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INEQUALITIES FOR AVERAGES OF DIVIDED DIFFERENCES

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Abstract. We provide upper bounds for the averages of divided differences in terms of the norms of an appropriate derivative. These generalize a result of Ostrowski.

1. Introduction

Ostrowski [1] proved that if f is absolutely continuous and $\lambda \geqslant 1$ then

$$\int_0^1 \int_0^1 \left| \frac{f(x) - f(y)}{x - y} \right|^{\lambda} dxdy \leqslant (\log 4) \int_0^1 |(f'(x)|^{\lambda} dx. \tag{1}$$

If $\lambda = 1$ then log 4 is the best constant.

On another occasion, Fink [2] considered lower bounds for the average of $f[t, u_n] = f[t, \underbrace{u, \ldots, u}_{n \text{ times}}]$, the nth divided difference at the named points. Here we look at upper bounds for the average of general divided differences as Ostrowski did.

2. The setting

There are various ways to write formulae for the divided difference of a function f which has n continuous derivatives. Here $f[x_0, x_1] \equiv \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ and $f[x_0, \ldots, x_k] \equiv \frac{f[x_0, \ldots, x_{k-1}] - f[x_1, \ldots, x_k]}{x_0 - x_k} k = 2, \ldots, n$. Following Schoenberg [3], let $0 \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant 1$, then

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} \int_0^1 M[t; x_0, \ldots, x_n] f^{(n)}(t) dt, \tag{2}$$

where $f[x_0, x_1, \ldots, x_n]$ is the nth divided difference of f at $x_0, \ldots, x_n$ and $M[t; x_0, \ldots, x_n]$ is a basic B-spline obtained by taking the nth divided difference of the function

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$M(t;x) = (x-t)_+^{n-1}$ with respect to the variable x at the points $x_0,\ldots,x_n$. The basic facts about M are, that for distinct x_i,

$$M[t;x_0,\ldots,x_n] = \sum_{k=0}^{n} \frac{n(x_k-t)_+^{n-1}}{w'(x_k)},$$

where $w(t) = \prod_{i=0}^{n}(t-x_i)$;
$M[t;x_0,\ldots,x_n] > 0$ on (x_0,x_n) and zero elsewhere;

$$\int_0^1 M[t;x_0,\ldots,x_n]dt = 1;$$

and

$$\int_0^1 tM[t;x_0,\ldots,x_n]dt = \frac{1}{n+1}\sum_{l=0}^{n} x_l.$$

It is easy to see that M is symmetric in its arguments $x_0,\ldots,x_n$ so the order is immaterial. One can also verify that in the limit as two x_i take a common value, one merely differentiates $M[t;x_0,\ldots,x_n]$ with respect to x_i and evaluates at the common value, etc. In this way $M[t;x_0,\ldots,x_n]$ is defined on the cube $[0,1]^{n+1}$ in the $t-x$ space.

Consider now the averages

$$Av(f;n,\lambda) = \int_0^1 \ldots \int_0^1 |f[x_0,\ldots,x_n]|^\lambda dx_0\ldots dx_n. \tag{3}$$

We want to obtain upper bounds of this quantity in terms of the various norms of $f^{(n)}$ on $[0,1]$. Ostrowski's result is that $Av(f;1,\lambda) \leqslant \log 4\|f'\|_\lambda^\lambda$.

A standing hypothesis is that f *has the required derivatives,* n *is a positive integer and* $\lambda > 0$.

3. Results for general differences

THEOREM 1. $Av(f;n;\lambda) \leqslant \frac{1}{n!}\|f^{(n)}\|_\infty^\lambda$ *with equality when* $f^{(n)}$ *is a constant, i.e.* f *is a polynomial of degree* n, *so the constant* $\frac{1}{n!}$ *cannot be improved.*

Proof. Combining (2) and (3), we have

$$Av(f;n,\lambda) = \frac{1}{n!}\int_0^1 \ldots \int_0^1 \left| \int_0^1 M[t;x_0,\ldots x_n]f^{(n)}(t)dt\right|^\lambda dx_0\ldots dx_n. \tag{4}$$

We estimate $f^{(n)}(t)$ by $\|f^{(n)}\|_\infty$ (remembering that $M \geqslant 0$), and the remaining integrals are all 1. Since $M \geqslant 0$, equality will hold if $f^{(n)}$ is a constant.

We want to prove inequalities of the form

$$Av(f;n,\lambda) \leqslant K(\lambda,n,p)\|f^{(n)}\|_p^{\alpha(\lambda,p)} \tag{5}$$

for appropriate powers $\alpha(\lambda,p)$. The result of Theorem 1 is that $K(\lambda,n,\infty)=\frac{1}{n!}$ and $\alpha(\lambda,\infty)=\lambda$.

PROPOSITION 1. *To prove inequalities of the type (5) we may assume that $f^{(n)} \geqslant 0$ a.e.*

Proof. Let f be given and define g to be any function such that $g^{(n)}(x)=|f^{(n)}(x)|$. We have

$$|f[x_0,\dots,x_n]| = \frac{1}{n!}\left|\int_0^1 M[t;x_0,\dots,x_n]f^{(n)}(t)dt\right|$$

$$\leqslant \frac{1}{n!}\int_0^1 M[t;x_0,\dots,x_n]|f^{(n)}(t)|dt = \frac{1}{n!}\int_0^1 M[t;x_0,\dots,x_n]g^{(n)}(t)dt$$

$$= g[x_0,\dots,x_n] = |g[x_0,\dots,x_n]|$$

since by its representation, $g[x_0,\dots,x_n] \geqslant 0$. Since $g^{(n)}$, and $f^{(n)}$ have the same L_p norms, the proposition follows.

The case $\lambda = 1$ is the most definitive here.

THEOREM 2. *For $1 < p < \infty$, (5) holds with $\alpha(\lambda,p)=1$ and $K(1,n,p) = \|g_n\|_{p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$ and $g_n(t) = \int\limits_{0\leqslant x_0\leqslant x_1}\dots\int\limits_{\leqslant\dots\leqslant x_n} M[t;x_0,\dots,x_n]dx_0\dots dx_n$ also $K(1,n,1) \leqslant \|g_n\|_\infty$.*

Proof. Since by Proposition 1 we may assume $f^{(n)} \geqslant 0$, we have

$$Av(f;n,1) = \frac{1}{n!}\int_0^1\dots\int_0^1 M[t;x_0,\dots,x_n]f^{(n)}(t)dtdx_0\dots dx_n$$

$$= \int_0^1 f^{(n)}(t)g_n(t)dt$$

where $g_n(t)=\frac{1}{n!}\int_0^1\dots\int_0^1 M[t;x_0,\dots,x_n]dx_0\dots dx_n$.

Now the cube $[0,1]^n$ is the union of $n!$ simplices, each corresponding to a given ordering of the x_i. Since the overlap is of measure zero and M is symmetric in its arguments, the integral over any one of the simplices is the same as any other. Hence g_n has the representation given.

We now use Hölders inequality to arrive at $Av(f;n,1) \leqslant \|g_n\|_{p'}\|f^{(n)}\|_p$ with equality if $f^{(n)}(t) = \mu g_n(t)^{p'-1}$ for some positive constant μ. For $p=1$ we have $Av(f;n,1) \leqslant \|g_n\|_\infty\|f^{(n)}\|_1$. This ends the proof.

In the case of $p = 1$, equality would hold for $f^{(n)}(t) = \delta(t_0)$ where t_0 is a maximizing point of $g_n(t)$, so it is likely that $\|g_n\|_\infty$ is the best constant also.

When $\lambda > 1$ we proceed in a different way and do not get best constants. We look at equation (4) remembering that $M \geqslant 0$ and that we may assume that $f^{(n)} \geqslant 0$. We note that the inner integral

$$\left(\int_0^1 M[t;x_0,\dots,x_n]f^{(n)}(t)dt\right)^\lambda \leqslant \int_0^1 M[t;x_0,\dots,x_n]f^{(n)}(t)^\lambda dt$$

by Jensen's inequality applied with the probability measure $d\mu = M[t;x_0,\dots,x_n]dt$. Consequently we have the basic inequality

$$\begin{aligned} Av(f;n,\lambda) &\leqslant \frac{1}{n!}\int_0^1 \dots \int_0^1 M[t;x_0,\dots,x_n]f^{(n)}(t)^\lambda dt dx_0 \dots, x_n \qquad (6) \\ &= \int_0^1 f^{(n)}(t)^\lambda g_n(t)dt \end{aligned}$$

where g_n is defined in Theorem 2. In general (6) is a strict inequality if $\lambda > 1$.

THEOREM 3. *For $\lambda < p < \infty$ we have*

$$K(\lambda,n,p) \leqslant \|g_n\|_{\frac{p}{p-\lambda}} \text{ and } \alpha(\lambda,p) = \lambda, \text{ and}$$
$$K(\lambda,n,\infty) \leqslant \frac{1}{n!} \text{ with } \alpha(\lambda,p) = \lambda.$$

Proof. For $p < \infty$ this is immediate from (6) if one applies Hölder's inequality with index $r = \frac{p}{\lambda}$. For $p = \infty$ the result follows from (4).

4. The two point case

We now want to do the special case where the x_i take on only two values; Ostrowski's inequality being an example. We do not use the above representation. Instead we begin with

$$\frac{f(t) - f(u)}{t - u} = \int_0^1 f'(\theta t + (1-\theta)u)d\theta.$$

We let

$$f[t_j;u_k] \equiv f[t,\dots,t,u,\dots,u], t\ (j+1) \text{ times and } u\ (k+1) \text{ times.}$$

Then we can compute this by taking j derivatives of $\frac{f(t)-f(u)}{t-u}$ with respect to t and k derivatives with respect to u. We have

$$f[t_j;u_k] = \int_0^1 \theta^j(1-\theta)^k f^{(n)}(\theta t+(1-\theta)u)dt \tag{7}$$

with $n=j+k+1$, we look at $Av(f[t_j,u_k]) = \int_0^1\int_0^1 f[t_j;u_k]dtdu$.

PROPOSITION 2. *For $j,k \geqslant 0$ and $n=j+k+1$,*

$$Av(f[t_j,u_k]) = \int_0^1 f^{(n)}(s)g_{jk}(s)ds$$

where $g_{jk}(s) = \int_s^1 dt\int_0^{\frac{s}{t}}[x^j(1-x)^{k-1}+x^k(1-x)^{j-1}]dx$.

Proof. Using (7) we have

$$Av(f[t_j,u_k]) = \int_0^1\int_0^1 dt\,du\int_0^1 \theta^j(1-\theta)^k f^{(n)}(\theta t+(1-t)u)d\theta.$$

Consider, for continuous $p \geqslant 0$, the integral

$$I = \int_0^1\int_0^1 dt\,du\int_0^1 \theta^j(1-\theta)^k p(\theta t+(1-t)u)d\theta = I_1+I_2 \tag{8}$$

where I_1 is the integral over $t \geqslant u$ and I_2 the integral over $t \leqslant u$. For I_1 we let $s=\theta t+(1-t)u$ in the inner integral so that

$$I_1 = \int_0^1 dt\int_0^t du\left(\int_u^t \frac{(s-u)^j(t-s)^k}{(t-u)^{j+k+1}}p(s)ds\right)$$

The integrand is bounded by $\frac{1}{t-u}\int_u^t p(s)ds \leqslant \|p\|_\infty$ so we may interchange order of integration to get

$$I_1 = \int_0^1 dt\int_0^t p(s)ds\int_0^s \frac{(s-u)^j(t-s)^k}{(t-u)^{j+k+1}}du.$$

In the inner integral let $x=\frac{s-u}{t-u}$ to get

$$I_1 = \int_0^1 dt\int_0^t p(s)ds\int_0^{\frac{s}{t}} x^j(1-x)^{k-1}dx = \int_0^1 p(s)\left(\int_s^1 dt\int_0^{\frac{s}{t}} x^j(1-x)^{k+1}dx\right)ds$$

In a similar way $I_2 = \int_0^1 p(s)\left(\int_s^1 dt \int_0^{\frac{s}{t}} x^k(1-x)^{j-1}dx\right) ds$. This completes the proof.

THEOREM 4. *For* $1 < p < \infty$, $Avf[t_j, u_k] \leqslant \|g_{jk}\|_{p'}\|f^{(n)}\|_p$ *with equality when* $f^{(n)} = \mu g_{jk}^{p'-1}$ *and* $Avf[t_j, j_k] \leqslant \|g_{jk}\|_\infty \|f^{(n)}\|_1$

Proof. This is immediate from Proposition 2 and Hölder's inequality.

The norms $\|g_{jk}\|_\infty$ can be computed. We note that

$$g'_{jk}(s) = -\int_0^1 (x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx$$

$$+\int_s^1 \left(\frac{s}{t}\right)^j \left(1-\frac{s}{t}\right)^{k-1} + \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{j-1} \frac{dt}{t}$$

$$= -\int_0^1 [x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx$$

$$+\int_s^1 [x^{j-1}(1-x)^{k-1} + x^{k-1}(1-x)^{j-1}]dx$$

and $g''_{jk}(s) \leqslant 0$. We claim that $g'_{jk}\left(\frac{1}{2}\right) = 0$ so that $g_{jk}\left(\frac{1}{2}\right)$ is $\|g_{jk}\|_\infty$. By writing the first integral as the sum of two integrals one on $\left[0, \frac{1}{2}\right]$ and one on $\left[\frac{1}{2}, 1\right]$, combining the latter with second integral we get

$$g'_{jk}\left(\frac{1}{2}\right) = -\int_0^{\frac{1}{2}} [x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx + \int_{\frac{1}{2}}^1 [(1-x)^k x^{j-1} + x^{k-1}(1-x)^j]dx$$

which is zero by letting $u = 1-x$ in the second integral.

Then

$$\|g_{jk}\|_\infty = g_{jk}\left(\frac{1}{2}\right) = \int_{\frac{1}{2}}^1 du \int_0^{\frac{1}{2u}} [x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx$$

$$= \int_0^{\frac{1}{2}} [x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx \tag{9}$$

by interchange of order.

5. Some explicit estimates

We note immediately that $\|g_{oo}\|_\infty = \int_0^{\frac{1}{2}} \frac{2}{1-x}dx = \log 4$ which is Ostrowski's result (1). Actually for all $\lambda \geqslant 1$ one can get this with use of Jensen's inequality as above. Other explicit computations lead to

$$\|g_{01}\|_\infty = \log 2, \|g_{1k}\|_\infty = \frac{1}{k(k+1)}\left[1-\left(\frac{1}{2}\right)^k\right], k \geqslant 1, \text{ and}$$

$$\|g_{0k}\|_\infty = \frac{1}{k} + \sum_{i=1}^{\infty}\left(\frac{1}{k+i}\right)\left(\frac{1}{2}\right)^{k+i}, k \geqslant 2.$$

For $j,k \geqslant 1$ one can use the binomial theorem to get an explicit expansion for $\|g_{jk}\|_\infty$. One can get good lower bounds by making the change of variables in (9) $x = \frac{u}{1+u}$ to arrive at

$$g_{jk}\left(\frac{1}{2}\right) = \int_0^1 \frac{uj + u^k}{(1+k)^{j+k+1}}$$

which is very nearly a Beta function. But $B(j+1,k+1) = \int_0^1 \frac{uj+u^k}{(1+u)^{j+k+2}} \leqslant g_{jk}\left(\frac{1}{2}\right)$ gives a nice lower bound.

To give some estimates for $p < \infty$ we change the order of integration in the representation of g_{jk} in Proposition 2 to get

$$g_{jk}(s) = (1-s)\int_0^s [x^j(1-x)^{k-1} + x^k(1-x)^{j-1}]dx + s\int_s^1 [x^{j-1}(1-x)^k + x^{k-1}(1-x)^j]dx. \tag{10}$$

Only a few are simple enough to compute neatly. For example $g_{00} = 2g_{01} = -2(1-s)\log(1-s) - 2s\log s$ and $g_{11} = s(1-s)$ but others are more complicated. The representation (10) is reminiscent of a Green's function. Indeed, we have for $j,k > 1$, $g_{jk}(0) = g_{jk}(1) = 0$ and $-g''_{jk} = s^{j-1}(1-s)^{k-1} + s^{k-1}(1-s)^{j-1}$. Thus for B the Beta function

$$\int_0^1 |g''_{jk}|ds = 2B(j,k). \tag{11}$$

We have

$$g_{jk}(s) = \int_0^1 G(s,t)[-g''_{jk}(t)]dt \tag{12}$$

for $G(s,t) = \begin{cases} t(1-s), t \leqslant s \\ s(1-t), s \leqslant t. \end{cases}$

PROPOSITION 3. *For $j,k>1$*

$$\|g_{jk}\|_{p'} \leqslant 2B(j,k)B(p'+1,p'+1)^{\frac{1}{p'}}. \tag{13}$$

Proof. From (12)

$$|g_{jk}(s)| \leqslant s(1-s)\int_0^1 -g''_{jk}dt = 2s(1-s)B(j,k) \text{ so}$$

$$\|g_{jk}\|_{p'} \leqslant 2B(j,k)\left(\int_0^1 s^{p'}(1-s)^{p'}\right)^{\frac{1}{p'}}$$

$$= 2B(j,k)B(p'+1,p'+1)^{\frac{1}{p'}}.$$

Note that the factor of 2 is sort of extra since $\|g_u\|_{p'} = B(p'+1,p'+1)^{\frac{1}{p'}}$ and $B(1,1)=1$.

The estimate (13) can be improved slightly. We have shown elsewhere Fink [4] the following. Let $C(p,1)$ be the best constant for the inequality

$$\|f\|_p \leqslant C(p,1)\|f''\|, \tag{14}$$

given that f has a zero at each end of $[0,1]$. Furthermore, let $C(\infty,p')$ be the best constant for the inequality

$$\|f\|_\infty \leqslant C(\infty,p')\|f''\|_{p'} \tag{15}$$

given that f has a zero at each end of $[0,1]$. Fink proved that $C(p,1)=C(\infty,p')$.

The constant $C(\infty,p')$ can be computed this way. With $G(s,t)$ as above

$$f(s) = \int_0^1 G(s,t)(-f''(t))dt$$

so that

$$f(s) \leqslant \left(\int_0^1 G(s,t)^p dt\right)^{\frac{1}{p}} \|f''\|_{p'}$$

$$= \left[\frac{s^{p+1}(1-s)^p + s^p(1-s)^{p+1}}{p+1}\right]^{\frac{1}{p}} \|f''\|_{p'}$$

$$\leqslant \frac{1}{4}\frac{1}{(p+1)^{\frac{1}{p}}}\|f''\|_{p'}$$

since the bracket has its max at $s=\frac{1}{2}$. One can argue that this is best possible by letting $-f''(t) = G\left(\frac{1}{2},t\right)^{p-1}$. So $C(p,1) = \frac{1}{4}\frac{1}{(p+1)^{\frac{1}{p}}}$.

PROPOSITION 4. *For $j, k > 1$*

$$\|g_{jk}\|_{p'} \leqslant \frac{1}{2}\frac{1}{(p+1)^{\frac{1}{p'}}}B(j,k).$$

Proof. We apply (11) and (14).

By Stirling's formula $B(p'+1,p'+1)^{\frac{1}{p'}} \sim \frac{\left(\sqrt{\pi p'}\right)^{\frac{1}{p'}}}{2}$ which is larger than $\frac{1}{2}\frac{1}{(p'+1)^{\frac{1}{p'}}}$ but asymptotically the same as p goes to ∞.

6. Final Remarks

One can improve the bounds in Theorem 3 by obtaining a stronger inequality in (6). In [5], Fink and Jodeit have shown that $\frac{\left(\int_0^1 g(x)d\mu\right)^{\lambda}}{\int_0^1 g(x)^{\lambda}d\mu}$ is decreasing in λ if μ is a probability measure and $g \geqslant 0$. From $p > \lambda$

$$\frac{\left(\int_0^1 gd\mu\right)^{\lambda}}{\int_0^1 g^{\lambda}d\mu} < \frac{\left(\int_0^1 gd\mu\right)^{p}}{\int_0^1 g^{p}d\mu}$$

one gets

$$Av(f;n,\lambda) \leqslant \frac{\left(\int_0^1 f^{(n)}M[t;x_0,\ldots,x_n]dt\right)^{p}}{\int_0^1 f^{(n)}(t)^p M[t;x_0,\ldots,x_n]dt}\int_0^1 f^{(n)}(t)^{\lambda}g_n(t)dt, \qquad (6')$$

with this extra factor < 1 by Jensen's Inequality.

Finally, although Fink [2] establishes the existence of a lower bound of $Av(f;n,1)$ in a specific case, the argument also applies to the cases of this paper. However we have not been able to get specific constants of the sort

$$Av(f;n,1) \geqslant C\|f^{(n)}\|_p.$$

We leave this an an open problem.

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