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A TRIGONOMETRIC INEQUALITY AND ITS GEOMETRIC APPLICATIONS

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Abstract. A sharp inequality for a linear combination of four cosines is obtained. It is used to sharpen inequalities due to Florian, Lenhard, Ozeki all of which extend the classical Erdös-Mordell inequality. Reverse isoperimetric inequalities for cyclic quadrilaterals are also obtained.

1. Introduction

Let $x_1, x_2, \ldots, x_n$ and $\delta_1, \delta_2, \ldots, \delta_n$ be two sets of positive real numbers such that

$$\delta_1 + \delta_2 + \cdots + \delta_n = \pi. \tag{1}$$

If we put $x_{n+1} = x_1$, then it was conjectured $[1, p.139]$ by L.Fejes Tóth that

$$\sum_{i=1}^{n} x_i x_{i+1} \cos \delta_i \leqslant \cos\left(\frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2. \tag{2}$$

A proof of (2) when $n = 3$ runs as follows:

Let $x, y, z, \alpha, \beta, \gamma$ be positive real numbers with $\alpha + \beta + \gamma = \pi$. Then

$$(xz\cos\alpha + yz\cos\beta - xy)^2 + (xz\sin\alpha - yz\sin\beta)^2 \geqslant 0$$

with equality holding if and only if

$$x\sin\alpha = y\sin\beta = z\sin\gamma.$$

Expansion and simplification give

$$x\cos\alpha + y\cos\beta + z\cos\gamma \leqslant \frac{x^2y^2 + y^2z^2 + z^2x^2}{2xyz}. \tag{3}$$

Now use $x = x_1x_2, y = x_2x_3, z = x_3x_1$ in (3) to obtain (2) for the case $n = 3$.

The first proof of (2) when $n = 4$ was given by Florian [4] and the proof for general n was obtained by Lenhard [5]. The interest in (2) was originally in connection with extension of the classical Erdös-Mordell inequality: *If P is interior to a triangle,*

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the sum of its distances to the sides is at most one half the sum of its distances to the vertices.

To see how (2) is used to obtain the extension let P be a point interior to a convex polygon $A_1A_2...A_n$. If the bisector of the angle $\sphericalangle A_kPA_{k+1} = 2\delta_k$ meets A_kA_{k+1} in W_k and has length w_k then

$$w_k = \frac{2R_kR_{k+1}}{R_k + R_{k+1}} \cos\delta_k, R_k = PA_k, R_{n+1} = R_1. \tag{4}$$

Since $2\sqrt{R_kR_{k+1}} \leqslant R_k + R_{k+1}$ and, here, $\cos\delta_k \geqslant 0$, it follows by (2) that

$$\sum_{k=1}^{n} w_k \leqslant \sum_{k=1}^{n} \sqrt{R_kR_{k+1}} \cos\delta_k \leqslant \cos(\frac{\pi}{n}) \sum_{k=1}^{n} R_k.$$

Since the distance from P to the side A_kA_{k+1} is smaller than w_k we obtain that *the sum of the distances of P to the sides of the polygon is at most $\cos(\frac{\pi}{n})$ times the sum of its distances to the vertices.* When $n = 3$ the polygon reduces to a triangle, $\cos(\frac{\pi}{n}) = \frac{1}{2}$ and we have the Erdös-Mordell inequality.

Lenhard's proof of (2) is completely different from Florian's proof of (2) for the case $n = 4$. But absent from both proofs is a preliminary inequality analogous to (3). This raises the question as to whether there is an inequality corresponding to (3) for $n \geqslant 4$. The purpose of this note is to obtain such an inequality for $n = 4$. (Theorem 1 below). As a consequence we are able to further extend the Erdös-Mordell inequality for a quadrilateral, and obtain new propositions some of which complement the isoperimetric inequality for cyclic quadrilaterals. We also include a cosine inequality for the case $n = 3$ which may be of independent interest.

2. Cosine inequalities

THEOREM 1. *Let $x, y, z, t, \delta_1, \delta_2, \delta_3, \delta_4$ be positive real numbers with $\delta_1 + \delta_2 + \delta_3 + \delta_4 = \pi$. Then*

$$x\cos\delta_1 + y\cos\delta_2 + z\cos\delta_3 + t\cos\delta_4 \leqslant \sqrt{\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}}. \tag{5}$$

Furthermore, equality holds if and only if

$$x\sin\delta_1 = y\sin\delta_2 = z\sin\delta_3 = t\sin\delta_4. \tag{6}$$

In particular, if x_1, x_2, x_3, x_4 are positive and $x_5 = x_1$, then

$$\sum_{k=1}^{4} x_kx_{k+1}\cos\delta_k \leqslant \sqrt{2(x_1^2 + x_3^2)(x_2^2 + x_4^2)}.$$

Proof. Let $s > 0$ be an auxiliary variable to be determined shortly. Since $\cos(\delta_1 + \delta_2) = -\cos(\delta_3 + \delta_4)$ we have using inequality (3)

$$\begin{aligned} & x\cos\delta_1 + y\cos\delta_2 + z\cos\delta_3 + t\cos\delta_4 \\ &\quad = x\cos\delta_1 + y\cos\delta_2 + s\cos(\delta_3+\delta_4) + z\cos\delta_3 + t\cos\delta_4 + s\cos(\delta_1+\delta_2) \\ &\quad \leqslant \frac{x^2y^2 + y^2s^2 + s^2x^2}{2xys} + \frac{z^2t^2 + t^2s^2 + s^2z^2}{2zts} \\ &\quad = (\frac{x^2+y^2}{2xy} + \frac{z^2+t^2}{2zt})s + (\frac{xy+zt}{2})\frac{1}{s} \\ &\quad = \sqrt{(\frac{x^2+y^2}{xy} + \frac{z^2+t^2}{zt})(xy+zt)} = \sqrt{\frac{(xy+zt)(xz+yt)(xt+yz)}{xyzt}} \end{aligned}$$

where we made a special choice for the value of s namely $s = \sqrt{\frac{\frac{xy+zt}{2}}{\frac{x^2+y^2}{2xy} + \frac{z^2+t^2}{2zt}}}$. This finishes the proof of the inequality. If equality holds in (5) then, by the case of equality in (3) we must have

$$x\sin\delta_1 = y\sin\delta_2 = s\sin(\delta_3+\delta_4), \quad \text{and} \quad z\sin\delta_3 = t\sin\delta_4 = s\sin(\delta_1+\delta_2).$$

PROPOSITION 1. *Let $a, b, c, \alpha, \beta, \gamma$ be positive real numbers such that $\alpha+\beta+\gamma = \pi$. Then*

$$\frac{4bc}{b+c}\cos\alpha + \frac{4ca}{c+a}\cos\beta + \frac{4ab}{a+b}\cos\gamma \leqslant a+b+c - \frac{ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2}{(a+b)(b+c)(c+a)}$$

Note that Proposition 1 implies, in the notation of the introduction, that

$$w_1 + w_2 + w_3 = \frac{2bc}{b+c}\cos\alpha + \frac{2ca}{c+a}\cos\beta + \frac{2ab}{a+b}\cos\gamma \leqslant \frac{1}{2}(a+b+c)$$

without assumption of *positivity* of the cosines.

The proof of Proposition 1 depends on the following algebraic identity which may be of independent interest.

PROPOSITION 2. *If a, b and c are three real numbers then*

$$2a^2(b+c)^2 + 2b^2(c+a)^2 + 2c^2(a+b)^2 - (a+b+c)(a+b)(b+c)(c+a)$$

$$= -ab(a-b)^2 - bc(b-c)^2 - ca(c-a)^2.$$

In particular, if a, b, c are all non-negative then

$$2a^2(b+c)^2 + 2b^2(c+a)^2 + 2c^2(a+b)^2 \leqslant (a+b+c)(a+b)(b+c)(c+a)$$

and equality holds if and only if $a = b = c$.

Proof. The proof is computational. We have, successively

$$(a+b)(b+c)(c+a) = 2abc + ab^2 + a^2b + ac^2 + a^2c + bc^2 + b^2c,$$
$$(a+b+c)(a+b)(b+c)(c+a) = 4a^2bc + 2b^2c^2 + b^3c + bc^3 + 4ab^2c + 2a^2c^2 + ac^3 + a^3c + 4abc^2 + 2a^2b^2 + ab^3 + a^3b.$$

Also

$$2a^2(b+c)^2 + 2b^2(c+a)^2 + 2c^2(a+b)^2 = 4a^2bc + 4ab^2c + 4abc^2 + 4a^2b^2 + 4b^2c^2 + 4c^2a^2.$$

Subtraction now gives

$$\begin{aligned} 2a^2(b+c)^2 + 2b^2(c+a)^2 + 2c^2(a+b)^2 - (a+b+c)(a+b)(b+c)(c+a) \\ = 2a^2b^2 - ab^3 - a^3b + 2b^2c^2 - b^3c - bc^3 + 2c^2a^2 - ca^3 - c^3a \\ = -ab(a-b)^2 - bc(b-c)^2 - ca(c-a)^2. \end{aligned}$$

This completes the proof of the identity. The inequality, as well as the case of equality in it, follow immediately.

Returning to Proposition 1 suppose that $a, b, c, \alpha, \beta, \gamma$ are positive and $\alpha + \beta + \gamma = \pi$. Applying (3) we obtain

$$\frac{4bc}{b+c}\cos\alpha + \frac{4ca}{c+a}\cos\beta + \frac{4ab}{a+b}\cos\gamma \leqslant \frac{2a^2(b+c)}{(c+a)(a+b)} + \frac{2b^2(c+a)}{(a+b)(b+c)} + \frac{2c^2(a+b)}{(b+c)(c+a)}$$

and the righthand side is $\leqslant (a+b+c) - \dfrac{ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2}{(a+b)(b+c)(c+a)}$ by Proposition 2 above.

3. Applications

3.1. Sharpening Florian's inequality for a quadrilateral

Let P be a point interior to a convex quadrilateral $A_1A_2A_3A_4$. If the bisector of the angle $\sphericalangle A_kPA_{k+1} = 2\delta_k$ meets A_kA_{k+1} in W_k and has length w_k then Florian's inequality gives

$$w_1 + w_2 + w_3 + w_4 \leqslant \frac{1}{\sqrt{2}}(R_1 + R_3 + R_2 + R_4).$$

If we use $w_k = \frac{2R_kR_{k+1}}{R_k+R_{k+1}}\cos\delta_k \leqslant \sqrt{R_kR_{k+1}}\cos\delta_k$ and Theorem 1, we obtain

$$w_1 + w_2 + w_3 + w_4 \leqslant \sqrt{2(R_1+R_3)(R_2+R_4)} \tag{7}$$

which is readily seen to be sharper than Florian's inequality.

3.2. A geometric inequality for quadrilaterals

PROPOSITION 3. *Let P be the point of intersection of the diagonals AC, BD of a convex quadrilateral $ABCD$. If the bisectors of the angles between the diagonals meet the opposite sides in R,S,T,U respectively then*

$$RT+SU \leqslant \sqrt{2\,AC\cdot BD}.$$

Furthermore this inequality is sharper than the corresponding Erdös-Mordell inequality.

The question of comparing lengths of bisectors and diagonals in this setting was considered by Demir and Bankoff $[3,p.131]$ who showed that

$$RT+SU \leqslant \frac{3}{4}(AC+BD).$$

But this inequality is not sharp.

Proof. In this case and with the same notation as above

$$\begin{aligned} RT+SU &= RP+PT+SP+PU = w_1+w_2+w_3+w_4 \\ &\leqslant \sqrt{2(PA+PC)(PB+PD)} = \sqrt{2\,AC\cdot BD}. \end{aligned}$$

since $2\sqrt{AC\cdot BD} \leqslant AC+BD$ this inequality implies that

$$RT+SU \leqslant \frac{\sqrt{2}}{2}(AC+BD)$$

which is the only available information from the Erdös-Mordell inequality.

3.3. Reverse isoperimetric inequalities for cyclic quadrilaterals

Let $ABCD$ be a convex cyclic quadrilateral inscribed in a circle of radius R. Denote the lengths of its sides by a,b,c,d the lengths of its diagonals by e,f,g and $L=2s=a+b+c+d$ its perimeter. The area Δ of the quadrilateral is given by Brahmagupta's formula $[2,p.57]$

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d).$$

Using the inequality between the arithmetic and geometric means of four positive numbers we obtain

$$L^2 \geqslant 16\Delta^2. \tag{8}$$

This is the isoperimetric inequality for cyclic quadrilaterals.

In the inequality (5) take $\delta_k = \frac{\pi}{4}$ to obtain

$$(a+b+c+d)\frac{1}{\sqrt{2}} \leqslant \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{abcd}}$$

or

$$\frac{L}{\sqrt{2}} \leqslant \sqrt{\frac{16R^2\Delta^2}{abcd}},$$

where we used $[2,p.60]$ the formula $\Delta^2 = \dfrac{(ab+cd)(ac+bd)(ad+bc)}{16R^2}$ for the area of $ABCD$. Thus

$$L \leqslant \frac{4\sqrt{2}R\Delta}{\sqrt{abcd}}. \tag{9}$$

This sharp inequality which goes in a direction opposite to (8) may be viewed as a reverse isoperimetric inequality for cyclic quadrilaterals.

3.4. Circumscribed cyclic quadrilaterals

If the cyclic quadrilateral $ABCD$ is also circumscribed about a circle, that is its sides are tangent to a circle then $[2,p.60]$ its area is given by $\Delta^2 = abcd$. Using (9) we obtain

$$L \leqslant 4R\sqrt{2} \quad \text{and} \quad L^2 \leqslant \frac{32\Delta'}{\pi}$$

where Δ' is the area of the circumcircle of $ABCD$.

REFERENCES

[1] BOTTEMA O., DJORDJEVIC R. Z., JANIC R. R., MITRINOVIC D. S., VASIC P. M., *Geometric Inequalities*, Wolters-Noordhoff publishing Groningen, 1969, The Netherlands.
[2] COXETER H. S. M., GREITZER S. L., *Geometry Revisited*, The Mathematical Association Of America, 1967.
[3] MITRINOVIC D. S., PECARIC J. E., VOLENEC V., *Recent Advances In Geometric Inequalities*, Kluwer Academic publishers, 1989.
[4] FLORIAN A., *Zu einem Satz von P. Erdos*, Elem. Math. **13** (1958), 55–58.
[5] LENHARD H-C., *Verallgemeinerung und Verscharfung der Erdos-Mordellschen Ungleichung fur Polygone*, Arch. Math. Vol. XII, 1961, 311–314.

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