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MINMAX PROBLEMS FOR FRACTIONAL PARTS OF REAL NUMBERS

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Abstract. The view-obstruction problem for the n-dimensional cube with side 1 can be interpreted as the problem of evaluating the function $\kappa(n) = \inf \max_{0\leqslant x\leqslant 1} \min_{1\leqslant k\leqslant n} \| v_k x \|$, where the infimum is taken over all n-tuples $v_1, \cdots, v_n$ of positive integers. So the following could perhaps be called "generalized view-obstruction problems": given a periodic function ϕ, an interval I and a set of integers $\mathscr{S}$, find

$$(i)\ \ \min_{x\in I}\max_{s\in\mathscr{S}} \phi(sx), \qquad (ii)\ \ \max_{x\in I}\min_{s\in\mathscr{S}} \phi(sx).$$

We study minmax problems of this nature where

$$\phi(x) = \{x\}^{\alpha}(1-\{x\}) \text{ and } \left\{(x-1/2)^{\alpha}\right\},$$

and

$$I = [0,1],\ \ \mathscr{S} = \{1, \cdots, N\}.$$

Here $\{x\}$ denotes the fractional part of x, and $N \geqslant 2$ and $\alpha \geqslant 1$ are integers.

1. Introduction

In 1973, T. W. Cusick introduced a problem in geometry of numbers which he called the "view-obstruction problem". This problem in its original form has attracted sustained interest over the years. The view-obstruction problem for the n-dimensional cube with side 1 can be interpreted as the problem of evaluating the function

$$\kappa(n) = \inf \max_{0\leqslant x\leqslant 1} \min_{1\leqslant k\leqslant n} \| v_k x \|,$$

where the infimum is taken over all n-tuple $v_1, \cdots, v_n$ of positive integers (see [1], [2]; the function $\kappa(n)$ was already introduced by Wills [3]). This view-obstruction problem has a geometrical interpretation (see [1]) which motivates its name. The answer of the view-obstruction problem for the n-dimensional cube with side 1 is in fact $1-2\kappa(n)$, $n \geqslant 2$. Hence the following could be perhaps called "generalized view-obstruction problems": given a periodic function ϕ, an interval I and a set of integers $\mathscr{S}$, find

$$(i)\ \ \min_{x\in I}\max_{s\in\mathscr{S}} \phi(sx), \qquad (ii)\ \ \max_{x\in I}\min_{s\in\mathscr{S}} \phi(sx). \tag{1.1}$$

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In this paper, we will study minmax problems of this nature where

$$\phi(x)=\{x\}^{\alpha}(1-\{x\}) \text{ and } \left\{(x-1/2)^{\alpha}\right\}$$

and

$$I=[0,1],\ \mathscr{S}=\{1,\cdots,N\}.$$

Here $\{x\}$ denotes the fractional part of x, and $N\geqslant 2$ and $\alpha\geqslant 1$ are integers. First, we have

$$\min_{0\leqslant x\leqslant 1}\max_{1\leqslant k\leqslant N}\left(\{kx\}^{\alpha}(1-\{kx\})\right)=0, \tag{1.1}$$

$$t_{\alpha}(N):=\max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}\left\{(kx-1/2)^{\alpha}\right\}=(1/2)^{\alpha}\quad(\alpha\text{ even}), \tag{1.2}$$

since, for (1.1), at $x=0$, $\{kx\}^{\alpha}(1-\{kx\})=0$ for all k, $1\leqslant k\leqslant N$, and for (1.2),

$$\max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}\left\{(kx-1/2)^{\alpha}\right\}\geqslant\min_{1\leqslant k\leqslant N}\left\{(k\cdot 1-1/2)^{\alpha}\right\}=(1/2)^{\alpha}$$

and

$$\max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}\{(kx-1/2)^{\alpha}\}\leqslant\max_{0\leqslant x\leqslant 1}\{(x-1/2)^{\alpha}\}=(1/2)^{\alpha}.$$

However if we switch min and max in (1.1) and (1.2), the extremal values are not so obvious. The main purpose of this paper will be the study of the following minmax problems:

$$\begin{aligned} r_{\alpha}(N) &:= \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}\left(\{kx\}^{\alpha}(1-\{kx\}\right), \\ s_{\alpha}(N) &:= \min_{0\leqslant x\leqslant 1}\max_{1\leqslant k\leqslant N}\left\{(kx-1/2)^{\alpha}\right\}\quad(\alpha\text{ even}), \\ t_{\alpha}(N) &= \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}\left\{(kx-1/2)^{\alpha}\right\}\quad(\alpha\text{ odd}). \end{aligned}$$

Let $N\geqslant 2$ and $\alpha\geqslant 1$ be integers. In Section 2, we will obtain the extremal values of $r_1(N)$, $s_1(N)$ and $t_{\alpha}(N)$ (α odd).

In Section 3, for

$$\begin{cases} N=2,3,4, & \alpha\geqslant 1, \\ N\geqslant 5, & \alpha\geqslant\frac{\log N}{\log(2+2/(N-3))}, \end{cases}$$

we will show that, as $N\to\infty$,

$$r_{\alpha}(N)=\frac{N\left(\frac{N-1}{N^{1/\alpha}-1}\right)^{\alpha}}{\left(\frac{N^{(1+\alpha)/\alpha}-1}{N^{1/\alpha}-1}\right)^{\alpha+1}}\sim\frac{1}{N}$$

and that this extremal value is attained at

$$x=\frac{N^{1/\alpha}(N-1)}{N^{(\alpha+1)/\alpha}-1}=1-\frac{1}{N}+O(N^{-1-1/\alpha}).$$

Suppose that α is even. Then, in Section 4, we will show that if certain conditions are satisfied then we have

$$s_\alpha(N) = (1/2 - 1/(N+1))^\alpha$$

and that this extremal value is attained at $x = 1/(N+1)$. The conditions are: for $1 \leqslant m \leqslant \lfloor \left(\frac{3}{2}\right)^\alpha - 1 \rfloor$, there is $k_0 = k_0(m)$ with $3 \leqslant k_0 \leqslant N$ such that

$$2\left(\sqrt[\alpha]{\left\lfloor \left(\frac{k_0}{2}\sqrt[\alpha]{m} + \frac{1}{2}\left(\frac{k_0}{2} - 1\right)\right)^\alpha - \left(\frac{N-1}{2(N+1)}\right)^\alpha \right\rfloor + 1} + \frac{1}{2}\right)$$
$$\geqslant k_0\left(\sqrt[\alpha]{\left(\frac{N-1}{2(N+1)}\right)^\alpha + m} + \frac{1}{2}\right),$$

and

$$\sqrt[\alpha]{\left\lfloor \left(\frac{3}{2}\right)^\alpha - 1 \right\rfloor + 1} \geqslant \frac{3N-1}{2(N+1)}.$$

Moreover, we will see some examples of this.

2. $r_1(N)$, $s_1(N)$ and $t_\alpha(N)$ (α odd)

Let N be an integer $\geqslant 2$. For any real number x, let $\| x \|$ and $\{x\}$ denote the distance from x to the nearest integer and the fractional part of x, respectively. In this section, we will obtain the extremal values of $r_1(N)$, $s_1(N)$ and $t_\alpha(N)$ (α odd). Here is the answer for $r_1(N)$.

PROPOSITION 2.1. *We have*

$$r_1(N) = \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}(\{kx\}(1-\{kx\}) = \frac{1}{N+1}\left(1 - \frac{1}{N+1}\right).$$

Proof. By Dirichlet's box principle,

$$\max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N} \| kx \| = 1/(N+1).$$

Moreover,

$$\min_{0\leqslant x\leqslant 1}\max_{1\leqslant k\leqslant N} \| kx - 1/2 \| = 1/2 - 1/(N+1), \tag{2.1}$$

since

$$\begin{aligned}1/2 - \min_{0\leqslant x\leqslant 1}\max_{1\leqslant k\leqslant N} \| kx - 1/2 \| &= 1/2 + \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}(-\| kx - 1/2 \|)\\ &= \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}(1/2 - \| kx - 1/2 \|)\\ &= \max_{0\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N} \| kx \|\\ &= 1/(N+1).\end{aligned}$$

By the identities

$$\{u\}(1-\{u\}) = \| u \| (1 - \| u \|)$$

and

$$1/2 = \| u \| + \| u + 1/2 \| = \| u \| + \| u - 1/2 \|,$$

we obtain

$$r_1(N) = 1/4 - \min_{0 \leqslant x \leqslant 1} \max_{1 \leqslant k \leqslant N} \| kx - 1/2 \|^2 .$$

It follows that

$$r_1(N) = \frac{1}{N+1}\left(1 - \frac{1}{N+1}\right)$$

and that the extremal values are attained for $x = 1/(N+1)$ and $x = 1 - 1/(N+1)$. □

Now we replace $\| \ \|$ with "$\{\ \}$" in (2.1). Then

PROPOSITION 2.2. *We have*

$$s_1(N) = \min_{0 \leqslant x \leqslant 1} \max_{1 \leqslant k \leqslant N} \{kx - 1/2\} = 1/2 - 1/(2N).$$

Proof. We have

$$\begin{aligned} s_1(N) &\leqslant \max_{1 \leqslant k \leqslant N} \left\{ k\left(\frac{2N-1}{2N}\right) - \frac{1}{2} \right\} = \max_{1 \leqslant k \leqslant N} \left\{ k + \frac{1}{2} - \frac{k}{2N} \right\} \\ &= \max_{1 \leqslant k \leqslant N} \left\{ \frac{1}{2} - \frac{k}{2N} \right\} = \frac{1}{2} - \frac{1}{2N}. \end{aligned}$$

We now prove that $s_1(N) \geqslant 1/2 - 1/(2N)$. Figure 2.1 is the graph of $e(u) = \{u - 1/2\}$.

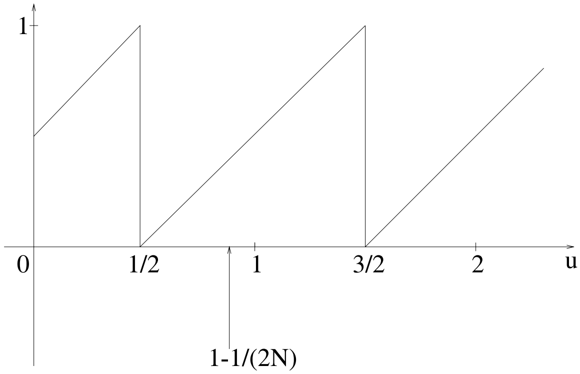

Figure 2.1

Reduce all $kx \pmod 1$ so that they lie in $[1/2, 3/2]$. Note that we are done if there is always a k such that $kx \pmod 1$ equals or lies to the right of $1 - 1/(2N)$. Thus we only consider x such that

$$x,\ 2x,\ 3x,\ \cdots,\ Nx$$

all lie in $[1/2, 1 - 1/(2N))$. Write $x = 1 - \epsilon$, where

$$\frac{1}{2N} < \epsilon \leqslant \frac{1}{2}.$$

Choose k such that

$$k\epsilon \leqslant \frac{1}{2} \quad \text{and} \quad \frac{1}{2} < (k+1)\epsilon.$$

The first inequality is true for $k = 1$ and the second for $k = N - 1$, so we may choose such a k with

$$1 \leqslant k \leqslant N - 1.$$

Now

$$(k+1)x = k + 1 - (k+1)\epsilon$$

and

$$\frac{1}{2} < (k+1)\epsilon = k\epsilon + \epsilon \leqslant \frac{1}{2} + \frac{1}{2} = 1,$$

so

$$(k+1) - 1 \leqslant (k+1)x < (k+1) - 1/2$$

and

$$1 \leqslant (k+1)x \pmod 1 < \frac{3}{2}$$

for an appropriate choice of $(k+1)x \pmod 1$. But since $1 \leqslant k + 1 \leqslant N$, this contradicts the assumption on x. □

The following is the answer for $t_\alpha(N)$ (α odd).

PROPOSITION 2.3. *Let $N \geqslant 2$ and $\alpha \geqslant 1$ be integers and α odd. Then*

$$t_\alpha(N) = \max_{0 \leqslant x \leqslant 1} \min_{1 \leqslant k \leqslant N} \left\{ (kx - 1/2)^\alpha \right\} = 1 - \left(\frac{N-1}{2N} \right)^\alpha .$$

This extremal value is attained at $x = 1/(2N)$.

Proof. We have

$$\begin{aligned} t_\alpha(N) &\geqslant \min_{1 \leqslant k \leqslant N} \left\{ \left(\frac{k}{2N} - \frac{1}{2} \right)^\alpha \right\} = \min_{1 \leqslant k \leqslant N} \left(1 + \left(\frac{k}{2N} - \frac{1}{2} \right)^\alpha \right) \\ &= 1 - \max_{1 \leqslant k \leqslant N} \left(\frac{1}{2} - \frac{k}{2N} \right)^\alpha = 1 - \left(\frac{1}{2} - \frac{1}{2N} \right)^\alpha \\ &= 1 - \left(\frac{N-1}{2N} \right)^\alpha . \end{aligned}$$

Now we prove that $t_\alpha(N) \leqslant 1-((N-1)(2N))^\alpha$. Observe that

$$t_\alpha(N) \leqslant \max\left(\max_{0\leqslant x\leqslant 1/(2N)}\left\{(x-1/2)^\alpha\right\},\ \max_{1/2\leqslant x\leqslant 1}\left\{(x-1/2)^\alpha\right\},\right.$$
$$\left.\max_{0\leqslant j\leqslant N-2}\ \max_{\frac{1}{2(N-j)}\leqslant x\leqslant\frac{1}{2(N-j-1)}}\left\{((N-j)x-1/2)^\alpha\right\}\right).$$

We see that the function $\left\{(x-1/2)^\alpha\right\} = 1+(x-1/2)^\alpha$ and the function

$$\left\{((N-j)x-1/2)^\alpha\right\} = ((N-j)x-1/2)^\alpha$$

are increasing on $[0,1/(2N)]$ and on $[1/(2(N-j)),1/(2(N-j-1))]$, respectively. Also it is clear that

$$\max_{1/2\leqslant x\leqslant 1}\left\{(x-1/2)^\alpha\right\} = (1/2)^\alpha.$$

Hence

$$t_\alpha(N) \leqslant \max\left(1+\left(\frac{1}{2N}-\frac{1}{2}\right)^\alpha,\ \left(\frac{1}{2}\right)^\alpha,\ \max_{0\leqslant j\leqslant N-2}\left(\frac{N-j}{2(N-j-1)}-\frac{1}{2}\right)^\alpha\right).$$

But

$$\max_{0\leqslant j\leqslant N-2}\left(\frac{N-j}{2(N-j-1)}-\frac{1}{2}\right)^\alpha = \left(\frac{N-(N-2)}{2(N-(N-2)-1)}-\frac{1}{2}\right)^\alpha = \left(\frac{1}{2}\right)^\alpha.$$

Since

$$1+\left(\frac{1}{2N}-\frac{1}{2}\right)^\alpha = 1-\left(\frac{N-1}{2N}\right)^\alpha \geqslant 1-\left(\frac{1}{2}\right)^\alpha \geqslant \left(\frac{1}{2}\right)^\alpha,$$

we have $t_\alpha(N) \leqslant 1-((N-1)(2N))^\alpha$. This completes the proof. □

3. $r_\alpha(N)$, where $\alpha \geqslant 2$

Theorem 3.1 is the answer for $r_\alpha(N)$, where $N \geqslant 2$ and α is a function of N. In Figure 3.1, we have plotted the functions involved in the definition of $r_4(4)$.

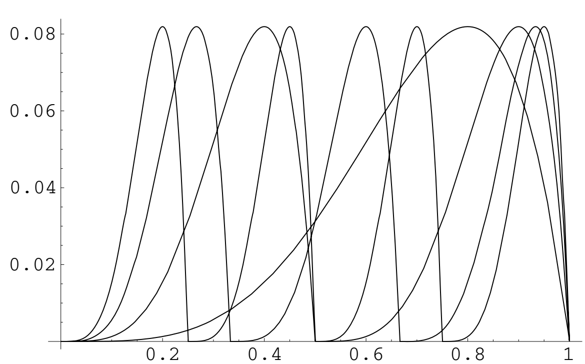

Figure 3.1 $r_4(4)$

THEOREM 3.1. *Let $N \geqslant 2$ and $\alpha \geqslant 1$ be integers. Then, for*

$$\begin{cases} N = 2, 3, 4, & \alpha \geqslant 1, \\ N \geqslant 5, & \alpha \geqslant \frac{\log N}{\log(2+2/(N-3))}, \end{cases}$$

we have

$$r_\alpha(N) = \frac{N\left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha}{\left(\frac{N^{(1+\alpha)/\alpha}-1}{N^{1/\alpha}-1}\right)^{\alpha+1}} \sim \frac{1}{N}. \tag{3.1}$$

This extremal value is attained at

$$x = \frac{N^{1/\alpha}(N-1)}{N^{(\alpha+1)/\alpha}-1} = 1 - \frac{1}{N} + O(N^{-1-1/\alpha}).$$

In order to prove Theorem 3.1, we need several lemmas. For the rest of this section, we let N and α be the same as in Theorem 3.1, and define the function

$$f(k,t;x) := (kx-(k-t))^\alpha(1-(kx-(k-t)))$$

for x in $[(k-t)/k, (k-t+1)/k]$, where k is an integer with $1 \leqslant k \leqslant N$ and $1 \leqslant t \leqslant k$. We observe that $(k-t)/k$ and $(k-t+1)/k$ are zeros of $f(k,t;x)$ and

$$(\{kx\}^\alpha(1-\{kx\})) = f(k,t;x) \quad \text{on} \quad \left(\frac{k-t}{k}, \frac{k-t+1}{k}\right).$$

Denote, for convenience,

$$\Theta(N,\alpha) := \frac{N\left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha}{\left(\frac{N^{(1+\alpha)/\alpha}-1}{N^{1/\alpha}-1}\right)^{\alpha+1}},$$

$$x_{k,t,1} := x_{k,t,1,N,\alpha} = \frac{(k-t+1)\,\eta(N,\alpha) - N}{k\,\eta(N,\alpha)},$$

$$x_{k,t,2} := x_{k,t,2,N,\alpha} = \frac{(k-t+1)\,\eta(N,\alpha) - 1}{k\,\eta(N,\alpha)},$$

where $\eta(N,\alpha) = \frac{N^{(1+\alpha)/\alpha}-1}{N^{1/\alpha}-1}$.

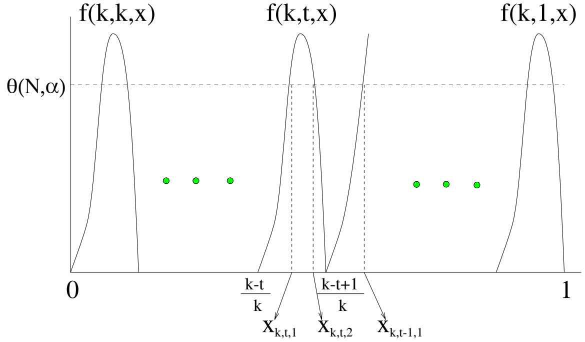

Figure 3.2 $\quad y = (\{kx\}^\alpha(1-\{kx\}))$

LEMMA 3.2. *Let k be an integer with $1 \leqslant k \leqslant N$.*
(a) On the interval $[x_{N,1,1}, 1)$, the equation $x^\alpha(1-x) = (Nx-(N-1))^\alpha(1-(Nx-(N-1)))$ has the real zero,

$$x(N,\alpha) := \frac{N^{1/\alpha}(N-1)}{N^{(\alpha+1)/\alpha}-1}.$$

Also

$$x(N,\alpha)^\alpha(1-x(N,\alpha)) = \Theta(N,\alpha).$$

(b) On each interval $[(k-t)/k, (k-t+1)/k]$, where $1 \leqslant t \leqslant k$, the real zeros of

$$f(k,t;x) = \Theta(N,\alpha). \tag{3.2}$$

are $x_{k,t,1}$ and $x_{k,t,2}$. Moreover, $(k-t)/k < x_{k,t,1} < x_{k,t,2} < (k-t+1)/k$, and $f(k,t;x)$ is

$$\begin{cases} \text{increasing on } \left(\frac{k-t}{k}, x_{k,t,1}\right), \\ \text{decreasing on } \left(x_{k,t,2}, \frac{k-t+1}{k}\right). \end{cases}$$

Proof. (a) We observe that

$$x^\alpha(1-x)-(Nx-(N-1))^\alpha(1-(Nx-(N-1))) = -(x-1)\left(x^\alpha - N(Nx-N+1)^\alpha\right).$$

Since $x = \frac{N^{1/\alpha}(N-1)}{N^{(\alpha+1)/\alpha}-1}$ implies that $x^\alpha = N(Nx-N+1)^\alpha$, (a) holds.
(b) The equation $f'(k,t;x) = -k(kx+t-k)^{\alpha-1}\left(k(\alpha+1)x-\alpha-(\alpha+1)(k-t)\right) = 0$ determines the critical points $x_0 = (k-t)/k$ and

$$x_1 = \frac{\alpha+(\alpha+1)(k-t)}{k(\alpha+1)}.$$

We compute that

$$\begin{aligned} f(k,t,;x_0) &= 0, \\ f(k,t,;x_1) &= \alpha^\alpha/(\alpha+1)^{\alpha+1} > 0. \end{aligned} \tag{3.3}$$

Observe that

$$\frac{df}{dx} = \begin{cases} <0, & x > x_1, \\ >0, & x_0 < x < x_1. \end{cases} \tag{3.4}$$

In order to show the first part, it suffices to show that the equation $f(k,t;x) = \Theta(N,\alpha)$ has two real zeros on $[(k-t)/k, (k-t+1)/k]$. We compute (e.g. by computer algebra) that

$$\begin{aligned} &(kx_{k,t,1}-(k-t))^\alpha(1-(kx_{k,t,1}-(k-t)))-\Theta(N,\alpha) \\ &= \frac{N}{\eta(N,\alpha)^{\alpha+1}}\left(\left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha - \left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha\right) = 0, \end{aligned}$$

$$\begin{aligned} &(kx_{k,t,2}-(k-t))^\alpha(1-(kx_{k,t,2}-(k-t)))-\Theta(N,\alpha) \\ &= \frac{1}{\eta(N,\alpha)^{\alpha+1}}\left(N\left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha - N\left(\frac{N-1}{N^{1/\alpha}-1}\right)^\alpha\right) = 0, \end{aligned}$$

and

$$x_{k,t,1} - \frac{k-t}{k} = \frac{N-1}{k(N^{(1+\alpha)/\alpha}-1)} > 0,$$
$$\frac{k-t+1}{k} - x_{k,t,2} = \frac{N(N^{1/\alpha}-1)}{k(N^{(1+\alpha)/\alpha}-1)} > 0.$$

Note that, by (3.3), $f(k,t;x_1)$ does not depend on k and t. So, by (3.4), for each k and t, the maximum values of $f(k,t;x)$ on $[(k-t)/k, (k-t+1)/k]$ are both equal to $\alpha^\alpha/(\alpha+1)^{\alpha+1}$. It follows from the construction of $x_{k,t,1}$ and $x_{k,t,2}$ that $f(k,t;x)$ is

$$\begin{cases} \text{increasing on } \left(\frac{k-t}{k}, x_{k,t,1}\right) \\ \text{decreasing on } \left(x_{k,t,2}, \frac{k-t+1}{k}\right). \end{cases} \quad \square$$

LEMMA 3.3. *On the interval* $[x_{k,t,2},\, x_{k,t-1,1}]$,

$$(\{kx\}^\alpha(1-\{kx\})) \leqslant \Theta(N,\alpha).$$

Proof. Observe that, by the definition of $x_{k,t,2}$ and $x_{k,t-1,1}$, we have

$$f(k,t;x_{k,t,2}) = f(k,t-1;x_{k,t-1,1}) = \Theta(N,\alpha)$$

and

$$f\left(k,t;\frac{k-t+1}{k}\right) = f\left(k,t-1;\frac{k-t+1}{k}\right) = 0.$$

By (b) of Lemma 3.2, $f(k,t;x)$ is decreasing on $[x_{k,t,2}, (k-t+1)/k]$ and $f(k,t-1;x)$ is increasing on $[(k-t+1)/k, x_{k,t-1,1}]$. This proves the lemma. $\square$

LEMMA 3.4. *For* $2 \leqslant k \leqslant N-1$, *we have*

$$\begin{cases} x_{k,k,2} \leqslant x_{k+1,k,1}, \\ x_{k+1,2,2} \leqslant x_{k,1,1}. \end{cases}$$

Proof. We note that $x_{k,k,2} \leqslant x_{k+1,k,1}$ if and only if $\eta(N,\alpha) \geqslant N-1+\frac{N-2}{k-1}$, and $x_{k+1,2,2} \leqslant x_{k,1,1}$ if and only if $\eta(N,\alpha) \geqslant \left(1+\frac{1}{k}\right)N-1$. The largest possible number among $N-1+\frac{N-2}{k-1}$ and $\left(1+\frac{1}{k}\right)N-1$ is $\max\{2N-3, 3N/2-1\} = 2N-3$ (for $k=2$). But we can compute that $\eta(N,\alpha) \geqslant 2N-3$ if and only if

$$N^{1/\alpha}(N-3) \leqslant 2N-4.$$

This holds for $N=2,3,4$ for any $\alpha \geqslant 1$. But, for $N \geqslant 5$, if $\alpha \geqslant \frac{\log N}{\log(2+2/(N-3))}$, then $N^{1/\alpha}(N-3) \leqslant 2N-4$, i.e. $\eta(N,\alpha) \geqslant 2N-3$. This completes the proof. $\square$

LEMMA 3.5. *We have*

$$\max_{x_{N,N,2}\leqslant x\leqslant x_{N,1,1}} \min_{1\leqslant k\leqslant N} (\{kx\}^{\alpha}(1-\{kx\}) \leqslant \Theta(N,\alpha).$$

Proof. Let u be an integer with $3 \leqslant u \leqslant N$. Then, by Lemma 3.3, for

$$x_{u,u,2} \leqslant x \leqslant x_{u,u-1,1}, \tag{3.5}$$

we have

$$\{ux\}^{\alpha}(1-\{ux\} \leqslant \Theta(N,\alpha).$$

But, by Lemma 3.4,

$$(x_{u,u,2} \leqslant)\ x_{u-1,u-1,2} \leqslant x_{u,u-1,1}. \tag{3.6}$$

So, by (3.5) and (3.6), for

$$x_{u,u,2} \leqslant x \leqslant x_{u-1,u-1,2}, \tag{3.7}$$

we have

$$\{ux\}^{\alpha}(1-\{ux\} \leqslant \Theta(N,\alpha).$$

By Lemma 3.3 again, for

$$x_{u-1,u-1,2} \leqslant x \leqslant x_{u-1,u-2,1}, \tag{3.8}$$

we have

$$\{(u-1)x\}^{\alpha}(1-\{(u-1)x\} \leqslant \Theta(N,\alpha).$$

Hence, by (3.7) and (3.8), we have

$$\max_{x_{u,u,2}\leqslant x\leqslant x_{u-1,u-2,1}} \min_{1\leqslant k\leqslant N} (\{kx\}^{\alpha}(1-\{kx\}) \leqslant \Theta(N,\alpha).$$

Since $x_{u,u,2} \leqslant x_{u-1,u-1,2} \leqslant x_{u-1,u-2,1}$ for each u, by considering those u with $3 \leqslant u \leqslant N$, we find that

$$\max_{x_{N,N,2}\leqslant x\leqslant x_{2,1,1}} \min_{1\leqslant k\leqslant N} (\{kx\}^{\alpha}(1-\{kx\}) \leqslant \Theta(N,\alpha). \tag{3.9}$$

Since, by Lemma 3.4,

$$x_{3,2,2} \leqslant x_{2,1,1}, \tag{3.10}$$

it suffices to consider the interval $[x_{3,2,2}, x_{N,1,1}]$. By Lemma 3.3, for

$$x_{u,2,2} \leqslant x \leqslant x_{u,1,1}, \tag{3.11}$$

we have

$$\{2x\}^{\alpha}(1-\{2x\} \leqslant \Theta(N,\alpha).$$

But Lemma 3.4 again,

$$(x_{u,2,2} \leqslant)\ x_{u+1,2,2} \leqslant x_{u,1,1}. \tag{3.12}$$

So, by (3.11) and (3.12), for

$$x_{u,2,2} \leqslant x \leqslant x_{u+1,2,2}, \tag{3.13}$$

we have

$$\{2x\}^{\alpha}(1-\{2x\}\leqslant \Theta(N,\alpha).$$

By Lemma 3.3 again, for

$$x_{u+1,2,2}\leqslant x\leqslant x_{u+1,1,1}, \tag{3.14}$$

we have

$$\begin{aligned}
&\max_{x_{u,2,2}\leqslant x\leqslant x_{u+1,1,1}}\min_{1\leqslant k\leqslant N}(\{kx\}^{\alpha}(1-\{kx\})\\
\leqslant &\max_{x_{u,u,2}\leqslant x\leqslant x_{u-1,u-2,1}}(\{2x\}^{\alpha}(1-\{2x\})\\
\leqslant &\ \Theta(N,\alpha).
\end{aligned}$$

Since $x_{u,2,2}\leqslant x_{u+1,2,2}\leqslant x_{u+1,1,1}$ for each u, by considering those u with $3\leqslant u\leqslant N$, we find that

$$\max_{x_{3,2,2}\leqslant x\leqslant x_{N,1,1}}\min_{1\leqslant k\leqslant N}(\{kx\}^{\alpha}(1-\{kx\})\leqslant \Theta(N,\alpha). \tag{3.15}$$

By (3.9), (3.10) and (3.15), the result follows. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let $N\geqslant 2$ and $\alpha\geqslant 1$ be integers. Suppose that N and α satisfy the following:

$$\begin{cases} N=2,3,4, & \text{if } \alpha\geqslant 1,\\ N\geqslant 5, & \text{if } \alpha\geqslant \frac{\log N}{\log(2+2/(N-3))}. \end{cases}$$

By Lemma 3.2 (a) (note $x_{N,1,1}=x(N,\alpha)$) and Lemma 3.5, it suffices to show that

$$\max_{0\leqslant x\leqslant x_{N,N,2}}\min_{1\leqslant k\leqslant N}(\{kx\}^{\alpha}(1-\{kx\})\leqslant \Theta(N,\alpha), \tag{a}$$

$$\max_{x_{N,1,1}\leqslant x\leqslant 1}\min_{1\leqslant k\leqslant N}(\{kx\}^{\alpha}(1-\{kx\})\leqslant \Theta(N,\alpha). \tag{b}$$

For (a), we see that

$$\begin{aligned}
&\max_{0\leqslant x\leqslant x_{N,N,2}}\min_{1\leqslant k\leqslant N}(\{kx\}^{\alpha}(1-\{kx\})\\
\leqslant &\max_{0\leqslant x\leqslant x_{N,N,2}}(\{x\}^{\alpha}(1-\{x\})=\max_{0\leqslant x\leqslant x_{N,N,2}}f(1,1;x).
\end{aligned}$$

So it is enough to show that $x_{N,N,2}\leqslant x_{1,1,1}$, since $f(1,1;x)$ is increasing on $[0,x_{1,1,1}]$ and $f(1,1;x_{1,1,1})=\Theta(N,\alpha)$. In fact, we have

$$x_{1,1,1}-x_{N,N,2}=\frac{\eta(N,\alpha)-N}{\eta(N,\alpha)}-\frac{\eta(N,\alpha)-1}{N\,\eta(N,\alpha)}=\frac{(N-1)(N-N^{1/\alpha})}{N(N^{(1+\alpha)/\alpha}-1)}>0,$$

since $\alpha\geqslant 2$. Part (b) is clear, since $f(1,1;x_{1,1,2})=f(1,1;x(N,\alpha))=\Theta(N,\alpha)$ and $f(1,1;x)$ is decreasing on $[x_{1,1,2},1]$. This completes the proof of Theorem 3.1. □

REMARK 3.6. (a) Naturally, we may conjecture that Theorem 3.1 is true for all $\alpha \geqslant 1$ and $N \geqslant 2$. For an integer $\alpha \geqslant 1$,

$$\max_{0 \leqslant x \leqslant 1} x^{\alpha}(1-x) = \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}.$$

If $N = q^{\alpha}$ in (3.1), then

$$r_{\alpha}(q^{\alpha}) = \frac{q^{\alpha}\left(\frac{q^{\alpha}-1}{q-1}\right)^{\alpha}}{\left(\frac{q^{1+\alpha}-1}{q-1}\right)^{\alpha+1}}.$$

Since

$$\lim_{q \to 1} \frac{q^{\alpha}\left(\frac{q^{\alpha}-1}{q-1}\right)^{\alpha}}{\left(\frac{q^{1+\alpha}-1}{q-1}\right)^{\alpha+1}} = \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}},$$

we can say that

$$r_{\alpha}(N) = \max_{0 \leqslant x \leqslant 1} \min_{1 \leqslant k \leqslant N} \left(\{kx\}^{\alpha}(1-\{kx\}\right)$$

is a q-analogue of the problem

$$\max_{0 \leqslant x \leqslant 1} x^{\alpha}(1-x),$$

provided that the conjecture is true.
(b) Let, with the same assumption as in Theorem 3.1,

$$u_{\alpha}(N) := \max_{0 \leqslant x \leqslant 1} \min_{1 \leqslant k \leqslant N} \left(\{kx\}(1-\{kx\})^{\alpha}\right).$$

Then it is obvious that $u_{\alpha}(N) = r_{\alpha}(N)$ and $u_{\alpha}(N)$ is extremal at $x = \frac{N^{1/\alpha}-1}{N^{(1+\alpha)/\alpha}-1}$.

4. $s_{\alpha}(N)$, where $\alpha \geqslant 2$

In this section, we obtain some answers for $s_{\alpha}(N)$ (α positive even integer). In Figure 4.1, we have plotted the functions involved in the definition of $s_2(4)$.

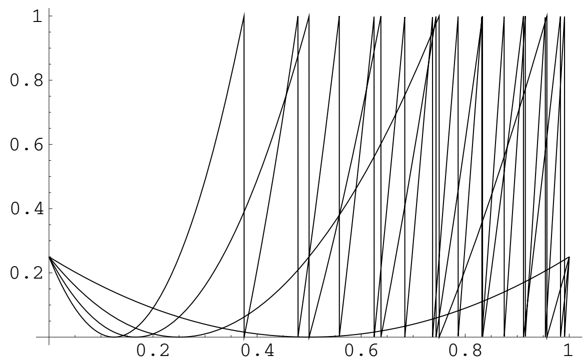

Figure 4.1 $s_2(4)$

PROPOSITION 4.1. *Let $N \geqslant 2$ and α be positive integers, and α even. Then*

$$\min_{0\leqslant x\leqslant 3/4}\max_{1\leqslant k\leqslant N}\{(kx-1/2)^\alpha\} = (1/2-1/(N+1))^\alpha .$$

This extremal value is attained at $x = 1/(N+1)$.

Proof. If $0 \leqslant x \leqslant 3/(2N)$, then $\{(kx-1/2)^\alpha\} = (kx-1/2)^\alpha$ for any $1 \leqslant k \leqslant N$. We partition $[0,3/4]$ into

$$\left[0,\frac{3}{2N}\right] \quad \text{and} \quad \left[\frac{3}{2N},\frac{3}{4}\right].$$

Then it suffices to show that (i) and (ii) below hold.
(i) We have

$$\min_{0\leqslant x\leqslant 3/(2N)}\max_{1\leqslant k\leqslant N}\{(kx-1/2)^\alpha\} = (1/2-1/(N+1))^\alpha .$$

This extremal value is attained at $x = 1/(N+1)$.
(ii) For any integer j, $0 \leqslant j \leqslant N-3$,

$$\min_{\frac{3}{2(N-j)}\leqslant x\leqslant \frac{3}{2(N-j-1)}}\max_{1\leqslant k\leqslant N}\{(kx-1/2)^\alpha\} \geqslant (1/2-1/(N+1))^\alpha .$$

Proof of (i)
If we show that, for $2 \leqslant k \leqslant N$,

$$\left(x-\frac{1}{2}\right)^\alpha \geqslant \left(kx-\frac{1}{2}\right)^\alpha, \tag{4.1}$$

where $0 \leqslant x \leqslant 1/(N+1)$, and, for $1 \leqslant k \leqslant N-1$,

$$\left(Nx-\frac{1}{2}\right)^\alpha \geqslant \left(kx-\frac{1}{2}\right)^\alpha, \tag{4.2}$$

where $1/(N+1) \leqslant x \leqslant 3/(2N)$, then (i) holds. This is because

$$\begin{cases} y = \left(x-\frac{1}{2}\right)^\alpha \text{ is decreasing for } 0 \leqslant x \leqslant \frac{1}{N+1}, \\ y = \left(Nx-\frac{1}{2}\right)^\alpha \text{ is increasing for } \frac{1}{N+1} \leqslant x \leqslant \frac{3}{2N}, \end{cases}$$

and $(x-1/2)^\alpha = (Nx-1/2)^\alpha = (1/2-1/(N+1))^\alpha$ when $x = 1/(N+1)$. To do this, it suffices to do it for the case $\alpha = 2$, since α is even. But

$$\left(x-\frac{1}{2}\right)^2 - \left(kx-\frac{1}{2}\right)^2 = -(k-1)x\left((k+1)x-1\right)$$

and

$$\left(Nx-\frac{1}{2}\right)^2 - \left(kx-\frac{1}{2}\right)^2 = (N-k)x\left((N+k)x-1\right).$$

Hence the extremal value $(1/2-1/(N+1))^{\alpha}$ is attained at $x=1/(N+1)$.
Proof of (ii)
Partition $[3/(2N),3/4]$ as follows:

$$\left[\frac{3}{2N},\frac{3}{2(N-1)}\right],\ \left[\frac{3}{2(N-1)},\ \frac{3}{2(N-2)}\right],\ \dots,\ \left[\frac{3}{6},\frac{3}{4}\right].$$

Let $0\leqslant j\leqslant N-3$. Then, for $0\leqslant x\leqslant 3/(2(N-j-1))$,

$$\left\{\left((N-j-1)x-\frac{1}{2}\right)^{\alpha}\right\}=\left((N-j-1)x-\frac{1}{2}\right)^{\alpha}.$$

So it suffices to show that, for $3/(2(N-j))\leqslant x\leqslant 3/(2(N-j-1))$,

$$((N-j-1)x-1/2)^{\alpha}\geqslant(1/2-1/(N+1))^{\alpha}.$$

Define $g(x):=g(N,\alpha,j;x)=((N-j-1)x-1/2)^{\alpha}$ on $[0,3/(2(N-j-1))]$. Then $g'(x)=\alpha(N-j-1)((N-j-1)x-1/2)^{\alpha+1}$ has the critical point $1/(2(N-j-1))$, and

$$\frac{3}{2(N-j)}-\frac{1}{2(N-j-1)}=\frac{2N-2j-3}{2(N-j)(N-j-1)}>0,$$

since $N-j\geqslant 3$. Moreover,

$$g\left(\frac{3}{2(N-j)}\right)>\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha},$$

since

$$g\left(\frac{3}{2(N-j)}\right)=\left(\frac{2N-2j-3}{2N-2j}\right)^{\alpha}$$

and

$$\left(\frac{2N-2j-3}{2N-2j}\right)-(1/2-1/(N+1))=\frac{N(N-j)-3(j+1)}{2(N+1)(N-j)}>0.$$

Hence for $3/(2(N-j))\leqslant x\leqslant 3/(2(N-j-1))$ we have

$$g(x)\geqslant(1/2-1/(N+1))^{\alpha}$$

which implies (ii) and completes the proof of the proposition. □

THEOREM 4.2. *Let $N\geqslant 3$ and $\alpha\geqslant 2$ be integers, and α even. If, for $1\leqslant m\leqslant\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\rfloor$, there is $k_0=k_0(m)$ with $3\leqslant k_0\leqslant N$ such that*

$$\begin{aligned}&2\left(\sqrt[\alpha]{\left\lfloor\left(\frac{k_0}{2}\sqrt[\alpha]{m}+\frac{1}{2}\left(\frac{k_0}{2}-1\right)\right)^{\alpha}-\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}\right\rfloor+1}+\frac{1}{2}\right)\\&\geqslant k_0\left(\sqrt[\alpha]{\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}+m}+\frac{1}{2}\right)\end{aligned}\tag{4.3}$$

and

$$\sqrt[\alpha]{\left\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\right\rfloor+1} \geqslant \frac{3N-1}{2(N+1)}, \tag{4.4}$$

then $s_\alpha(N)$ *has the extremal value*

$$(1/2-1/(N+1))^{\alpha},$$

and this extremal value is attained at $x=1/(N+1)$.

Proof. We partition $[0,1]$ into the intervals

$$\left[0,\frac{3}{2N}\right],\ \left[\frac{3}{2N},\frac{3}{4}\right],\ \left[\frac{3}{4},\frac{1}{2}\left(\sqrt[\alpha]{\left\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\right\rfloor+1}+\frac{1}{2}\right)\right],$$
$$\left[\frac{1}{2}\left(\sqrt[\alpha]{\left\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\right\rfloor+1}+\frac{1}{2}\right),1\right].$$

Then by Proposition 4.1 it is enough to show that (i) and (ii) below;

$$\min_{\frac{3}{4}\leqslant x\leqslant\frac{1}{2}\left(\sqrt[\alpha]{\lfloor(\frac{3}{2})^{\alpha}-1\rfloor+1}+\frac{1}{2}\right)} \max_{1\leqslant k\leqslant N}\{(kx-1/2)^{\alpha}\} \geqslant (1/2-1/(N+1))^{\alpha}, \tag{i}$$

$$\min_{\frac{1}{2}\left(\sqrt[\alpha]{\lfloor(\frac{3}{2})^{\alpha}-1\rfloor+1}+\frac{1}{2}\right)\leqslant x\leqslant 1} \max_{1\leqslant k\leqslant N}\{(kx-1/2)^{\alpha}\} \geqslant (1/2-1/(N+1))^{\alpha}. \tag{ii}$$

Proof of (i)
We observe that, for $3/4 \leqslant x \leqslant 1$,

$$\left\{\left(2x-\frac{1}{2}\right)^{\alpha}\right\} = \left(2x-\frac{1}{2}\right)^{\alpha} - m$$

for some positive integer $m \left(=\lfloor(2x-1/2)^{\alpha}\rfloor\right)$, and the equation $(2x-1/2)^{\alpha}-m=1$ has the positive solution

$$\frac{1}{2}\left(\sqrt[\alpha]{m+1}+\frac{1}{2}\right). \tag{4.5}$$

Here $m+1=(2x-1/2)^{\alpha} \leqslant (3/2)^{\alpha}$, i.e.

$$m \leqslant \left(\frac{3}{2}\right)^{\alpha}-1. \tag{4.6}$$

From (4.5) and (4.6), we may write

$$\left[\frac{3}{4},\frac{1}{2}\left(\sqrt[\alpha]{\left\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\right\rfloor+1}+\frac{1}{2}\right)\right] = \bigcup_{1\leqslant m\leqslant\lfloor(\frac{3}{2})^{\alpha}-1\rfloor} \left[\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right),\frac{1}{2}\left(\sqrt[\alpha]{m+1}+\frac{1}{2}\right)\right]$$

Then it suffices to show that

$$\min_{\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right)\leqslant x\leqslant\frac{1}{2}\left(\sqrt[\alpha]{m+1}+\frac{1}{2}\right)}\max_{1\leqslant k\leqslant N}\{(kx-1/2)^{\alpha}\}\geqslant(1/2-1/(N+1))^{\alpha}.$$

Consider, for any m, $1\leqslant m\leqslant\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\rfloor$, the interval

$$\left[\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right),\frac{1}{2}\left(\sqrt[\alpha]{m+1}+\frac{1}{2}\right)\right]. \tag{4.7}$$

Let m be an integer with $1\leqslant m\leqslant\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\rfloor$. Then $h(x):=(2x-1/2)^{\alpha}-m$ is increasing on the interval (4.7) and

$$h\left(\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right)\right)=0,\quad h\left(\frac{1}{2}\left(\sqrt[\alpha]{m+1}+\frac{1}{2}\right)\right)=1.$$

On the other hand, the equation $h(x)=(1/2-1/(N+1))^{\alpha}$ has the positive zero

$$\frac{1}{2}\left(\sqrt[\alpha]{\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}+m}+\frac{1}{2}\right).$$

Hence it is enough to consider the subinterval

$$\left[\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right),\frac{1}{2}\left(\sqrt[\alpha]{\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}+m}+\frac{1}{2}\right)\right] \tag{4.8}$$

instead of (4.7). Suppose that there is $k_0=k_0(m)$ with $3\leqslant k_0\leqslant N$ such that

$$\begin{aligned} x_0:=&\frac{1}{k_0}\left(\sqrt[\alpha]{\left\lfloor\left(\frac{k_0}{2}\sqrt[\alpha]{m}+\frac{1}{2}\left(\frac{k_0}{2}-1\right)\right)^{\alpha}-\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}\right\rfloor+1}+\frac{1}{2}\right)\\ \geqslant&\frac{1}{2}\left(\sqrt[\alpha]{\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}+m}+\frac{1}{2}\right). \end{aligned} \tag{4.9}$$

Consider a function in x defined by

$$j(x):=\left(k_0x-\frac{1}{2}\right)^{\alpha}-\left\lfloor\left(\frac{k_0}{2}\sqrt[\alpha]{m}+\frac{1}{2}\left(\frac{k_0}{2}-1\right)\right)^{\alpha}-\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}\right\rfloor.$$

Then, since

$$\begin{aligned} j\left(\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right)\right)=&\left(\frac{k_0}{2}\sqrt[\alpha]{m}+\frac{1}{2}\left(\frac{k_0}{2}-1\right)\right)^{\alpha}-\\ &\left\lfloor\left(\frac{k_0}{2}\sqrt[\alpha]{m}+\frac{1}{2}\left(\frac{k_0}{2}-1\right)\right)^{\alpha}-\left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha}\right\rfloor, \end{aligned}$$

we have

$$j\left(\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right)\right) \geqslant \left(\frac{1}{2}-\frac{1}{N+1}\right)^{\alpha},$$
$$j(x_0) = 1.$$

Moreover, $j(x)$ is increasing on

$$\left[\frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right), \infty\right],$$

since $j'(x) > 0$ on $(1/(2k_0), \infty)$, and

$$\frac{1}{2k_0} \leqslant \frac{1}{6} < \frac{3}{4} \leqslant \frac{1}{2}\left(\sqrt[\alpha]{m}+\frac{1}{2}\right).$$

Hence, by (4.9), (i) is proved.
Proof of (ii)
The curves $y = \{(x-1/2)^{\alpha}\} = (x-1/2)^{\alpha}$ and $y = (1/2 - 1/(N+1))^{\alpha}$ meet at $x = 1/(N+1)$ and $x = 1-1/(N+1)$ on $[0,1]$. So if we show that

$$\frac{1}{2}\left(\sqrt[\alpha]{\left\lfloor\left(\frac{3}{2}\right)^{\alpha}-1\right\rfloor+1}+\frac{1}{2}\right) \geqslant 1-\frac{1}{N+1}, \tag{4.10}$$

the result holds. In fact, we can compute that (4.10) holds if and only if (4.4) holds. This proves (ii) and completes the proof of Theorem 4.2. □

α	N
2	$3 \leqslant N \leqslant 7$
4	$3 \leqslant N \leqslant 53$
6	$3 \leqslant N \leqslant 11$
8	$3 \leqslant N \leqslant 7$
10	$3 \leqslant N \leqslant 38$
$\vdots$	$\vdots$

Table 4.1

EXAMPLE 4.3. For each α even, by computer algebra, we can check the hypotheses in Theorem 4.2. Here we only check the cases $\alpha = 2, 4, 6, 8, 10$ and, for each α, find N's to get the conclusion of Theorem 4.2. The detailed calculation of this is omitted. Table 4.1 shows the pairs (α, N) for which $s_{\alpha}(N)$ has the extremal value

$$(1/2 - 1/(N+1))^{\alpha}.$$

Moreover, for $\alpha = 2$, we have

COROLLARY 4.4. *The function* $s_2(N)$ *has the extremal value*

$$s_2(N) = (1/2 - 1/(N+1))^2,$$

and this extremal value is attained at $x = 1/(N+1)$.

Proof. This is easy for $1 \leqslant N \leqslant 3$. We assume that $N \geqslant 4$. By the proof of Theorem 4.2 and $\frac{1}{2}\left(\sqrt{\left\lfloor\left(\frac{3}{2}\right)^2 - 1\right\rfloor + 1} + \frac{1}{2}\right) = \frac{1+2\sqrt{2}}{4}$, it suffices to show that

$$\min_{\frac{3}{4} \leqslant x \leqslant \frac{1+2\sqrt{2}}{4}} \max_{1 \leqslant k \leqslant N} \{(kx - 1/2)^2\} \geqslant (1/2 - 1/(N+1))^2.$$

Now, for any $N \geqslant 1$, $(1/2 - 1/(N+1))^2 < 1/4$, and we can check that

$$3/4 \leqslant x < (1+2\sqrt{7})/8 \Rightarrow \{(4x - 1/2)^2\} \geqslant 1/4,$$
$$(1+2\sqrt{7})/8 \leqslant x < 5/6 \Rightarrow \{(3x - 1/2)^2\} \geqslant (61 - 12\sqrt{7})/64 = 0.457047\cdots,$$
$$5/6 \leqslant x < (1+2\sqrt{2})/4 \Rightarrow \{(2x - 1/2)^2\} \geqslant 1/3.$$

So the result is proved. □

For any integers $N \geqslant 3$ and $\alpha \geqslant 2$, the function $s_\alpha(N)$ (α even) seems to have the extremal value

$$(1/2 - 1/(N+1))^\alpha$$

at $x = 1/(N+1)$. But we have not resolved this question. For $\alpha \geqslant 3$ odd, it seems hard to find the value of $s_\alpha(N)$ by looking at the graphs generated by computer algebra. The value does not seem to have a simple form.

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