Mathematical
Inequalities
& Applications
Volume 3, Number 4 (2000), 485–496

ON SOME INTEGRAL INEQUALITIES OF OPIAL TYPE

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(*communicated by R. P. Agarwal*)

Abstract. Integral inequalities of the form

$$\int_I s|h|^p|h'|dt \leqslant \int_I r|h'|^{p+1}dt, \qquad h \in H,$$

are derived, where $p > 0$, $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, r and s are given real functions of the variable t, H is the class of functions h, which are absolutely continuous on I and satisfy the integral condition $\int_I r|h'|^{p+1}dt < \infty$, as well as one of the following boundary conditions: $h(\alpha) = 0$ or $h(\beta) = 0$.

In [13], [11], [14] and [12] a uniform method of obtaining and investigating various types of integral inequalities involving a function and its first derivative are introduced. In [13] and [11] Hardy type integral inequalities of the form

$$\int_I s|h|^p\,dt \leqslant \int_I r|h'|^p\,dt, \qquad h \in H \tag{1}$$

$(p = 2$ [13] and $p > 1$ [11]) are obtained. In [14] quadratic Opial type integral inequalities of the form

$$\int_I s|hh'|\,dt \leqslant \int_I rh'^2\,dt, \qquad h \in H \tag{2}$$

are derived and in [12] quadratic integral inequalities of the general form

$$\int_I (rh'^2 + 2shh' + uh^2)\,dt \geqslant 0, \qquad h \in H \tag{3}$$

are derived. Here $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, r, s and u are real functions and H is a class of absolutely continuous functions on I. The method of obtaining these integral inequalities as follows: given any weight function r and an auxiliary function φ the weight function s for the inequalities (1) and (2) is selected or the weight functions s and u for the inequality (3) are selected, such that a suitable differential identity is satisfied. In the case of the inequality (1), the weight function s is determined directly. In the cases of the inequalities (2) and (3), the weight function s or the weight functions

Mathematics subject classification (1991): 26D10.
Key words and phrases: absolutely continuous function, integral inequality, Opial inequality.

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Paper MIA-03-47

s and u satisfy appropriate differential inequalities. Next, for such determined weight functions, the calculated differential identity is then used to construct as wide a class H as possible of functions h, for which the integral inequalities considered hold.

In this paper, we will generalize the above method of obtaining integral inequalities in the case of Opial type integral inequalities of the form

$$\int_I s|h|^p|h'|\,dt \leqslant \int_I r|h'|^{p+1}\,dt, \qquad h \in H, \tag{4}$$

where $p > 0$. The method allows us, for a given weight function r and an auxiliary function φ, to determine the weight function s as the solution of the appropriate differential equation. Then we can directly calculate the auxiliary function v and use these functions to construct the class H of functions h, for which the inequality (4) holds.

Integral inequalities of the form (4) have also been obtained by other methods (see notably Boyd and Wong [9] and Beesack [5], for an extensive bibliography see the books of Mitrinović, Pečarić and Fink [16] and Agarwal and Pang [2]). Numerous authors have studied generalizations and extensions of such integral inequalities, notably Shum [20], Sinnamon [21], Agarwal and Pang [3], Bloom [7]. These inequalities are obtained as particular cases of Opial-type integral inequalities involving higher order derivatives (see Li [15], Agarwal [1], Pachpatte [18], Bloom [7]).

Let p be any positive real number and let $I = (\alpha,\beta), -\infty \leqslant \alpha < \beta \leqslant \infty$, be an arbitrary open interval. We denote the class of absolutely continuous real functions on I by $AC(I)$. Let $\mathscr{A}$ denote the set of the pairs of functions (r,φ), such that $r \in AC(I)$ and $\varphi \in AC(I)$, $r > 0$ and $\varphi > 0$ on I and $r|\varphi'|^p \operatorname{sgn}\varphi' \in AC(I)$.

Let $(r,\varphi) \in \mathscr{A}$ and let $s \in AC(I)$ be an arbitrary function satisfying the differential equation

$$\varphi^p s' - (p+1)(\, r|\varphi'|^p \operatorname{sgn}\varphi'\,)' = 0 \tag{5}$$

almost everywhere on I. We also assume

$$v = r|\varphi'|^p \operatorname{sgn}\varphi'\,\varphi^{-p} - (p+1)^{-1}s \tag{6}$$

on I. These assumptions imply that $v \in AC(I)$, since $\varphi^{-p} \in AC(I)$ ($\varphi \in AC(I)$, $\varphi > 0$ on I and $p > 0$).

Let $\hat{H}$ denote the class of functions $h \in AC(I)$ such that $h \geqslant 0$ on I and

$$\int_I r|h'|^{p+1}\,dt < \infty, \tag{7}$$

$$\int_I sh^p h'\,dt > -\infty, \tag{8}$$

$$\liminf_{t\to\alpha} vh^{p+1} < \infty, \qquad \limsup_{t\to\beta} vh^{p+1} > -\infty. \tag{9}$$

THEOREM 1. *Let* $(r,\varphi) \in \mathscr{A}$.
For every function $h \in \hat{H}$ *both limits in (9) are proper and finite, and*

$$\int_I sh^p h'\,dt + \lim_{t\to\beta} vh^{p+1} - \lim_{t\to\alpha} vh^{p+1} \leqslant \int_I r|h'|^{p+1}\,dt. \tag{10}$$

If $h \not\equiv 0$, *then equality holds in (10), if and only if* $\varphi \in \hat{H}$ *and* $h = c\varphi$, *where* $c = const > 0$.

Proof. Let $(r,\varphi) \in \mathscr{A}$ and let $h \in AC(I)$ and $h \geqslant 0$ on I. We assume

$$g = r|h'|^{p+1} + (\,r|\varphi'|^p \operatorname{sgn}\varphi'\,)'\varphi^{-p}h^{p+1} - (\,r|\varphi'|^p \operatorname{sgn}\varphi'\,\varphi^{-p}h^{p+1})' \tag{11}$$

almost everywhere on I. It follows from Lemma 1 in [11] that $g \geqslant 0$ a.e. on I. $g = 0$ a.e. on I, if and only if $h = c\varphi$, where $c = \text{const} \geqslant 0$. In view of (5) we have

$$\begin{aligned}(\,r|\varphi'|^p \operatorname{sgn}\varphi'\,)'\varphi^{-p}h^{p+1} &= (p+1)^{-1}s'h^{p+1} \\ &= [(p+1)^{-1}sh^{p+1}]' - sh^p h' \end{aligned} \tag{12}$$

a.e. on I. By virtue of (6) and (12) and from the equality (11) we get the following identity

$$r|h'|^{p+1} = sh^p h' + (vh^{p+1})' + g\,, \tag{13}$$

which is valid almost everywhere on I.

Now let $h \in \hat{H}$. The condition (7) implies that the function $r|h'|^{p+1}$ is summable on I since $r|h'|^{p+1} \geqslant 0$ on I. It follows from our assumptions, that the functions $sh^p h'$ and $(vh^{p+1})'$ are summable on each compact interval $[a,b] \subset I$. Thus by (13) we get the summability of the function g on $[a,b] \subset I$ and we obtain the equality

$$\int_a^b r|h'|^{p+1}\,dt = \int_a^b sh^p h'\,dt + vh^{p+1}\,|_a^b + \int_a^b g\,dt. \tag{14}$$

for arbitrary $\alpha < a < b < \beta$. In view of (9) there exist two sequences (a_n) and (b_n), such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty} vh^{p+1}\,|_{a_n} < \infty \qquad , \qquad \lim_{n\to\infty} (-vh^{p+1})\,|_{b_n} < \infty.$$

Thus there is a constant C, such that

$$(-vh^{p+1})\,|_{a_n}^{b_n} \leqslant C < \infty.$$

By virtue of the condition $g \geqslant 0$ a.e. on I and from the equality (14) we infer that

$$\int_{a_n}^{b_n} sh^p h'\,dt \leqslant \int_{a_n}^{b_n} r|h'|^{p+1}\,dt + C \leqslant \int_I r|h'|^{p+1}\,dt + C$$

and from this letting $n \to \infty$, we obtain

$$\int_I sh^p h'\,dt \leqslant \int_I r|h'|^{p+1}\,dt + C < \infty.$$

Using this estimate and the condition (8), we conclude that the function $sh^{p+1}h'$ is summable on I. Next, in a similar way, using (14) and the summability of the functions $r|h'|^{p+1}$ and $sh^p h'$ on I, we prove that the function g is summable on I. Thus all the integrals in the equality (14) have finite limits as $a \to \alpha$ or $b \to \beta$, and hence both of the limits in (9) are proper and finite. Now by (14), as $a \to \alpha$ and $b \to \beta$, we obtain the equality

$$\int_I r|h'|^{p+1}dt = \int_I sh^p h' dt + \lim_{t\to\beta} vh^{p+1} - \lim_{t\to\alpha} vh^{p+1} + \int_I g dt, \tag{15}$$

whence the inequality (10) follows, since $g \geqslant 0$ a. e. on I.

If the inequality (10) becomes an equality for a non-vanishing function $h \in \hat{H}$, then by (15) we have $\int_I g dt = 0$. As $g \geqslant 0$ a. e. on I we obtain $g = 0$ a. e. on I. In view of Lemma 1 in [11], $g = 0$ a. e. on I, if and only if $h = c\varphi$, where $c = \text{const} \geqslant 0$. Thus $\varphi \in \hat{H}$ and $h = c\varphi$ with $c > 0$.

Now let $\varphi \in \hat{H}$ and $h = c\varphi$, where $c = \text{const} > 0$. That implies $g = 0$ a. e. on I so that $\int_I g dt = 0$ and, in view of (15), the inequality (10) becomes an equality which completes the proof.

Let $h \in AC(I)$ and $\int_I r|h'|^{p+1}\,dt < \infty$. Using Hölder's inequality we obtain the estimate

$$|h(b) - h(a)| \leqslant \int_a^b |h'|\,dt \leqslant \left(\int_a^b r^{-1/p}\,dt\right)^{p/(p+1)} \left(\int_a^b r|h'|^{p+1}\,dt\right)^{1/(p+1)}, \tag{16}$$

where $\alpha < a < b < \beta$. If $\int_\alpha^t r^{-1/p}\,d\tau < \infty$ for some $t \in I$, then the Cauchy condition for the existence of the limit yields the existence of a proper and finite limit $\lim_{t\to\alpha} h \equiv h(\alpha)$. Furthermore, if $h(\alpha) = 0$ and $v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \alpha$, where v is an arbitrary measurable function on I, then from (16) as $a \to \alpha$ and with $b = t$ we get the estimate

$$0 \leqslant |v|h|^{p+1}| \leqslant \left|v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p\right| \int_\alpha^t r|h'|^{p+1}\,d\tau$$

and hence

$$\lim_{t\to\alpha} v|h|^{p+1} = 0. \tag{17}$$

Similarly, if $\int_t^\beta r^{-1/p}\,d\tau < \infty$, then there exists a proper and finite limit $\lim_{t\to\beta} h \equiv h(\beta)$ and moreover, if $h(\beta) = 0$ and $v(t)\left(\int_t^\beta r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \beta$, then $\lim_{t\to\beta} v|h|^{p+1} = 0$.

From (6) and (5) we have

$$v' = -pr|\varphi'|^{p+1}\varphi^{-(p+1)} \leqslant 0 \tag{18}$$

a. e. on I. Therefore v is a nonincreasing function on I. Hence there exist proper limits $\lim_{t\to\alpha} v \equiv v(\alpha)$ and $\lim_{t\to\beta} v \equiv v(\beta)$. Moreover $v(\alpha) > -\infty$, $v(\beta) < \infty$ and $v(\alpha) \geqslant v \geqslant v(\beta)$ on I.

We will denote by U_α (resp. U_β) some right-hand (resp. left-hand) neighbourhood of the point α (resp. β).

LEMMA 1. *(i) If $sv \geqslant 0$ on U_α and $v(\alpha) \neq 0$, then $\int_\alpha^t r^{-1/p}\,d\tau < \infty$ for some $t \in I$. Moreover, if $v(\alpha) = \infty$, then $v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \alpha$.*

(ii) If $sv \geqslant 0$ on U_β and $v(\beta) \neq 0$, then $\int_t^\beta r^{-1/p}\,d\tau < \infty$ for some $t \in I$. Moreover, if $v(\beta) = -\infty$, then $v(t)\left(\int_t^\beta r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \beta$.

Proof. We prove the lemma only for the point α. The proof for β is analogous.

Let $v(\alpha) \neq 0$ and consider some right-hand neighbourhood $U \subset U_\alpha$ of α, such that $v \neq 0$ on U.

Let $v > 0$ on U. Then $s \geqslant 0$ on U and by (6) we obtain $v \leqslant r|\varphi'|^p \operatorname{sgn}\varphi'\,\varphi^{-p}$ on U. Hence $\varphi' > 0$ on U, because $r > 0$ and $\varphi > 0$ on I. Thus we have

$$v \leqslant r|\varphi'|^p\varphi^{-p} \tag{19}$$

on U. By virtue of (18) and (19) we get

$$v' \leqslant -pr^{-1/p}v^{(p+1)/p}$$

a.e. on U. Thus $r^{-1/p} \leqslant -p^{-1}v^{-(p+1)/p}v'$ a.e. on U and we obtain the estimate

$$\int_a^t r^{-1/p}\,d\tau \leqslant -p^{-1}\int_a^t v^{-(p+1)/p}v'\,d\tau = [v(t)]^{-1/p} - [v(a)]^{-1/p} < [v(t)]^{-1/p}$$

for $\alpha < a < t < \beta$ on U. Hence as $a \to \alpha$ we obtain

$$\int_\alpha^t r^{-1/p}\,d\tau \leqslant [v(t)]^{-1/p} < \infty. \tag{20}$$

If $v(\alpha) = \infty$, then from (20) we get

$$0 < v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p \leqslant 1$$

for $t \in U$ and thus $v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \alpha$.

Let $v < 0$ on U. Then $s \leqslant 0$ on U and by (6) we obtain $v \geqslant r|\varphi'|^p \operatorname{sgn}\varphi'\,\varphi^{-p}$ on U. Hence $\varphi' < 0$ on U and

$$v \geqslant -r|\varphi'|^p\varphi^{-p} \tag{21}$$

on U. Similarly, by virtue of (18) and (21), we get the estimate

$$\int_a^t r^{-1/p}\,d\tau \leqslant -p^{-1}\int_a^t |v|^{-(p+1)/p}v'\,d\tau = |v(a)|^{-1/p} - |v(t)|^{-1/p} < [v(a)]^{-1/p}$$

for $\alpha < a < t < \beta$ on U. Letting $a \to \alpha$ shows that

$$\int_\alpha^t r^{-1/p}\,d\tau \leqslant [v(\alpha)]^{-1/p} < \infty.$$

We denote by H_0 (resp. H^0) the class of functions $h \in AC(I)$ satisfying the integral condition (7) and the limit condition

$$\liminf_{t\to\alpha} |h| = 0 \quad (\text{resp.} \liminf_{t\to\beta} |h| = 0). \tag{22}$$

In the cases considered hereafter, the condition (22) is equivalent to

$$h(\alpha) = 0 \quad (\text{resp.}\, h(\beta) = 0). \tag{23}$$

THEOREM 2. *Let* $(r,\varphi) \in \mathscr{A}$.

(i) If $s \geqslant 0$ *on* I *and* $v(\beta) \geqslant 0$ *then*

$$\int_I s|h|^p|h'|\,dt + \lim_{t\to\beta} v|h|^{p+1} \leqslant \int_I r|h'|^{p+1}\,dt \tag{24}$$

for every function $h \in H_0$.

Moreover, if $v(\beta) > 0$, *then there exists a finite limit value* $h(\beta)$ *and (24) takes the form*

$$\int_I s|h|^p|h'|\,dt + v(\beta)|h(\beta)|^{p+1} \leqslant \int_I r|h'|^{p+1}\,dt \tag{25}$$

If $s > 0$ *on* I *and* $h \not\equiv 0$, *then equality holds in (24), if and only if* $\varphi \in H_0$ *and* $h = c\varphi$, *where* $c = const \neq 0$.

(ii) If $s \leqslant 0$ *on* I *and* $v(\alpha) \leqslant 0$, *then*

$$\int_I |s||h|^p|h'|\,dt - \lim_{t\to\alpha} v|h|^{p+1} \leqslant \int_I r|h'|^{p+1}\,dt \tag{26}$$

for every function $h \in H^0$.

Moreover, if $v(\alpha) < 0$, *then there exists a finite limit value* $h(\alpha)$ *and (26) takes the form*

$$\int_I |s||h|^p|h'|\,dt - v(\alpha)|h(\alpha)|^{p+1} \leqslant \int_I r|h'|^{p+1}\,dt. \tag{27}$$

If $s < 0$ *on* I *and* $h \not\equiv 0$, *then equality holds in (26), if and only if* $\varphi \in H^0$ *and* $h = c\varphi$, *where* $c = const \neq 0$.

Proof. (i) Let $v(\beta) \geqslant 0$ and $s \geqslant 0$ on I. Then $v \geqslant 0$ on I and $\limsup_{t\to\beta} v|h|^{p+1} \geqslant 0$ for every $h \in AC(I)$. If $v(\alpha) = 0$, then $v \equiv 0$ and $s \equiv 0$ on I and it is trivial to show (24) holds. If $v(\alpha) > 0$, then by Lemma 1 (i) we have $\int_\alpha^t r^{-1/p}\,d\tau < \infty$ for some $t \in I$ and $v(t)\left(\int_\alpha^t r^{-1/p}\,d\tau\right)^p = O(1)$ as $t \to \alpha$, since $sv \geqslant 0$ on I and $v(\alpha) > 0$. Thus, if $h \in H_0$, then there exists a finite limit value $h(\alpha)$ and for every $h \in H_0$ we have $\lim_{t\to\alpha} v|h|^{p+1} = 0$.

Furthermore, let $h_+ \in H_0$ be such that $h_+ \geqslant 0$ on I and $h'_+ \geqslant 0$ a.e. on I. Then $\int sh_+^p h'_+ \geqslant 0$ and so the condition (8) is satisfied. It follows from previous cosiderations that $\lim_{t\to\alpha} vh_+^{p+1} = 0$ and $\limsup_{t\to\beta} vh_+^{p+1} \geqslant 0$ and so h_+ satisfies (9). Thus $h_+ \in \hat{H}$ and by Theorem 1 we get

$$\int_I sh_+^p h'_+\,dt + \lim_{t\to\beta} vh_+^{p+1} \leqslant \int_I rh'^{\,p+1}_+\,dt. \tag{28}$$

Now, let $h \in H_0$ and set $h_+ = \int_\alpha^t |h'|\,d\tau$ on I. Then $h_+ \in AC(I)$, $h_+(\alpha) = 0$, $h_+ \geqslant 0$ on I, $h'_+ = |h'| \geqslant 0$ a. e. on I and

$$\int_I r {h'_+}^{p+1}\,dt = \int_I r|h'|^{p+1}\,dt < \infty. \tag{29}$$

Thus $h_+ \in H_0$ and satisfies the inequality (28). Notice that

$$|h| = \left|\int_\alpha^t h'\,d\tau\right| \leqslant \int_\alpha^t |h'|\,d\tau = h_+$$

on I. Hence we obtain

$$\int_I s|h|^p|h'|\,dt + \lim_{t\to\beta} v|h|^{p+1} \leqslant \int_I s h_+^p h'_+\,dt + \lim_{t\to\beta} v h_+^{p+1}, \tag{30}$$

since $s \geqslant 0$ and $v \geqslant 0$ on I. By virtue of (28) - (30) we get the inequality (24).

If $v(\beta) > 0$, then by Lemma 1 (ii) we obtain $\int_t^\beta r^{-1/p}\,d\tau < \infty$ and there exists a finite limit value $h(\beta)$. Hence the inequality (24) takes the form of the inequality (25).

Finally, let $s > 0$ on I. Then by virtue of (6) we have $\varphi' > 0$ on I, since $v \geqslant 0$ on I.

If both sides of (24) are equal for some non-vanishing function $h \in H_0$, then by (28) – (30) it follows that for $h_+ = \int_\alpha^t |h'|\,d\tau$ equalities hold in (28) and (30). Since equality holds in (28), Theorem 1 now yields $h_+ = c\varphi$, where $c =$ const > 0 and $\varphi \in \hat{H}$. Hence it follows that $\varphi \in H_0$. Furthermore, we have

$$\int_I s|h|^p|h'|\,dt = \int_I s h_+^p h'_+\,dt,$$

because equality holds in (30), and $s > 0$, $v \geqslant 0$, $|h| \leqslant h_+$ and $|h'| = h'_+$ on I. Therefore

$$\int_I s\varphi'[|h|^p - (c\varphi)^p]\,dt = 0$$

and we obtain $s\varphi'[|h|^p - (c\varphi)^p] = 0$ a. e. on I. From this it follows that $|h| = c\varphi$ on I, since $s > 0$ and $\varphi' > 0$ on I. We thus get $h = c\varphi$, where $c =$ const $\neq 0$, since $h \in AC(I)$ and $\varphi > 0$ on I.

Conversely, if $h = c\varphi$, where $c =$ const $\neq 0$ and $\varphi \in H_0$, then it follows from previous considerations that $\lim_{t\to\alpha} v|h|^{p+1} = 0$ and $\varphi \in \hat{H}$. Thus by Theorem 1 it follows that for the function $h_+ = |c|\varphi$ equality holds in (10). Hence the inequality (24) becomes an equality for h since $h_+ = |h|$ and $h'_+ = |h'|$.

From Theorem 2 we directly obtain, for any weight function s satisfying the conditions $s \geqslant 0$ on I and $v(\beta) \geqslant 0$ or $s \leqslant 0$ on I and $v(\alpha) \leqslant 0$, Opial type integral inequalities of the form

$$\int_I s|h|^p|h'|\,dt \leqslant \int_I r|h'|^{p+1}\,dt, \qquad h \in H, \tag{31}$$

where $H = H_0$ if $s \geqslant 0$ on I and $v(\beta) \geqslant 0$, or $H = H^0$ if $s \leqslant 0$ on I and $v(\alpha) \leqslant 0$.

Inequality (31) was derived by Beesack (Theorem 4.1 in [5]) with some additional assumptions.

Now we determine an optimal or near optimal weight function s in the inequality (31).

Let $(r,\varphi)\in\mathscr{A}$ and let $s\in AC(I)$ satisfy the differential equation (5). Then

$$s(t)=s_0+(p+1)\int_{t_0}^{t}(\,r|\varphi'|^p\operatorname{sgn}\varphi'\,)'\varphi^{-p}\,d\tau,\qquad t\in I, \tag{32}$$

where $t_0\in I$ and s_0 is an arbitrary real number. From (6) and (32) integrating by parts we obtain

$$v(t)=-(p+1)^{-1}s_0+(\,r|\varphi'|^p\operatorname{sgn}\varphi'\,\varphi^{-p})(t_0)-p\int_{t_0}^{t}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau \tag{33}$$

for $t\in I$.

If the function s satisfies the conditions $s\geqslant 0$ on I and $v(\beta)\geqslant 0$, then $v\geqslant 0$ on I, since v is nonincreasing on I, also it follows from (6) that $\varphi'\geqslant 0$ on I. Moreover, by virtue of (33), from the condition $v(\beta)\geqslant 0$ we obtain

$$\int_{t_0}^{\beta}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau<\infty.$$

Now let r and φ be the functions such that $\varphi'\geqslant 0$ on I,

$$\int_{t}^{\beta}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau<\infty$$

for some $t\in I$ and

$$r|\varphi'|^p\varphi^{-p}-p\int_{t}^{\beta}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau\geqslant 0$$

on I. Further let us set

$$\hat{s}=(p+1)(r|\varphi'|^p\varphi^{-p}-p\int_{t}^{\beta}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau) \tag{34}$$

on I. The function $\hat{s}$ satisfies the differential equation (5) as well as the conditions $\hat{s}\geqslant 0$ on I and $\hat{v}(\beta)=0$. If $s\in AC(I)$ is an arbitrary weight function such that $s\geqslant 0$ on I and $v(\beta)\geqslant 0$, then, by (32), (33) and (34), after simple calculations we get

$$\hat{s}(t)-s(t)=v(\beta)\geqslant 0$$

for $t\in I$. Therefore the weight function $\hat{s}$ is the maximum weight function in the class of weight functions satisfying the conditions $s\geqslant 0$ on I and $v(\beta)\geqslant 0$.

In an analogous way we show that if $\varphi'\leqslant 0$ on I, together with

$$\int_{\alpha}^{t}r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau<\infty$$

for some $t \in I$ and

$$r|\varphi'|^p\varphi^{-p} - p\int_\alpha^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau \geqslant 0$$

on I, then the function

$$\tilde{s} = (p+1)(r|\varphi'|^p\varphi^{-p} - p\int_\alpha^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau) \geqslant 0$$

is the maximum weight function in the class of weight functions $|s|$ such that $s \leqslant 0$ on I and $v(\alpha) \leqslant 0$, e.g.

$$\tilde{s}(t) - |s(t)| = |v(\alpha)| \geqslant 0$$

for $t \in I$.

From the above considerations and Theorem 2 we directly obtain:

THEOREM 3. *Let* $(r,\varphi) \in \mathscr{A}$.

(i) If $\varphi' \geqslant 0$ *on* I*, together with* $\int_t^\beta r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau < \infty$ *for some* $t \in I$ *and*

$$s = (p+1)(r|\varphi'|^p\varphi^{-p} - p\int_t^\beta r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau) \geqslant 0 \tag{35}$$

on I*, then*

$$\int_I s|h|^p|h'|\,dt \leqslant \int_I r|h'|^{p+1}\,dt \tag{36}$$

for every function $h \in AC(I)$ *such that* $h(\alpha) = 0$ *and* $\int_I r|h'|^{p+1}\,dt < \infty$.

If $s > 0$ *on* I *and* $h \not\equiv 0$*, then equality holds in (36), if and only if* $\varphi(\alpha) = 0$, $\int_I r|\varphi'|^{p+1}\,dt < \infty$ *and, moreover,*

$$\lim_{t\to\beta}\varphi^{p+1}\int_t^\beta r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau = 0$$

provided $\varphi(\beta) = \infty$ *and* $h = c\varphi$*, where* $c = const \neq 0$.

(ii) If $\varphi' \leqslant 0$ *on* I, $\int_\alpha^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau < \infty$ *for some* $t \in I$ *and*

$$s = (p+1)(r|\varphi'|^p\varphi^{-p} - p\int_\alpha^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau) \geqslant 0 \tag{37}$$

on I*, then for every function* $h \in AC(I)$ *such that* $h(\beta) = 0$ *and*

$$\int_I r|h'|^{p+1}\,dt < \infty$$

the inequality (36) holds.

If $s > 0$ *on* I *and* $h \not\equiv 0$*, then equality holds in (36), if and only if* $\varphi(\beta) = 0$, $\int_I r|\varphi'|^{p+1}\,dt < \infty$ *and, moreover,*

$$\lim_{t\to\alpha}\varphi^{p+1}\int_\alpha^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau = 0$$

provided $\varphi(\alpha)=\infty$ *and* $h=c\varphi$, *where* $c=const\neq 0$.

EXAMPLE 1. Take $I=(0,\beta)$, $0<\beta\leqslant\infty$, and let r be an arbitrary function absolutely continuous on I such that $r>0$ on I and $\int_0^\beta r^{-1/p}<\infty$.

Let $\varphi=\int_0^t r^{-1/p}\,d\tau$ on I. Then $\varphi'=r^{-1/p}>0$ on I and from (35) we get

$$\begin{aligned} s &= (p+1)\left\{\left(\int_0^t r^{-1/p}d\tau\right)^{-p}-p\int_t^\beta\left[r^{-1/p}\left(\int_0^\sigma r^{-1/p}d\tau\right)^{-(p+1)}\right]d\sigma\right\}\\ &= (p+1)\left\{\left(\int_0^t r^{-1/p}d\tau\right)^{-p}+\int_t^\beta\left[\left(\int_0^\sigma r^{-1/p}d\tau\right)^{-p}\right]'d\sigma\right\}\\ &= (p+1)\left(\int_0^\beta r^{-1/p}\,d\tau\right)^{-p}>0. \end{aligned}$$

Simultaneously, we have $\varphi(\alpha)=0$ and

$$\int_0^\beta r|\varphi'|^{p+1}\,dt=\int_0^\beta r^{-1/p}\,dt<\infty.$$

Now let $\varphi=\int_t^\beta r^{-1/p}\,d\tau$ on I. Then $\varphi'=-r^{-1/p}<0$ on I and, in a similar way, by virtue of (37) we obtain

$$s=(p+1)\left(\int_0^\beta r^{-1/p}\,d\tau\right)^{-p}>0.$$

We also have $\varphi(\beta)=0$ and $\int_0^\beta r|\varphi'|^{p+1}\,dt<\infty.$

Now, applying Theorem 3 we get:

For an arbitrary $p>0$ *and every function* h *absolutely continuous on the interval* $(0,\beta)$, $0<\beta\leqslant\infty$, *satisfying the integral condition*

$$\int_0^\beta r|h'|^{p+1}\,dt<\infty$$

and one of the limit conditions $h(0)=0$ *or* $h(\beta)=0$ *the inequality*

$$\int_0^\beta |h|^p|h'|\,dt\leqslant\frac{1}{p+1}\left(\int_0^\beta r^{-1/p}\,dt\right)^p\int_0^\beta |h'|^{p+1}\,dt \tag{38}$$

holds.

The inequality (38) becomes an equality, if and only if $h=c\int_0^t r^{-1/p}\,d\tau$ or $h=c\int_t^\beta r^{-1/p}\,d\tau$, where c is an arbitrary constant (cf. Beesack [4], Calvert [10], Beesack and Das [6]).

In the case $0<\beta<\infty$ and $r=1$ on $(0,\beta)$ we obtain the inequality

$$\int_0^\beta |h|^p|h'|\,dt\leqslant\frac{\beta^p}{p+1}\int_0^\beta |h'|^{p+1}\,dt, \tag{39}$$

which holds for all $h \in AC((0,\beta))$, such that $\int_0^\beta |h'|^{p+1}\,dt < \infty$ and either $h(0)=0$ or $h(\beta)=0$. Equality holds in (39), if and only if $h=ct$ or $h=c(\beta-t)$, where $c=\text{const}$ (cf. Yang [23], Wong [22], Boyd [8], Shum [19]).

If $p=1$, then the inequality (39) becomes the original Opial Inequality ([17]).

EXAMPLE 2. Let $I=(0,\infty)$. Let $r=t^a$ and $\varphi=t^k$ on I, where $a\neq p$ and $k\neq 0$ are arbitrary constants. Then we have $\varphi'>0$, if $k>0$ and $\varphi'<0$, if $k<0$. $\int_t^\infty r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau<\infty$, if $a<p$ and $\int_0^t r|\varphi'|^{p+1}\varphi^{-(p+1)}\,d\tau<\infty$, if $a>p$. Futhermore from (35) or (37) we get

$$s=(p+1)\left(|k|^p-\frac{p}{|a-p|}|k|^{p+1}\right)t^{a-p}$$

on I. The expression $|k|^p-\frac{p}{|a-p|}|k|^{p+1}$ takes its maximum value when $|k|=\frac{|a-p|}{p+1}$ and then

$$s=\left(\frac{|a-p|}{p+1}\right)^p t^{a-p}>0.$$

Now, applying Theorem 3 we infer the following:

For arbitrary $p>0$ and $a\neq p$ and for every function $h\in AC((0,\infty))$ and $h\not\equiv 0$ satisfying the integral condition

$$\int_0^\infty t^a|h'|^{p+1}\,dt<\infty$$

and the limit condition $h(0)=0$, if $a<p$ and $h(\infty)=0$, if $a>p$, the inequality

$$\left(\frac{|a-p|}{p+1}\right)^p\int_0^\infty t^{a-p}|h|^p|h'|\,dt<\int_0^\infty t^a|h'|^{p+1}\,dt \tag{40}$$

is valid.

The inequality (40) is some new Opial type inequality which is a homologue of the well-known Hardy Integral Inequality ([16]).

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(Received January 8, 2000)

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