GENERALIZATION THEOREM ON CONVERGENCE
AND INTEGRABILITY FOR SINE SERIES

ŽIVORAD TOMOVSKI

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Abstract. In this paper a generalization of the Zahid's theorem for convergence and integrability for sine series (see [9]) is made by considering the class $S_p(\delta)$, $p > 1$ instead of $S(\delta)$.

1. Introduction

1. A sequence $\{A_k\}$ is said to be δ–quasi–monotone if $A_k \to 0$, $A_k > 0$ ultimately and $\Delta A_k \geqslant -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

Now, we say that a sequence $\{a_k\}$ belongs to the class $S(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is δ–quasi-monotone, $\sum_{k=1}^{\infty} k\,\delta_k < \infty$, $\sum_{k=1}^{\infty} A_k < \infty$, and $|\Delta a_k| \leqslant A_k$, for all k.

In [8] we defined a new class $S_p(\delta)$, $p > 1$ as follows:

A sequence $\{a_k\}$ of numbers belongs to the class $S_p(\delta)$ or $a_k \in S_p(\delta)$, if $a_k \to 0$ as $k \to \infty$ and there exists a sequence $\{A_k\}$ such that:

(a) $\{A_k\}$ is δ–quasi–monotone and $\sum_{k=1}^{\infty} k\,\delta_k < \infty$.

(b) $\sum_{k=1}^{\infty} A_k < \infty$.

(c) $\dfrac{1}{n} \sum_{k=1}^{n} \dfrac{|\Delta a_k|^p}{A_k^p} = O(1)$.

2. Preliminaries

2. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

and

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

be trigonometric series.

In [8] for the cosine series we proved the following theorem.

THEOREM A. *Let the coefficients of the series $f(x)$ belongs to the class $S_p(\delta)$, $1 < p \leqslant 2$. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|dx \leqslant C_p \sum_{n=0}^{\infty} A_n ,$$

where C_p is a positive constant depends on p and on the condition (c) of $S_p(\delta)$.

On the other hand, S. Zahid [9] proved the following theorem.

THEOREM B. ([9]) *Let the coefficients of the series $g(x)$ belongs to the class $S(\delta)$. Then the series converges to a function and the following relation holds for $p = 1, 2, 3, \ldots$*

$$\int_{\pi/(p+1)}^{\pi} \left| \sum_{n=1}^{\infty} a_n \sin nx \right| dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left(\sum_{n=1}^{\infty} A_n \right) .$$

3. For the sine series, we shall prove the following main results.

COROLLARY. *Let the coefficients of the series $f(x)$ belongs to the class $S_p(\delta)$, $1 < p \leqslant 2$. Then $\dfrac{1}{x}\sum_{k=0}^{\infty} \Delta a_k \sin\left(k + \dfrac{1}{2}\right)x = \dfrac{h(x)}{x}$ converges for $x \in (0, \pi]$, and $\dfrac{h(x)}{x} \in L^1[0, \pi]$.*

THEOREM. *Let the coefficients of the series $g(x)$ belongs to the class $S_p(\delta)$, $1 < p \leqslant 2$. Then the series converges to a function and the following inequality holds for $m = 1, 2, 3, \ldots$,*

$$\int_{\pi/(m+1)}^{\pi} \left| \sum_{n=1}^{\infty} a_n \sin nx \right| dx \leqslant \sum_{n=1}^{m} \frac{|a_n|}{n} + K_p M \left(\sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} n\,\delta_n \right) ,$$

where

$$M \geqslant \max_n \left(\frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} \right)$$

and K_p is a constant depending on p.

4. For the proof of our results we require the following lemmas.

LEMMA 1. ([1], [9] case $v = 1$). *If $\{a_n\}$ is a δ-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n\delta_n < \infty$, *then the convergence of* $\sum_{n=1}^{\infty} a_n$ *implies that* $na_n = o(1)$, $n \to \infty$.

LEMMA 2. ([1], [9]) *Let $\{a_n\}$ be a δ-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n\delta_n < \infty$.

If $\sum_{n=1}^{\infty} a_n < \infty$, *then* $\sum_{n=1}^{\infty} (n+1) |\Delta a_n| < \infty$.

5. *Proof of the corollary.* Since

$$
\begin{aligned}
2 \sin \frac{x}{2} f(x) &= a_0 \sin \frac{x}{2} + \sum_{k=1}^{\infty} a_k \left(2 \sin \frac{x}{2} \cos kx \right) \\
&= a_0 \sin \frac{x}{2} + \sum_{k=1}^{\infty} a_k \left[\sin \left(k + \frac{1}{2} \right) x - \sin \left(k - \frac{1}{2} \right) x \right] \\
&= (a_0 - a_1) \sin \frac{x}{2} + (a_1 - a_2) \sin \frac{3x}{2} + (a_2 - a_3) \sin \frac{5x}{2} + \cdots \\
&= \sum_{k=0}^{\infty} \Delta a_k \sin(2k+1) \frac{x}{2} = h(x)
\end{aligned}
$$

by theorem A, proof is obvious.

Proof of the theorem. By summation by parts, we have:

$$
\begin{aligned}
\sum_{k=1}^{n} |\Delta a_k| &= \sum_{k=1}^{n} A_k \frac{|\Delta a_k|}{A_k} \\
&\leqslant \sum_{k=1}^{n-1} |\Delta A_k| \sum_{j=1}^{k} \frac{|\Delta a_j|}{A_j} + A_n \sum_{j=1}^{n} \frac{|\Delta a_j|}{A_j} \\
&\leqslant \sum_{k=1}^{n-1} k |\Delta A_k| \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} + n A_n \left(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta a_j|^p}{A_j^p} \right)^{1/p} \\
&= O(1) \left[\sum_{k=1}^{n-1} k |\Delta A_k| + n A_n \right].
\end{aligned}
$$

$(*)$

Then by lemma 1 and lemma 2, $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$.

Thus $\sum_{n=1}^{\infty} a_n \sin nx$ converges to $g(x)$ for every x.

Let $a_0 = 0$, and we define

$$A_0 = \max\left(|a_1|, \sum_{k=1}^{\infty} k\,\delta_k\right).$$

Then

$$\int_{\pi/(m+1)}^{\pi} \left|\sum_{k=1}^{\infty} a_k \sin kx\right| dx = \int_{\pi/(m+1)}^{\pi} \left|\sum_{k=0}^{\infty} \Delta a_k \overline{D}_k(x)\right| dx$$

$$\leqslant \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k(x)\right| + O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k(x)\right| dx\right),$$

where

$$\overline{D}_k(x) = -\frac{\operatorname{ctg} \frac{x}{2}}{2} + \sin x + \sin 2x + \sin 3x + \cdots + \sin kx = -\frac{\cos\left(k + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}.$$

Let

$$I_1 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=j}^{\infty} \Delta a_k \overline{D}_k(x)\right| dx, \qquad I_2 = \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k \overline{D}_k(x)\right| dx.$$

For all $x \in [0, \pi]$ $k = 0, 1, 2, \ldots$

$$\overline{D}_k(x) = -\frac{1}{x} + O(k+1).$$

We have:

$$I_2 \leqslant \sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left|\sum_{k=0}^{j-1} \Delta a_k\right| \frac{dx}{x} + O\left(\sum_{j=1}^{m} \int_{\pi/(j+1)}^{\pi/j} \left[\sum_{k=0}^{j-1} |\Delta a_k|(k+1)\right] dx\right)$$

$$= \sum_{j=1}^{m} |a_j| \ln\left(1 + \frac{1}{j}\right) + O\left(\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)\,|\Delta a_k|}{j(j+1)}\right)$$

$$\leqslant \sum_{j=1}^{m} \frac{|a_j|}{j} + O\left(\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)\,|\Delta a_k|}{j(j+1)}\right).$$

But

$$\sum_{j=1}^{m} \sum_{k=0}^{j-1} \frac{(k+1)\,|\Delta a_k|}{j(j+1)} = \sum_{j=1}^{m} \frac{1}{j(j+1)} \sum_{k=0}^{j-1} (k+1)\,|\Delta a_k|$$

$$\leqslant \sum_{k=0}^{\infty}(k+1)\,|\Delta a_k| \left(\sum_{j=k+1}^{\infty} \frac{1}{j(j+1)} \right)$$

$$= \sum_{k=0}^{\infty}(k+1)\,|\Delta a_k| \frac{1}{k+1}$$

$$= \sum_{k=0}^{\infty}|\Delta a_k|$$

$$= |\Delta a_0| + \sum_{k=1}^{\infty}|\Delta a_k|$$

$$= |a_1| + \sum_{k=1}^{\infty}|\Delta a_k|$$

$$\leqslant \sum_{k=1}^{\infty}|\Delta a_k| + \sum_{k=1}^{\infty}|\Delta a_k| = 2\sum_{k=1}^{\infty}|\Delta a_k|.$$

Using the same technique as in the proof of $(*)$, and applying the Lemma 1, we obtain:

$$\sum_{k=1}^{\infty}|\Delta a_k| = O\left(\sum_{k=1}^{\infty} k|\Delta A_k| \right).$$

Since $|\Delta A_k| \leqslant \Delta A_k + 2\delta_k$, by Lemma 1, we obtain

$$\sum_{k=1}^{\infty}|\Delta a_k| = O\left(\sum_{k=1}^{\infty} A_k \right) + O\left(\sum_{k=1}^{\infty} k\,\delta_k \right).$$

Thus

$$\sum_{j=1}^{m}\sum_{k=0}^{j-1} \frac{(k+1)\,|\Delta a_k|}{j(j+1)} = O\left(\sum_{k=1}^{\infty} A_k \right) + O\left(\sum_{k=1}^{\infty} k\,\delta_k \right).$$

Therefore,

$$I_2 = \sum_{j=1}^{m} \frac{|a_j|}{j} + O\left(\sum_{k=1}^{\infty} A_k \right) + O\left(\sum_{k=1}^{\infty} k\,\delta_k \right).$$

Application of Abel's transformation and Lemma 1, yield:

$$\sum_{k=j}^{\infty} \Delta a_k\,\overline{D}_k(x) = \sum_{k=j}^{\infty} A_k \frac{\Delta a_k}{A_k}\,\overline{D}_k(x) = \sum_{k=j}^{\infty} \Delta A_k \sum_{i=0}^{k} \frac{\Delta a_i}{A_i}\,\overline{D}_i(x) - A_j \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\,\overline{D}_i(x).$$

Let us estimate the second integral:

$$I_1 \leqslant \sum_{j=1}^{m} \left[\sum_{k=j}^{\infty} |\Delta A_k| \int_{\pi/(j+1)}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i}\,\overline{D}_i(x) \right| dx + A_j \int_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i}\,\overline{D}_i(x) \right| dx \right].$$

Let

$$
J_k = \int\limits_{\pi/(j+1)}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \, \overline{D}_i(x) \right| dx = \int\limits_{\pi/(j+1)}^{\pi} \frac{1}{\sin\dfrac{x}{2}} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \, \cos\left(i+\frac{1}{2}\right) x \right| dx .
$$

We shall first apply the Hölder inequality, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$
J_k \leqslant \left[\int\limits_{\pi/(j+1)}^{\pi} \left(\frac{1}{\sin\dfrac{x}{2}} \right)^p dx \right]^{1/p} \left[\int\limits_{0}^{\pi} \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} \, \cos\left(i+\frac{1}{2}\right) x \right|^q dx \right]^{1/q} .
$$

Then by Hausdorff-Young inequality, where $k \geqslant j$, we obtain

$$
J_k \leqslant \alpha_p (j+1)^{1/q} \left(\sum_{i=0}^{k} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p}
$$

$$
\leqslant \alpha_p (k+1) \left[\frac{1}{k+1} \sum_{i=0}^{k} \frac{|\Delta a_i|^p}{A_i^p} \right]^{1/p} = O_p(k+1)
$$

where O_p depends on p and also on the condition (c) of $S_p(\delta)$.

But,

$$
I_j = \int\limits_{\pi/(j+1)}^{\pi/j} \left| \sum_{i=0}^{j-1} \frac{\Delta a_i}{A_i} \, \frac{\cos\left(i+\dfrac{1}{2}\right) x}{\sin\dfrac{x}{2}} \right| dx
$$

$$
\leqslant \int\limits_{\pi/(j+1)}^{\pi/j} \frac{dx}{\sin\dfrac{x}{2}} \left(\sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i} \right)
$$

$$
\leqslant O\left(\frac{1}{j}\right) \sum_{i=0}^{j-1} \frac{|\Delta a_i|}{A_i}
$$

$$
\leqslant \beta \left(\frac{1}{j} \sum_{i=0}^{j-1} \frac{|\Delta a_i|^p}{A_i^p} \right)^{1/p} = O(1)
$$

where O depends on the condition (c) of $S_p(\delta)$.

Then by Lemma 1, we obtain

$$I_1 \leqslant \sum_{j=1}^{m} \sum_{k=j}^{\infty} |\Delta A_k| J_k + O\left(\sum_{j=1}^{\infty} A_j\right)$$

$$\leqslant O_p(1) \sum_{k=1}^{\infty} (\Delta A_k)(k+1) + O_p\left(\sum_{k=1}^{\infty} k\delta_k\right) + O\left(\sum_{j=1}^{\infty} A_j\right)$$

$$= O_p\left(\sum_{k=1}^{\infty} A_k\right) + O_p\left(\sum_{k=1}^{\infty} k\delta_k\right) + O\left(\sum_{j=1}^{\infty} A_j\right)$$

$$= O_p\left(\sum_{k=1}^{\infty} A_k\right) + O_p\left(\sum_{k=1}^{\infty} k\delta_k\right),$$

where O_p depends on p and also on the condition (c) of $S_p(\delta)$.
Thus

$$\int_{\pi/(m+1)}^{\pi} \left| \sum_{k=0}^{\infty} a_k \sin kx \right| dx = \sum_{j=1}^{m} \frac{|a_j|}{j} + O_p\left(\sum_{n=1}^{\infty} A_n\right) + O_p\left(\sum_{n=1}^{\infty} n\delta_n\right),$$

where O_p depends on p and on the condition (c) of $S_p(\delta)$.

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Tomovski Živorad
Faculty of Mathematical and Natural Sciences
P.O. BOX 162
91000 Skopje
Macedonia
e-mail: tomovski@iunona.pmf.ukim.edu.mk

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com