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ON SOME INEQUALITIES FOR GENERALIZED BETA FUNCTION

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Abstract. Some integral representations and inequalities for generalized beta function are given.

1. Introduction

In the recent survey paper [1], about gamma and beta functions, S. S. Dragomir, R. P. Agarwal and N. S. Barnett presented relations, identities, integral representations and inequalities for the gamma and beta functions. Various aspects of the topic are presented, from integral representations, fundamental relations and identities, classical inequalities, convexity and logarithmic convexity to some very recent results on Ostrowski type inequalities and cubature formulae.

As a little contribution to the topic, in this paper we look at the natural generalization $B_n,\ n \geqslant 2,$ of the beta function, defined by

$$B_n(\mathbf{x}) = \frac{\Gamma(x_1)\cdot\cdot\Gamma(x_n)}{\Gamma(x_1+\cdot\cdot+x_n)} \tag{1.1}$$

where $\mathbf{x} \in \mathbf{R}_+^n,\ \mathbf{x} = (x_1, .., x_n),$ and $\mathbf{R}_+^n$ is the set of all vectors $\mathbf{x} \in \mathbf{R}^n$ with positive coordinates.

This function has been considered earlier, e.g. in the monograph [3], by H. Federer. See Remark 2 below.

We give some integral representations for $B_n,$ and prove that B_n is strictly decreasing and logarithmically convex on $\mathbf{R}_+^n.$ We also prove some inequalities for $B_n,$ and for its partial derivatives.

2. Some integral representations

In this section we shall give some integral representations for B_n, suitable in proving inequalities. To simplify notation we first introduce some standard notions and symbols. For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ the relation $\mathbf{x} \leqslant \mathbf{y}$ means $x_k \leqslant y_k$ for all $k = 1, .., n.$ Further, $(\mathbf{x}|\mathbf{y}) = x_1y_1 + .. + x_ny_n$ is the standard scalar product, $\mathbf{xy} = (x_1y_1, .., x_ny_n)$ is the product by coordinates and $\mathbf{u} = (1, .., 1).$ Also we write $|\mathbf{x}| = (|x_1|, .., |x_n|)$ and

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$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{a} \geqslant \mathbf{0}$ or for $\mathbf{x} > \mathbf{0}$ and $\mathbf{a} \in \mathbf{R}^n$. The standard basis of $\mathbf{R}^n$ is denoted by $\{\mathbf{e}_1, .., \mathbf{e}_n\}$, where $\mathbf{e}_1 = (1, 0, .., 0), .., \mathbf{e}_n = (0, .., 0, 1)$.

By $H(n)$ we denote the set of all continuous functions $\varphi : \mathbf{R}^n \to [0, \infty)$ such that φ is differentiable *a.e.* and $\varphi'(\mathbf{x}) \neq \mathbf{0}$ *a.e.* and

$$\varphi(t\mathbf{x}) = t\varphi(\mathbf{x}) \text{ for } t \in [0, \infty) \text{ and } \mathbf{x} \in \mathbf{R}^n$$

Here $\varphi'(\mathbf{x})$ means the gradient of φ.

For $\varphi \in H(n)$ we introduce

$$S_\varphi = \{\mathbf{x} \in \mathbf{R}^n; \varphi(\mathbf{x}) = 1\} \text{ and } D_\varphi = \{\mathbf{x} \in \mathbf{R}^n; \varphi(\mathbf{x}) < 1\}$$

the unit sphere and the unit disc of φ.

Every norm on $\mathbf{R}^n$ belongs to $H(n)$. If $\varphi_1, \varphi_2 \in H(n)$ then $\varphi_1 + \varphi_2 \in H(n)$, $\alpha\varphi_1 \in H(n)$ for positive α, and also $\max(\varphi_1, \varphi_2)$, $\min(\varphi_1, \varphi_2) \in H(n)$. If $\mathbf{a} \in \mathbf{R}^n$, $\mathbf{a} \neq \mathbf{0}$, and $\varphi(\mathbf{x}) = |(\mathbf{a}|\mathbf{x})|$ then $\varphi \in H(n)$. If $0 < p < \infty$ and

$$\|\mathbf{x}\|_p = (|x_1|^p + .. + |x_n|^p)^{1/p}$$

then $\|\cdot\|_p \in H(n)$ and we write

$$S_{\|\cdot\|_p} = S_n^p \text{ and } D_{\|\cdot\|_p} = D_n^p.$$

We denote by h the Hausdorff $(n-1)$-measure on $\mathbf{R}^n$. Restriction of h on a nice surface becomes the usual surface area measure from analysis. By $|D_\varphi|$ we denote the Lebesgue measure of D_φ, and we write $|S_\varphi|$ for the surface area of S_φ i.e. for $h(S_\varphi)$.

LEMMA 1. *For $\varphi \in H(n)$ and $F \in L_1(\mathbf{R}^n)$ we have*

$$\int_{\mathbf{R}^n} F(\mathbf{x})d\mathbf{x} = \int_0^\infty \int_{S_\varphi} F(t\mathbf{x})t^{n-1} \frac{dtdh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|},$$

where $\|\varphi'(\mathbf{x})\|$ is the standard euclidean norm of $\varphi'(\mathbf{x})$.

Proof. See [2, Theorem 7]. This formula is called the polar formula. □

COROLLARY 1. *Let $\varphi \in H(n)$ and $f : [0, \infty) \to \mathbf{R}$ be such that $F = f \circ \varphi \in L_1(\mathbf{R}^n)$. Then*

$$\int_{\mathbf{R}^n} f(\varphi(\mathbf{x}))d\mathbf{x} = n\,|D_\varphi| \int_0^\infty f(t)t^{n-1}dt.$$

Proof. By the polar formula for $F = f \circ \varphi$ we have

$$\begin{aligned}\int_{\mathbf{R}^n} f(\varphi(\mathbf{x}))d\mathbf{x} &= \int_0^\infty \int_{S_\varphi} f(t)t^{n-1} \frac{dtdh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|} \\ &= \int_{S_\varphi} \frac{dh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|} \int_0^\infty f(t)t^{n-1}dt.\end{aligned}$$

If we choose f to be the indicator function of $[0, 1)$, then we get

$$\int_{\mathbf{R}^n} f(\varphi(\mathbf{x}))d\mathbf{x} = |D_\varphi| = \int_{S_\varphi} \frac{dh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|} \int_0^1 t^{n-1}dt = \frac{1}{n} \int_{S_\varphi} \frac{dh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|}.$$

Therefore

$$\int_{S_\varphi} \frac{dh(\mathbf{x})}{\|\varphi'(\mathbf{x})\|} = n\left|D_\varphi\right|,$$

which proves our assertion. □

REMARK 1. There are functions $\varphi \in H(n)$ for which $\left|D_\varphi\right| = \infty$. For such a φ the conditions of the Corollary 1 are not fulfilled i.e. $F = f \circ \varphi$ is not in $L_1(\mathbf{R}^n)$ for every $f \neq 0$.

COROLLARY 2. *For* $0 < p < \infty$ *we have*

$$|D_n^p| = 2^n \frac{\Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)}.$$

Proof. Put $\varphi(\mathbf{x}) = \|\mathbf{x}\|_p$ and $f(t) = \exp(-t^p)$ in Corollary 1 to get

$$\left(\int_{\mathbf{R}} \exp(-|t|^p)dt\right)^n = n\left|D_\varphi\right| \int_0^\infty \exp(-t^p)t^{n-1}dt.$$

After simple calculation we have the result. □

THEOREM 1. *For* $\mathbf{x} \in \mathbf{R}_+^n$, $\varphi(\mathbf{y}) = \|\mathbf{y}\|_p$, *and* $0 < p < \infty$ *we have*

$$B_n(\mathbf{x}) = 2^{-n}p^{n-1} \int_{S_n^p} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}} \frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|}.$$

Proof. By the polar formula,

$$\begin{aligned} \int_{\mathbf{R}^n} f(\varphi(\mathbf{y}))\,|\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,d\mathbf{y} &= \int_0^\infty \int_{S_n^p} f(t)t^{p(\mathbf{x}|\mathbf{u})-1}\,|\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,dt\frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \\ &= \int_{S_n^p} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}} \frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \cdot \int_0^\infty f(t)t^{p(\mathbf{x}|\mathbf{u})-1}dt, \end{aligned}$$

for every $f : [0,\infty) \to \mathbf{R}$ such that the left hand side integral exists.

Let us apply this formula to $f(t) = \exp(-t^p)$. We get

$$\prod_{k=1}^n \int_{\mathbf{R}} \exp(-|t|^p)\,|t|^{px_k-1}\,dt = \int_{S_n^p} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}} \frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \cdot \int_0^\infty \exp(-t^p)t^{p(\mathbf{x}|\mathbf{u})-1}dt$$

and

$$\prod_{k=1}^n \frac{2}{p}\Gamma(x_k) = \int_{S_n^p} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}} \frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \cdot \frac{1}{p}\Gamma((\mathbf{x}|\mathbf{u})),$$

which gives our formula.

We also proved, as a side result, the integral formula

$$\int_{\mathbf{R}^n} f(\varphi(\mathbf{y}))\,|\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,d\mathbf{y} = 2^n p^{1-n} B_n(\mathbf{x}) \int_0^\infty f(t) t^{p(\mathbf{x}|\mathbf{u})-1} dt.$$

□

COROLLARY 3. *For $\mathbf{x} \in \mathbf{R}^n_+$ we have*

$$B_n(\mathbf{x}) = \frac{1}{2^n\sqrt{n}} \int_{S^1_n} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}}\,dh(\mathbf{y})$$

and

$$B_n(\mathbf{x}) = \frac{1}{2} \int_{S^2_n} |\mathbf{y}|^{2\mathbf{x}-\mathbf{u}}\,dh(\mathbf{y}). \tag{2.1}$$

Proof. Put $p = 1$ in Theorem 1. Then $\varphi(\mathbf{y}) = \|\mathbf{y}\|_1$ and $\|\varphi'(\mathbf{y})\| = \sqrt{n}$, *a.e.*, which gives the first formula. Further, put $p = 2$ in Theorem 1. Then $\varphi(\mathbf{y}) = \|\mathbf{y}\|_2 = \|\mathbf{y}\|$ and $\|\varphi'(\mathbf{y})\| = 1$ for $\mathbf{y} \neq \mathbf{0}$, which proves the second formula. □

REMARK 2. The formula (2.1) from Corollary 3 can be found in the monograph [3, 3.2.13]. More precisely, in [3] the formula (2.1) has been used as the definition formula for B_n, while the formula (1.1) has been proved.

COROLLARY 4. *Let Δ_n be the $(n-1)$-dimensional simplex with vertices $\mathbf{e}_1,..,\mathbf{e}_n$, where $\{\mathbf{e}_1,..,\mathbf{e}_n\}$ is the standard basis of $\mathbf{R}^n$. Then*

$$B_n(\mathbf{x}) = \frac{1}{\sqrt{n}} \int_{\Delta_n} \mathbf{y}^{\mathbf{x}-\mathbf{u}} dh(\mathbf{y}).$$

Proof. Follows from the first formula of Corollary 3 and

$$\Delta_n = \{\mathbf{x} \in S^1_n; \mathbf{x} \geqslant \mathbf{0}\}.$$

□

THEOREM 2. *For $\mathbf{x} \in \mathbf{R}^n_+$ and $0 < p < \infty$ we have*

$$B_n(\mathbf{x}) = 2^{-n} p^n (\mathbf{x}|\mathbf{u}) \int_{D^p_n} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,d\mathbf{y}.$$

Proof. By the polar formula and Theorem 1, for $\varphi(\mathbf{y}) = \|\mathbf{y}\|_p$ we have

$$\begin{aligned} \int_{D^p_n} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,d\mathbf{y} &= \int_0^1 \int_{S^p_n} t^{p(\mathbf{x}|\mathbf{u})-1}\,|\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,dt \frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \\ &= \frac{1}{p(\mathbf{x}|\mathbf{u})} \int_{S^p_n} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\,\frac{dh(\mathbf{y})}{\|\varphi'(\mathbf{y})\|} \\ &= \frac{1}{p(\mathbf{x}|\mathbf{u})} 2^n p^{1-n} B_n(\mathbf{x}), \end{aligned}$$

which gives our formula. □

COROLLARY 5. *For* $\mathbf{x} \in \mathbf{R}_+^n$ *we have*

$$B_n(\mathbf{x}) = 2^{-n}(\mathbf{x}|\mathbf{u}) \int_{D_n^1} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}}\, d\mathbf{y}$$

and

$$B_n(\mathbf{x}) = (\mathbf{x}|\mathbf{u}) \int_{D_n^2} |\mathbf{y}|^{2\mathbf{x}-\mathbf{u}}\, d\mathbf{y}.$$

Proof. Put $p = 1$ and $p = 2$ in Theorem 2. □

COROLLARY 6. *Let* Δ_n *be the open* n*-dimensional simplex with vertices* $\mathbf{0}, \mathbf{e}_1, .., \mathbf{e}_n$, *where* $\{\mathbf{e}_1, .., \mathbf{e}_n\}$ *is the standard basis of* $\mathbf{R}^n$. *Then*

$$B_n(\mathbf{x}) = (\mathbf{x}|\mathbf{u}) \int_{\Delta_n} \mathbf{y}^{\mathbf{x}-\mathbf{u}} d\mathbf{y}.$$

Proof. Follows from the first formula of Corollary 5 and

$$\Delta_n = \{\mathbf{y} \in D_n^1; \mathbf{y} > \mathbf{0}\}.$$

□

3. Some inequalities for generalized beta function

In this section we shall prove some inequalities for B_n, and its derivatives, using the integral representations from the preceding section. All these inequalities are valid for every $n \geqslant 2$, and for $n = 2$ they become the inequalities for the beta function. Some of these inequalities are given in [1], and some are new even for $n = 2$.

THEOREM 3. *The function* B_n *is strictly decreasing on* $\mathbf{R}_+^n$ *i.e. if* $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^n$, $\mathbf{x} \leqslant \mathbf{y}$ *and* $\mathbf{x} \neq \mathbf{y}$, *then* $B_n(\mathbf{x}) > B_n(\mathbf{y})$.

Proof. By Corollary 3 we have

$$B_n(\mathbf{x}) = \frac{1}{2^n\sqrt{n}} \int_{S_n^1} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}}\, dh(\mathbf{y}).$$

Let A_n be the set of all $\mathbf{y} \in S_n^1$ such that $|y_k| = 0$ or 1, for some $k = 1, .., n$. Then $h(A_n) = 0$ and

$$B_n(\mathbf{x}) = \frac{1}{2^n\sqrt{n}} \int_{S_n^1 \setminus A_n} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}}\, dh(\mathbf{y}).$$

For every $\mathbf{y} \in S_n^1 \setminus A_n$ we have $\mathbf{0} < |\mathbf{y}| < \mathbf{u}$ i.e. $0 < |y_k| < 1$, for every $k = 1, .., n$. Therefore, the map $\mathbf{x} \to |\mathbf{y}|^{\mathbf{x}-\mathbf{u}}$ is strictly decreasing on $\mathbf{R}_+^n$, for every $\mathbf{y} \in S_n^1 \setminus A_n$. Hence, by integrating over $S_n^1 \setminus A_n$, we conclude that B_n is strictly decreasing on $\mathbf{R}_+^n$.
□

REMARK 3. From the definition of the function B_n we see that B_n is analytic on $\mathbf{R}_+^n$. The partial derivatives $\partial_k B_n$, $k = 1,..,n$, can be represented by integrals using any integral representation of B_n from the previous section e.g.

$$\partial_k B_n(\mathbf{x}) = \frac{1}{2^n\sqrt{n}}\int_{S_n^1} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}} \log|y_k|\, dh(\mathbf{y}) = \frac{1}{2^n\sqrt{n}}\int_{S_n^1\setminus A_n} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}} \log|y_k|\, dh(\mathbf{y}).$$

From this formula it follows

$$\partial_k B_n(\mathbf{x}) < 0,$$

for every $\mathbf{x} \in \mathbf{R}_+^n$ and every $k = 1,..,n$. This gives another proof of Theorem 3.

THEOREM 4. *For* $0 < p < \infty$ *and* $\mathbf{x} \in \mathbf{R}_+^n$, $\mathbf{x} \geqslant \frac{1}{p}\mathbf{u}$ *we have*

$$B_n(\mathbf{x}) \leqslant \frac{\Gamma(\frac{1}{p})^n}{\Gamma(\frac{n}{p}+1)}(\mathbf{x}|\mathbf{u}).$$

Proof. By Theorem 2 we have

$$B_n(\mathbf{x}) = 2^{-n}p^n(\mathbf{x}|\mathbf{u})\int_{D_n^p} |\mathbf{y}|^{p\mathbf{x}-\mathbf{u}}\, d\mathbf{y}.$$

If $\mathbf{x} \geqslant \frac{1}{p}\mathbf{u}$, then $p\mathbf{x}-\mathbf{u} \geqslant \mathbf{0}$ and $|\mathbf{y}|^{p\mathbf{x}-\mathbf{u}} \leqslant 1$ for every $\mathbf{y} \in D_n^p$. Therefore

$$B_n(\mathbf{x}) \leqslant 2^{-n}p^n(\mathbf{x}|\mathbf{u})\int_{D_n^p} d\mathbf{y} = 2^{-n}p^n(\mathbf{x}|\mathbf{u})\,|D_n^p|\,.$$

Now, by Corollary 2,

$$|D_n^p| = 2^n\frac{\Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)}$$

and we get

$$B_n(\mathbf{x}) \leqslant 2^{-n}p^n(\mathbf{x}|\mathbf{u})2^n\frac{\Gamma(\frac{1}{p}+1)^n}{\Gamma(\frac{n}{p}+1)} = \frac{\Gamma(\frac{1}{p})^n}{\Gamma(\frac{n}{p}+1)}(\mathbf{x}|\mathbf{u}),$$

which proves our inequality. $\square$

THEOREM 5. *Let* μ *be a positive finite Borel measure on* S_φ, *with a density with respect to the Hausdorff* $(n-1)$*-measure* h, *where* $\varphi \in H(n)$ *is such that* S_φ *is compact, and* $\mathbf{a} \in \mathbf{R}_+^n$. *Then the function* $f : \mathbf{R}_+^n \to \mathbf{R}_+$ *defined by*

$$f(\mathbf{x}) = \int_{S_\varphi} |\mathbf{z}|^{\mathbf{a}\mathbf{x}-\mathbf{u}}\, d\mu(\mathbf{z}),$$

is logarithmically convex on $\mathbf{R}_+^n$, *i.e.*

$$f(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) \leqslant f(\mathbf{x})^\alpha f(\mathbf{y})^{1-\alpha} \tag{3.1}$$

for every $\alpha \in [0,1]$ *and* $\mathbf{x},\mathbf{y} \in \mathbf{R}_+^n$. *Further, if* $\alpha \in \mathbf{R}$, $\alpha \leqslant 0$ *or* $\alpha \geqslant 1$, *and* $\mathbf{x},\mathbf{y} \in \mathbf{R}_+^n$ *are such that*

$$\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in \mathbf{R}_+^n,$$

then

$$f(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) \geqslant f(\mathbf{x})^\alpha f(\mathbf{y})^{1-\alpha}. \tag{3.2}$$

Proof. The function f is well defined i.e. the integral exists for every $\mathbf{x} \in \mathbf{R}_+^n$, since S_φ is compact and μ has a density with respect to the Hausdorff $(n-1)$-measure h. The compactness of S_φ is equivalent to the condition:

$$\varphi(\mathbf{x}) = \mathbf{0} \text{ if and only if } \mathbf{x} = \mathbf{0}.$$

For $\alpha = 0$ or 1 the inequalities are trivial. Therefore, to prove the first inequality, we can assume that $0 < \alpha < 1$. We have

$$\begin{aligned} f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) &= \int_{S_\varphi} |\mathbf{z}|^{\alpha\mathbf{ax}+(1-\alpha)\mathbf{ay}-\mathbf{u}}\, d\mu(\mathbf{z}) \\ &= \int_{S_\varphi} |\mathbf{z}|^{\alpha\mathbf{ax}-\alpha\mathbf{u}} \cdot |\mathbf{z}|^{(1-\alpha)\mathbf{ay}-(1-\alpha)\mathbf{u}}\, d\mu(\mathbf{z}). \end{aligned}$$

Let us apply the Hölder inequality to this relation with

$$p = \frac{1}{\alpha} \text{ and } q = \frac{1}{1-\alpha}.$$

We get

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leqslant \left(\int_{S_\varphi} |\mathbf{z}|^{\mathbf{ax}-\mathbf{u}}\, d\mu(\mathbf{z})\right)^{\alpha} \left(\int_{S_\varphi} |\mathbf{z}|^{\mathbf{ay}-\mathbf{u}}\, d\mu(\mathbf{z})\right)^{1-\alpha} = f(\mathbf{x})^\alpha f(\mathbf{y})^{1-\alpha},$$

which proves the first inequality.

The second inequality follows by applying the reverse Hölder inequality. □

COROLLARY 7. *Let $g : \mathbf{R}_+ \to \mathbf{R}_+$, $g(t) = f(t\mathbf{u})$, where f is from Theorem 5. Then g is logarithmically convex on $\mathbf{R}_+$ i.e.*

$$g(\alpha s + (1-\alpha)t) \leqslant g(s)^\alpha g(t)^{1-\alpha} \tag{3.3}$$

for every $\alpha \in [0,1]$ and $s,t \in \mathbf{R}_+$. Further, if $\alpha \in \mathbf{R}$, $\alpha \leqslant 0$ or $\alpha \geqslant 1$ and $s,t \in \mathbf{R}_+$ are such that

$$\alpha s + (1-\alpha)t \in \mathbf{R}_+,$$

then

$$g(\alpha s + (1-\alpha)t) \geqslant g(s)^\alpha g(t)^{1-\alpha}. \tag{3.4}$$

Proof. Follows immediately from Theorem 5 for $\mathbf{x} = t\mathbf{u}$. □

THEOREM 6. *Let μ be a positive finite Borel measure on D_φ, with a density with respect to the Lebesgue measure, where $\varphi \in H(n)$ is such that S_φ is compact, and $\mathbf{a} \in \mathbf{R}_+^n$. Then the function $f : \mathbf{R}_+^n \to \mathbf{R}_+$ defined by*

$$f(\mathbf{x}) = \int_{D_\varphi} |\mathbf{z}|^{\mathbf{ax}-\mathbf{u}}\, d\mu(\mathbf{z}),$$

satisfies the inequalities (3.1) *and* (3.2).

Proof. Similar to the proof of Theorem 5. $\square$

COROLLARY 8. *Let* $g:\mathbf{R}_+\to\mathbf{R}_+$, $g(t)=f(t\mathbf{u})$, *where* f *is from Theorem 6. Then* g *satisfies the inequalities* (3.3) *and* (3.4).

Proof. Follows from Theorem 6 for $\mathbf{x}=t\mathbf{u}$. $\square$

THEOREM 7. *The function* B_n *satisfies the inequalities* (3.1) *and* (3.2).

Proof. By corollary 3 we have

$$B_n(\mathbf{x})=\frac{1}{2}\int_{S_n^2}|\mathbf{z}|^{2\mathbf{x}-\mathbf{u}}\,dh(\mathbf{z}).$$

The assertion follows from Theorem 5 for $\mathbf{a}=2\mathbf{u}$, $\varphi(\mathbf{x})=\|\mathbf{x}\|$ and $\mu=\frac{1}{2}h$. $\square$

THEOREM 8. *The function* $f:\mathbf{R}_+^n\to\mathbf{R}_+$ *defined by*

$$f(\mathbf{x})=\frac{B_n(\mathbf{ax})}{(\mathbf{a}|\mathbf{x})}$$

is strictly decreasing on $\mathbf{R}_+^n$, *for every* $\mathbf{a}\in\mathbf{R}_+^n$, *and satisfies the inequalities* (3.1) *and* (3.2).

Proof. By Corollary 5

$$f(\mathbf{x})=\frac{B_n(\mathbf{ax})}{(\mathbf{a}|\mathbf{x})}=\int_{D_n^2}|\mathbf{y}|^{2\mathbf{ax}-\mathbf{u}}\,d\mathbf{y}.$$

Repeat now the argument of the proof of Theorem 3 to conclude that this function is strictly decreasing on $\mathbf{R}_+^n$, for every $\mathbf{a}\in\mathbf{R}_+^n$.

The last assertion follows from Theorem 6 for $\varphi(\mathbf{x})=\|\mathbf{x}\|$ and $d\mu(\mathbf{y})=d\mathbf{y}$. $\square$

COROLLARY 9. *Let* $\mathbf{a}\in\mathbf{R}_+^n$ *and* $g:\mathbf{R}_+\to\mathbf{R}_+$ *be defined by* $g(t)=B_n(t\mathbf{a})$. *Then* g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4).

Proof. Follows from Theorem 3 and Corollary 7. $\square$

COROLLARY 10. *The function* $g:\mathbf{R}_+\to\mathbf{R}_+$ *defined by*

$$g(t)=\frac{\Gamma(t)^n}{\Gamma(nt)}$$

is strictly decreasing on $\mathbf{R}_+$, *for every* $n\geqslant 2$, *and satisfies the inequalities* (3.3) *and* (3.4).

Proof. Put $\mathbf{a}=\mathbf{u}$ in Corollary 9. $\square$

COROLLARY 11. *Let* $\mathbf{a}\in\mathbf{R}_+^n$ *and* $g:\mathbf{R}_+\to\mathbf{R}_+$ *be defined by*

$$g(t)=\frac{B_n(t\mathbf{a})}{t}.$$

Then g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4).

Proof. Follows from Theorem 8 for $\mathbf{x} = t\mathbf{u}$. □

COROLLARY 12. *The function* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *defined by*

$$g(t) = \frac{\Gamma(t)^n}{t\Gamma(nt)}$$

is strictly decreasing on $\mathbf{R}_+$, *for every* $n \geqslant 2$, *and satisfies the inequalities* (3.3) *and* (3.4).

Proof. Put $\mathbf{a} = \mathbf{u}$ in Corollary 11. □

THEOREM 9. *Let* $f : \mathbf{R}_+^n \to \mathbf{R}_+$ *be defined by*

$$f(\mathbf{x}) = (-\mathbf{u})^{\mathbf{m}} \partial^{\mathbf{m}} B_n(\mathbf{x}),$$

where $\mathbf{m}$ *is a multi-index and* $\partial^{\mathbf{m}} = \partial_1^{m_1} \cdots \partial_n^{m_n}$ *is the* $\mathbf{m}$*-partial derivative operator. Then* f *is strictly decreasing on* $\mathbf{R}_+^n$, *and satisfies the inequalities* (3.1) *and* (3.2).

Proof. By remark 3 we have

$$f(\mathbf{x}) = \frac{1}{2^n\sqrt{n}} \int_{S_n^1} |\mathbf{y}|^{\mathbf{x}-\mathbf{u}} (-\log|\mathbf{y}|)^{\mathbf{m}}\, dh(\mathbf{y}),$$

where

$$\log|\mathbf{y}| = (\log|y_1|, .., \log|y_n|).$$

The first assertion is proved as in Theorem 3. The last assertion follows from Theorem 5 for $\varphi(\mathbf{x}) = \|\mathbf{x}\|_1$, $\mathbf{a} = \mathbf{u}$ and

$$d\mu(\mathbf{y}) = \frac{1}{2^n\sqrt{n}} (-\log|\mathbf{y}|)^{\mathbf{m}}\, dh(\mathbf{y}).$$

□

THEOREM 10. *Let* $\mathbf{a} \in \mathbf{R}_+^n$ *and* $f : \mathbf{R}_+^n \to \mathbf{R}_+$ *be defined by*

$$f(\mathbf{x}) = (-\mathbf{u})^{\mathbf{m}} \partial^{\mathbf{m}} \frac{B_n(\mathbf{a}\mathbf{x})}{(\mathbf{a}|\mathbf{x})}.$$

Then f *is strictly decreasing on* $\mathbf{R}_+^n$, *and satisfies the inequalities* (3.1) *and* (3.2).

Proof. We have

$$f(\mathbf{x}) = \int_{D_n^2} |\mathbf{y}|^{2\mathbf{a}\mathbf{x}-\mathbf{u}} (-2\mathbf{a}\log|\mathbf{y}|)^{\mathbf{m}}\, d\mathbf{y}.$$

The first assertion is proved as in Theorem 3. The last assertion follows from Theorem 6 for $\varphi(\mathbf{x}) = \|\mathbf{x}\|_2$, and

$$d\mu(\mathbf{y}) = (-2\mathbf{a}\log|\mathbf{y}|)^{\mathbf{m}}\, d\mathbf{y}.$$

□

COROLLARY 13. *Let* $\mathbf{a} \in \mathbf{R}_+^n$ *and* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *be defined by*

$$g(t) = (-1)^k \frac{d^k}{dt^k} B_n(t\mathbf{a}).$$

Then g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4), *for every* $k \in \mathbf{N}$ *and* $n \geqslant 2$.

Proof. All the assertions follow from Theorem 9 for $\mathbf{x} = t\mathbf{a}$ and Corollary 7, since

$$g(t) = \frac{1}{2^n\sqrt{n}} \int_{S_n^1} |\mathbf{y}|^{t\mathbf{a}-\mathbf{u}} (\mathbf{a}| - \log|\mathbf{y}|)^k \, dh(\mathbf{y}) > 0.$$

□

COROLLARY 14. *Let* $\mathbf{a} \in \mathbf{R}_+^n$ *and* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *be defined by*

$$g(t) = (-1)^k \frac{d^k}{dt^k} \frac{B_n(t\mathbf{a})}{t}.$$

Then g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4), *for every* $k \in \mathbf{N}$ *and* $n \geqslant 2$.

Proof. All the assertions follow from Theorem 10 for $\mathbf{x} = t\mathbf{u}$ and Corollary 8, since

$$g(t) = (\mathbf{a}|\mathbf{u}) \int_{D_n^2} |\mathbf{y}|^{2t\mathbf{a}-\mathbf{u}} (2\mathbf{a}| - \log|\mathbf{y}|)^k \, d\mathbf{y} > 0.$$

□

COROLLARY 15. *Let* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *be defined by*

$$g(t) = (-1)^k \frac{d^k}{dt^k} \frac{\Gamma(t)^n}{\Gamma(nt)}.$$

Then g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4), *for every* $k \in \mathbf{N}$ *and* $n \geqslant 2$.

Proof. Put $\mathbf{a} = \mathbf{u}$ in Corollary 13. □

COROLLARY 16. *Let* $g : \mathbf{R}_+ \to \mathbf{R}_+$ *be defined by*

$$g(t) = (-1)^k \frac{d^k}{dt^k} \frac{\Gamma(t)^n}{t\Gamma(nt)}.$$

Then g *is strictly decreasing on* $\mathbf{R}_+$, *and satisfies the inequalities* (3.3) *and* (3.4), *for every* $k \in \mathbf{N}$ *and* $n \geqslant 2$.

Proof. Put $\mathbf{a} = \mathbf{u}$ in Corollary 14. □

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