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NEW PROOFS OF WEIGHTED POWER MEAN INEQUALITIES AND MONOTONICITY FOR GENERALIZED WEIGHTED MEAN VALUES

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Abstract. In the article, a new proof of the weighted power mean inequalities is given using Cauchy-Schwarz-Buniakowski's inequality, and another two simple and short proofs of monotonicity for the generalized weighted mean values with two parameters are showed.

1. Introduction

For positive sequences $a = (a_1, a_2, \ldots, a_n)$ and $q = (q_1, q_2, \ldots, q_n)$ satisfying $\sum_{k=1}^{n} q_k = 1$, the weighted arithmetic, geometric, and harmonic mean values are defined respectively by

$$A_n(a,q) = \sum_{i=1}^{n} q_i a_i, \qquad G_n(a,q) = \prod_{i=1}^{n} a_i^{q_i}, \qquad H_n(a,q) = \frac{1}{\sum_{i=1}^{n} \frac{q_i}{a_i}}. \tag{1}$$

For positive integrable functions f and p defined on $[x,y]$, their integral analogues of (1) are given by

$$\begin{aligned} A(f,p) &= \int_x^y p(t)f(t)\,\mathrm{d}\,t, \\ G(f,p) &= \exp\left(\int_x^y p(t)\ln f(t)\,\mathrm{d}\,t\right), \\ H(f,p) &= \frac{1}{\int_x^y \frac{p(t)\,\mathrm{d}\,t}{f(t)}}, \end{aligned} \tag{2}$$

where $\int_x^y p(t)\,\mathrm{d}\,t = 1$ holds.

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It is well-known that

$$A_n(a,q) \geqslant G_n(a,q) \geqslant H_n(a,q), \qquad A(f,p) \geqslant G(f,p) \geqslant H(f,p) \tag{3}$$

are called the weighted arithmetic mean-geometric mean-harmonic mean inequalities.

For the sake of brevity, the inequality between the arithmetic and geometric means will be called A-G inequality, while the inequality between the geometric and harmonic means will be called G-H inequality.

The A-G inequality has found much interest among many mathematicians, and there are numerous new proofs, extensions, refinements, and variants of it. The study of the A-G inequality has a rich literature, for details, please refer to [2, 3, 8, 16, 21], and so on. Recently, H. Alzer [1], and J. Pečarić and S. Varošanec [10] gave two new proofs of the A-G inequality.

The concepts of mean values have been generalized, extended in many directons. A recent development concerning the mean values has simply been introduced in [9, 11, 13, 14].

The generalized weighted mean values $M_{p,f}(r,s;x,y)$ with two parameters r and s are defined by the first author in [11, 20] as follows:

Let $x,y,r,s \in \mathbb{R}$, $p(u) \not\equiv 0$ be a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between x and y, then

$$M_{p,f}(r,s;x,y) = \left(\frac{\int_x^y p(u) f^s(u)\,\mathrm{d}\,u}{\int_x^y p(u) f^r(u)\,\mathrm{d}\,u} \right)^{1/(s-r)}, \qquad (r-s)(x-y) \neq 0; \tag{4}$$

$$M_{p,f}(r,r;x,y) = \exp\left(\frac{\int_x^y p(u) f^r(u) \ln f(u)\,\mathrm{d}\,u}{\int_x^y p(u) f^r(u)\,\mathrm{d}\,u} \right), \quad x-y \neq 0; \tag{5}$$

$$M(r,s;x,x) = f(x).$$

For our own convenience, we write

$$M_{p,f}(r,s;x,y) = M_{p,f}(r,s) = M_{p,f}(x,y) = M_{p,f},$$

shifting notations to suit the context.

Most of two variable mean values are special cases of $M_{p,f}$. If $s = 0$, then $M_{p,f}(r,0;x,y) = M^{[r]}(f,p;x,y)$ is called the weighted mean of order r of the function f on the interval between x and y with weight p in [6, 7]. If take $p(u) \equiv 1$, $f(u) = u$ and $x,y > 0$, then $M_{p,f}(r-1,s-1;x,y) = E(r,s;x,y)$ are called the extended mean values in [4, 7].

The extended mean values E are increasing with r and s, or with x and y. It had been proven by many mathematicians, for instances [4, 5, 7, 15, 17, 22]. The study of E has a rich literature, for details, please see [11]. The monotonicity of $M_{p,f}$ was verified by the first author in [11, 20] using the Tchebycheff's integral inequality, the Cauchy-Schwarz-Buniakowski's integral inequality, and the mean value thoerem.

In this article, using Cauchy-Schwarz-Buniakowski's inequality, we will give a new proof of the weighted power mean inequalities, i.e., the weighted A-G-H inequalities.

From the ideas and viewpoints used in [12, 17, 18, 19], we will prove the monotonicity of $M_{p,f}(r,s;x,y)$ by two new and simple methods.

2. A new proof of the weighted power mean inequalities

For continuous functions f and p satisfying $\int_x^y p(t)\,\mathrm{d}\,t = 1$, define

$$\begin{aligned}\psi(r) &= \left(\int_x^y p(t)f^r(t)\,\mathrm{d}\,t\right)^{1/r}, \quad r \neq 0;\\ \psi(0) &= G(f,p).\end{aligned} \tag{6}$$

For positive sequences $a = (a_1, a_2, \dots, a_n)$ and $q = (q_1, q_2, \dots, q_n)$ satisfying $\sum_{i=1}^n q_i = 1$, define

$$\begin{aligned}\varphi(r) &= \left(\sum_{i=1}^n q_i a_i^r\right)^{1/r}, \quad r \neq 0;\\ \varphi(0) &= G_n(a,q).\end{aligned} \tag{7}$$

THEOREM 1. *The functions $\psi(r)$ and $\varphi(r)$ are increasing with $r \in \mathbb{R}$, respectively.*

Proof. Simple calculation yields

$$\begin{aligned}\ln\psi(r) &= \frac{\ln\int_x^y p(t)f^r(t)\,\mathrm{d}\,t}{r}\\ &= \frac{\ln\int_x^y p(t)f^r(t)\,\mathrm{d}\,t - \ln\int_x^y p(t)f^0(t)\,\mathrm{d}\,t}{r}\\ &= \frac{1}{r}\int_0^r \frac{\int_x^y p(t)f^s(t)\ln f(t)\,\mathrm{d}\,t}{\int_x^y p(t)f^s(t)\,\mathrm{d}\,t}\,\mathrm{d}\,s.\end{aligned}$$

It is well-known [6, 11] that, if f is a continuous and increasing function on a given interval I, then the arithmetic mean $\Psi(r,s)$ of f defined as

$$\begin{aligned}\Psi(r,s) &= \frac{1}{s-r}\int_r^s f(t)\,\mathrm{d}\,t, \quad r-s \neq 0,\\ \Psi(r,r) &= f(r)\end{aligned} \tag{8}$$

is also increasing with both r and s on I.

Therefore, it is sufficient to verify that

$$\mathcal{F}(s) \triangleq \frac{\int_x^y p(t)f^s(t)\ln f(t)\,\mathrm{d}\,t}{\int_x^y p(t)f^s(t)\,\mathrm{d}\,t}$$

is increasing in $s \in \mathbb{R}$. By a simple differentiation as in the standard proofs and then also use Cauchy-Schwarz-Buniakowski's integral inequality, it is easy to see that the function $\mathcal{F}(s)$ increases with s if and only if

$$\left(\int_x^y p(t)f^s(t)\ln f(t)\,\mathrm{d}\,t\right)^2 \leqslant \int_x^y p(t)f^s(t)\,\mathrm{d}\,t\int_x^y p(t)f^s(t)\big[\ln f(t)\big]^2\,\mathrm{d}\,t. \tag{9}$$

Since

$$\int_x^y p(t)f^s(t)\ln f(t)\,\mathrm{d}\,t = \int_x^y \left[p^{1/2}(t)f^{s/2}(t)\right]\left[p^{1/2}(t)f^{s/2}(t)\ln f(t)\right]\mathrm{d}\,t,$$

from Cauchy-Schwarz-Buniakowski's integral inequality, the inequality (9) follows. The equality in (9) is valid if and only if f is a constant function. The function $\psi(r)$ is increasing with r.

By straightforward computation, we have

$$\begin{aligned}\ln\varphi(r) &= \frac{1}{r}\ln\sum_{i=1}^n q_i a_i^r \\ &= \frac{1}{r}\left(\ln\sum_{i=1}^n q_i a_i^r - \ln\sum_{i=1}^n q_i a_i^0\right) \\ &= \frac{1}{r}\int_0^r \left(\sum_{i=1}^n q_i a_i^s \ln a_i \Big/ \sum_{i=1}^n q_i a_i^s\right)\mathrm{d}\,s.\end{aligned} \tag{10}$$

Using Cauchy-Schwarz-Buniakowski's inequality in discrete form, by the similar arguments as proving the monotonicity of $\psi(r)$, we can easily obtain that the function $\varphi(r)$ increases with r. The proof of Theorem 1 follows.

□

COROLLARY. *For positive continuous functions f and p such that $\int_x^y p(t)\,\mathrm{d}\,t = 1$, or positive sequences $a = (a_1, a_2, \dots, a_n)$ and $q = (q_1, q_2, \dots, q_n)$ satisfying $\sum_{i=1}^n q_i = 1$, we have the following weighted A-G-H inequalities:*

$$A(f,p) \geqslant G(f,p) \geqslant H(f,p), \qquad A_n(a,q) \geqslant G_n(a,q) \geqslant H_n(a,q). \tag{11}$$

Equalities in (11) hold if and only if f is a constant function or $a_1 = a_2 = \cdots = a_n$, respectively.

Proof. It is easy to see that $\psi(1) = A(f,p)$, $\psi(-1) = H(f,p)$, $\varphi(1) = A_n(a,q)$ and $\varphi(-1) = H_n(a,q)$. Thus, the weighted A-G-H inequalities in integral form follows from the monotonicity of $\psi(r)$, the weighted A-G-H inequalities in discrete form follows from the monotonicity of $\varphi(r)$. The proof is complete.

□

3. New proofs of monotonicity for generalized weighted mean values

In this section, we will prove the following

THEOREM 2. *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive, increasing (or decreasing, respectively) and continuous function, then the generalized weighted mean values $M_{p,f}(r,s;x,y)$ increases (or decreases, respectively) with respect to either x or y.*

3.1. The first proof

Now let

$$h_{p,f}(t;x,y)=\int_x^y p(u)f^t(u)\,\mathrm{d}\,u,\quad t\in\mathbb{R}, \tag{12}$$

where x, y, p and f are defined as in Theorem 2.

It is easy to see that

$$\frac{\partial^n h_{p,f}(t;x,y)}{\partial t^n}=\int_x^y p(u)f^t(u)[\ln f(u)]^n\,\mathrm{d}\,u. \tag{13}$$

Set $Q_{p,f}(r,s;x,y)=\ln M_{p,f}(r,s;x,y)$, then

$$Q_{p,f}(r,s;x,y)=\frac{1}{s-r}\int_r^s\frac{\partial h_{p,f}(t;x,y)/\partial t}{h_{p,f}(t;x,y)}\,\mathrm{d}\,t,\quad (r-s)(x-y)\neq 0; \tag{14}$$

$$Q_{p,f}(r,r;x,y)=\frac{\partial h_{p,f}(r;x,y)/\partial r}{h_{p,f}(r;x,y)},\qquad x-y\neq 0. \tag{15}$$

To verify the monotonicity of $M_{p,f}(r,s;x,y)$ with x and y, it is sufficient to prove the monotonicity of $[\partial h_{p,f}(t;x,y)/\partial t]/h_{p,f}(t;x,y)$ in $Q_{p,f}(r,s;x,y)$ with x and y for any t. This is a special case of the following

LEMMA. *The functions*

$$\frac{\partial^{2(k+i)+1}h_{p,f}(t;x,y)/\partial t^{2(k+i)+1}}{\partial^{2k}h_{p,f}(t;x,y)/\partial t^{2k}} \tag{16}$$

are increasing (or decreasing, respectively) with x and y if $f(u)$ is increasing (or decreasing, respectively) for i and k being nonnegative integers.

Proof. Using the integral expressions (12) and (13) of $h_{p,f}(t;x,y)$, by standard arguments, we have

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{\partial^{2(k+i)+1}h_{p,f}(t;x,y)/\partial t^{2(k+i)+1}}{\partial^{2k}h_{p,f}(t;x,y)/\partial t^{2k}}\right)\\
&\quad=\left[\frac{\partial}{\partial y}\left(\frac{\partial^{2(k+i)+1}h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}}\right)\cdot\frac{\partial^{2k}h_{p,f}(t;x,y)}{\partial t^{2k}}\right.\\
&\qquad\left.-\frac{\partial^{2(k+i)+1}h_{p,f}(t;x,y)}{\partial t^{2(k+i)+1}}\cdot\frac{\partial}{\partial y}\left(\frac{\partial^{2k}h_{p,f}(t;x,y)}{\partial t^{2k}}\right)\right]\cdot\frac{1}{\left[\partial^{2k}h_{p,f}(t;x,y)/\partial t^{2k}\right]^2} \qquad (17)\\
&\quad=\frac{p(y)f^t(y)\left[\ln f(y)\right]^{2k}}{\left[\partial^{2k}h_{p,f}(t;x,y)/\partial t^{2k}\right]^2}\cdot\left[\left(\ln f(y)\right)^{2i+1}\int_x^y p(u)f^t(u)\left[\ln f(u)\right]^{2k}\mathrm{d}\,u\right.\\
&\qquad\left.-\int_x^y p(u)f^t(u)\left[\ln f(u)\right]^{2(i+k)+1}\mathrm{d}\,u\right].
\end{aligned}$$

When $f(u)$ increases (or decreases, respectively), the derivatives (17) are nonnegative (or nonpositive, respectively); hence, the desired monotonicity of (16) with respect to x and y follows, since the discussed functions (16) are symmetric in x and y. This completes the proof of lemma.

□

3.2. The second proof

Let

$$\alpha(t) = \frac{p(y)f^t(y)}{\int_x^y p(u)f^t(u)\,\mathrm{d}\,u}. \tag{18}$$

Straightforward computation yields

$$\alpha'(t) = \frac{p(y)f^t(y)\int_x^y p(u)f^t(u)\ln\frac{f(y)}{f(u)}\,\mathrm{d}\,u}{\left(\int_x^y p(u)f^t(u)\,\mathrm{d}\,u\right)^2} \geqslant 0. \tag{19}$$

By straightforward computation, from the mean-value theorem, we know that there is at least one point ξ between r and s such that

$$\frac{\partial M_{p,f}(r,s;x,y)/\partial y}{M_{p,f}(r,s;x,y)} = \frac{\alpha(s)-\alpha(r)}{s-r} = \alpha'(\xi) \geqslant 0, \tag{20}$$

thus, we obtain that the generalized weighted mean values $M_{p,f}(r,s;x,y)$ increases in y and x, since $M_{p,f}(r,s;x,y)$ is symmetric with x and y.

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