant 0$,

$$\frac{n}{2}\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \leqslant \left(\int_0^{2\pi}|p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \leqslant nC_\delta\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{8}$$

where $C_\delta = \left(\frac{2\pi}{\int_0^{2\pi}|1+e^{i\theta}|^\delta d\theta}\right)^{\frac{1}{\delta}}$. *The result is best possible and equality holds on the right-hand side for polynomials* $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$ *and on the left-hand side for polynomials* $p(z) = \beta z^{\frac{n}{2}}$, *where* n *is a positive even integer and* $\beta \neq 0$.

REMARK 1. Note that Corollary 1 can also be obtained from Theorem 1 by setting $\alpha = 0$ in (7).

The special case of Corollary 1 when $\delta \geqslant 1$ was proved by Govil and Jain [9, Corollary 1] (also see Dewan and Govil [5] and Govil [8]). The inequality on the left-hand side of (8) is due to Aziz and Rather [3].

Since $\lim_{\delta\to\infty} C_\delta = \dfrac{1}{2}$, if we make $\delta \to \infty$ in (6) and (7), we get

COROLLARY 2. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n\overline{p(\frac{1}{\bar{z}})}$, *then for all* α *with* $|\alpha| \geqslant 1$,

$$\frac{n(|\alpha|-1)}{2}\|p\| \leqslant \|D_\alpha p\| \leqslant \frac{n(|\alpha|+1)}{2}\|p\|, \tag{9}$$

and for all α *with* $|\alpha| \leqslant 1$,

$$\frac{n(1-|\alpha|)}{2}\|p\| \leqslant \|D_\alpha p\| \leqslant \frac{n(|\alpha|+1)}{2}\|p\|. \tag{10}$$

The inequality on the right-hand side of (9) *and* (10) *becomes equality for* $p(z) = z^n + 1$ *and the inequality on the left-hand side of* (9) *becomes equality for* $p(z) = (z-1)^n$, *where* n *is a positive even integer and* $\alpha \geqslant 1$. *On the left-hand side of* (10) *equality holds for* $p(z) = (z-1)^n$, *where* n *is a positive even integer and* $0 < \alpha \leqslant 1$.

The inequality on the right-hand side of (9) and (10) is also obtainable from Aziz [1, Corollary 3].

For polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$, we are only able to prove

THEOREM 2. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n p(\frac{1}{z})$, *then for each* $\delta \geqslant 1$ *and all* α *with* $|\alpha| \geqslant 1$,

$$\left(\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant \frac{n(|\alpha|-1)}{2}\left(\int_0^{2\pi}|p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{11}$$

and for all α *with* $|\alpha| \leqslant 1$,

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant \frac{n(1-|\alpha|)}{2}\left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}. \tag{12}$$

Dividing (11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result of Aziz and Rather [3].

COROLLARY 3. *If* $p(z)$ *is a self-reciprocal polynomial of degree* n, *then for every* $\delta \geqslant 1$,

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant \frac{n}{2}\left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}. \tag{13}$$

The inequality in (13) *is sharp and becomes equality for* $p(z) = \beta z^{\frac{n}{2}}$, *where* n *is a positive even integer and* $\beta \neq 0$.

REMARK 2. Note that Corollary 3 can also be obtained from Theorem 2 by setting $\alpha = 0$ in (12).

If in (11) and (12) we make $\delta \to \infty$, we get

COROLLARY 4. *If* $p(z)$ *is a self-reciprocal polynomial of degree* n, *then for every* α *with* $|\alpha| \geqslant 1$,

$$\|D_\alpha p\| \geqslant \frac{n(|\alpha|-1)}{2}\|p\| \tag{14}$$

and for every α *with* $|\alpha| \leqslant 1$,

$$\|D_\alpha p\| \geqslant \frac{n(1-|\alpha|)}{2}\|p\|. \tag{15}$$

If $\alpha \geqslant 1$, *then the inequality (14) becomes equality for* $p(z) = (z-1)^n$, *where* n *is a positive even integer. Equality holds in* (15) *for* $p(z) = (z-1)^n$, *where* n *is a positive even integer and* $0 < \alpha \leqslant 1$.

This first part of result is due to Aziz and Rather [2].

2. Lemmas

We need the following lemmas.

LEMMA 1. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, *then for* $\delta \geqslant 0$,

$$\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta \leqslant n^\delta C_\delta{}^\delta \int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta. \tag{16}$$

Lemma 1 is due to Govil [8].

LEMMA 2.. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$, *then for* $0 \leqslant \theta < 2\pi$,

$$|p'(e^{i\theta})| \geqslant \frac{n}{2}|p(e^{i\theta})|. \tag{17}$$

Proof of Lemma 2. If $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$, then

$$|p'(z)| = |-z^{n-2}\ \overline{p'\left(\frac{1}{\bar{z}}\right)} + nz^{n-1}\ \overline{p\left(\frac{1}{\bar{z}}\right)}|.$$

Therefore for $0 \leqslant \theta < 2\pi$,

$$\begin{aligned} |p'(e^{i\theta})| &= |ne^{i(n-1)\theta}\overline{p(e^{i\theta})} - e^{i(n-2)\theta}\overline{p'(e^{i\theta})}| \\ &\geqslant n|p(e^{i\theta})| - |p'(e^{i\theta})|, \end{aligned} \tag{18}$$

which implies (17).

LEMMA 3. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$, *then for* $0 \leqslant \theta < 2\pi$,

$$|p'(e^{i\theta})| = |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})|. \tag{19}$$

This follows immediately from (18).

LEMMA 4. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n p(\frac{1}{z})$, *then for* $0 \leqslant \theta < 2\pi$,

$$|p'(e^{-i\theta})| = |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})|. \tag{20}$$

The proof of this lemma is similar to that of Lemma 3.

LEMMA 5.. *If* $p(z)$ *is a polynomial of degree* n *satisfying* $p(z) \equiv z^n p(\frac{1}{z})$, *then for every* $\delta \geqslant 1$,

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant \frac{n}{2}\left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}. \tag{21}$$

This lemma is due to Aziz and Rather [3, Theorem 2].

3. Proofs of Theorems

Proof of Theorem 1. If $p(z)$ is a polynomial of degree n, then for any α and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned} |D_\alpha p(e^{i\theta})| &= |np(e^{i\theta}) + (\alpha - e^{i\theta})p'(e^{i\theta})| \\ &\leqslant |\alpha||p'(e^{i\theta})| + |np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})| \\ &= |\alpha||p'(e^{i\theta})| + |p'(e^{i\theta})|, \quad \text{by Lemma 3,} \\ &= (|\alpha| + 1)|p'(e^{i\theta})|, \end{aligned}$$

which implies that

$$\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta \leqslant (|\alpha|+1)^\delta \int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta. \tag{22}$$

Since $p(z)$ satisfies $p(z) \equiv z^n \overline{p(\frac{1}{\bar{z}})}$, if we combine (22) with Lemma 1, we get for any α and $\delta \geqslant 0$,

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \leqslant n(|\alpha|+1) C_\delta \left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{23}$$

and thus the inequalities on the right-hand side of (6) and (7) are established.

To prove the inequalities on the left-hand side of (6) and (7), note that for $\alpha \in \mathscr{C}$,

$$\begin{aligned} |D_\alpha p(e^{i\theta})| &= |\alpha p'(e^{i\theta}) + np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})| \\ &\geqslant |\alpha||p'(e^{i\theta})| - |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})| \\ &= |\alpha||p'(e^{i\theta})| - |p'(e^{i\theta})|, \quad \text{by Lemma 3,} \\ &= (|\alpha|-1)|p'(e^{i\theta})|. \end{aligned}$$

Thus for $|\alpha| \geqslant 1$,

$$|D_\alpha p(e^{i\theta})| \geqslant (|\alpha|-1)|p'(e^{i\theta})|. \tag{24}$$

Similarly, for $|\alpha| \leqslant 1$, we will get

$$|D_\alpha p(e^{i\theta})| \geqslant (1-|\alpha|)|p'(e^{i\theta})|. \tag{25}$$

Inequalities (24) and (25) when combined with Lemma 2, give

$$|D_\alpha p(e^{i\theta})| \geqslant \frac{(|\alpha|-1)}{2}|p(e^{i\theta})|, \quad \text{for } |\alpha| \geqslant 1, \tag{26}$$

and

$$|D_\alpha p(e^{i\theta})| \geqslant \frac{(1-|\alpha|)}{2}|p(e^{i\theta})|, \quad \text{for } |\alpha| \leqslant 1, \tag{27}$$

from which the inequalities on the left-hand side of (6) and (7) follow. This completes the proof of Theorem 1.

Proof of Theorem 2. If $p(z)$ is a polynomial of degree n, then for any α and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned} |D_\alpha p(e^{i\theta})| &= |np(e^{i\theta}) + (\alpha - e^{i\theta}) p'(e^{i\theta})| \\ &\geqslant |\alpha||p'(e^{i\theta})| - |np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})|. \end{aligned} \tag{28}$$

Since $p(z)$ satisfies $p(z) \equiv z^n p(\frac{1}{z})$, the inequality (28) when combined with Lemma 4, gives for $|\alpha| \geqslant 1$,

$$|D_\alpha p(e^{i\theta})| \geqslant |\alpha||p'(e^{i\theta})| - |p'(e^{-i\theta})|. \tag{29}$$

The above inequality is clearly equivalent to

$$|D_\alpha p(e^{i\theta})| + |p'(e^{-i\theta})| \geqslant |\alpha||p'(e^{i\theta})|, \quad |\alpha| \geqslant 1$$

which implies that

$$\left(\int_0^{2\pi} \left\{|D_\alpha p(e^{i\theta})| + |p'(e^{-i\theta})|\right\}^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant |\alpha| \left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}. \tag{30}$$

On applying Minkowski's inequality to the right-hand side of (30), we get that for $\delta \geqslant 1$,

$$\begin{aligned}\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} &\geqslant |\alpha| \left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} - \left(\int_0^{2\pi} |p'(e^{-i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \\ &= |\alpha| \left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} - \left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}.\end{aligned}$$

Thus

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant (|\alpha| - 1) \left(\int_0^{2\pi} |p'(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{31}$$

and this inequality when combined with Lemma 5, gives that for $\delta \geqslant 1$,

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}} \geqslant \frac{n(|\alpha| - 1)}{2} \left(\int_0^{2\pi} |p(e^{i\theta})|^\delta d\theta\right)^{\frac{1}{\delta}}, \tag{32}$$

which is (11). The proof of the case when $|\alpha| \leqslant 1$ is similar and we omit the details. This completes the proof of Theorem 2.

REFERENCES

[1] A. AZIZ, *Inequalities for the polar derivative of a polynomial*, J. Approx. Theory **55** (1988), 183–193.

[2] A. AZIZ AND N. A. RATHER, *A refinement of a theorem of Paul Turan concerning polynomials*, Math. Ineq. and Appl. **1** (1998), 231–238.

[3] A. AZIZ AND N. A. RATHER, *L^p inequalities for polynomials*, Glas. Mat. **32** (1997), 39–43.

[4] N. G. DE BRUIJN, *Inequalities concerning polynomials in the complex domain*, Nederl. Akad. Wetensch. Proc. **50** (1947), 1265–1272.

[5] K. K. DEWAN AND N. K. GOVIL, *An inequality for self-inversive polynomials*, J. Math. Anal. Appl. **95** (1983), 490.

[6] C. FRAPPIER AND Q. I. RAHMAN, *On an inequality of S. Bernstein,* Can. J. Math. **34** (1982), 932–944.

[7] C. FRAPPIER, Q. I. RAHMAN AND ST. RUSCHEWEYH, *New inequalities for polynomials,* Trans. Amer. Math. Soc. **288** (1985), 69–99.
[8] N. K. GOVIL, *An L^p inequality for self-inversive polynomials,* Glas. Mat. **32** (1997), 213–215.
[9] N. K. GOVIL AND V. K. JAIN, *An integral inequality for entire functions of exponential type*, Ann. Univ. Mariae Curie-Sklowdowska, Section A, **39** (1985), 57–60.
[10] N. K. GOVIL, V. K. JAIN AND G. LABELLE, *Inequalities for polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$*, Proc. Amer. Math. Soc. **57** (1976), 238–242.
[11] N. K. GOVIL AND D. H. VETTERLEIN, *Inequalities for a class of polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$*, Complex Variables, **31** (1996), 185–191.
[12] P. D. LAX, *Proof of a conjecture of of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.
[13] Q. I. RAHMAN AND G. SCHMEISSER, *L^p inequalities for polynomials*, J. Approx. Theory **53** (1988), 26–32.
[14] A. ZYGMUND, *A remark on conjugate series*, Proc. London Math. Soc. **34** (1932), 392–400.

(Received December 23, 1999)

N. K. Govil
Department of Mathematics
Auburn University
Auburn, AL 36849
U. S. A.

G. Nyuydinkong
Department of Mathematics
Auburn University
Auburn, AL 36849
U. S. A.