Mathematical
Inequalities
& Applications
Volume 4, Number 1 (2001), 35–52

RELATIVE BOUNDEDNESS–COMPACTNESS INEQUALITIES FOR A SECOND ORDER DIFFERENTIAL OPERATOR

DON B. HINTON AND SUZANNE COLLIER MELESCUE

(*communicated by M. K. Kwong*)

Abstract. For a one-term second order differential operator with power coefficients and in the limit circle case, we give necessary and sufficient conditions for perturbations to be relatively bounded or relatively compact. These conditions are expressed in terms of integrals of the coefficients of the perturbing operators and are easily verified in many cases. An application is given to the energy operator of the hydrogen atom.

1. Introduction

Two useful concepts in the spectral theory of differential operators are those of relative boundedness and compactness. If C and D are linear operators having domains $\mathscr{D}(C)$ and $\mathscr{D}(D)$ in a Hilbert space $\mathscr{H}$ with norm $\|\cdot\|$, then D is said to be *relatively bounded with respect to* C or C*-bounded* if $\mathscr{D}(C) \subset \mathscr{D}(D)$ and there exist nonnegative constants α and β such that for all $y \in \mathscr{D}(C)$, $\|Dy\| \leqslant \alpha\|y\| + \beta\|Cy\|$. The infimum of all such β is called the *relative bound of* D *with respect to* C. Further, D is said to be *relatively compact with respect to* C or C*-compact* if $\mathscr{D}(C) \subset \mathscr{D}(D)$ and D is a compact operator acting on the graph of C with graph norm $\|y\| + \|Cy\|$, denoted by $\|y\|_C$. Many important properties are preserved under relatively bounded or relatively compact perturbations, e.g., the essential spectrum is preserved under a relatively compact perturbation.

Many linear operators are associated with a given linear differential expression depending on boundary conditions. We consider a second order differential expression of the form

$$M[y] = \frac{1}{w}[-(py')' + qy]$$

on an interval $I = [c, \infty), c > 0$, or $I = (0, d], d > 0$, where the coefficient functions w, p, q are real, $w(x), p(x) > 0$ on I and $w, 1/p, q$ are locally Lebesgue integrable on I. The operator acts in the Hilbert space $\mathscr{L}^2(w, I)$ of functions f satisfying $\int_I w(x)|f(x)|^2\,dx < \infty$. The *maximal operator* M_1 associated with M is the restriction of M to the domain

$$\mathscr{D}(M_1) = \{y \in \mathscr{L}^2(w, I) : y, py' \in AC_{loc}(I), M[y] \in \mathscr{L}^2(w, I)\},$$

Mathematics subject classification (2000): 26D10, 47A55, 34L05, 34A40.

Key words and phrases: Sturm-Liouville operator; perturbation theory; relatively bounded; relatively compact; maximal and minimal operators.

Paper MIA-04-03

where $AC_{loc}(I)$ is the set of functions that are absolutely continuous on compact subintervals of I. The *unclosed minimal operator* M_0' associated with M is defined as the restriction of M_1 to the functions with compact support in the interior of I, and the *minimal operator* M_0 is defined as the closure of M_0'. When $I=[c,\infty)$ or $I=(0,d]$ with only one singular endpoint, M is said to be *limit circle* at the singular point ∞ or 0 if all solutions of $M[y]=0$ are in $\mathscr{L}^2(w,I)$; otherwise, M is said to be in the *limit point* case. A complete discussion of these properties may be found in the books of Naimark [6] and Weidmann [8].

We consider perturbation expressions A and B of M of the form

$$A[y]=\frac{1}{w}a\,y,\quad B[y]=\frac{1}{w}b\,y', \tag{1}$$

where the coefficient functions $a(x)$ and $b(x)$ are real or complex-valued functions on I and are locally Lebesgue integrable. The maximal operators A_1 and B_1 for A and B are defined with domains

$$\mathscr{D}(A_1)=\{y\in\mathscr{L}^2(w,I):A[y]\in\mathscr{L}^2(w,I)\}\text{ and}$$

$$\mathscr{D}(B_1)=\{y\in\mathscr{L}^2(w,I):y\in AC_{loc}(I),B[y]\in\mathscr{L}^2(w,I)\}.$$

The minimal operators A_0,B_0 are defined for A,B similarly to that of M. Since A is a multiplication operator, it can be shown that $A_0=A_1$. However, $M_0\neq M_1$, so it is possible for A_1 to be M_0-bounded without being M_1-bounded.

In the limit point case A_0 is M_0-bounded (compact) if and only if A_1 is M_1-bounded (compact) [2, 5]. A similar result holds for B_0 and B_1. The significance of this result is that relative boundedness is usually easier to prove for minimal operators but is more useful for maximal operators. For example, suppose A_1 is M_1-bounded with relative bound less than one and $a(x)$ is real. Then A_1 is self-adjoint, and if M_a is a self-adjoint operator obtained by appropriately restricting the domain of M_1, then A_1 is also M_a-bounded with relative bound less than one. Moreover, M_a+A_1 restricted to the domain of M_a is self-adjoint (Kato-Rellich Theorem).

In [1] an investigation was begun for the special case of M with $w(x)=x^\gamma,p(x)=x^\alpha$, i.e.,

$$L[y](x)=-x^{-\gamma}[x^\alpha y'(x)]'. \tag{2}$$

We can easily determine if L is in the limit point or limit circle case from the general solution of $L[y]=0$ [3, Theorem 9.11.2], which is

$$y(x)=\begin{cases} c_1+c_2x^{1-\alpha}, & \text{if } \alpha\neq 1\\ c_1+c_2\ln x, & \text{if } \alpha=1\end{cases}.$$

Thus if $I=[c,\infty)$, L is in the limit circle case if and only if $\gamma<\min\{-1,2\alpha-3\}$; and if $I=(0,d]$, L is in the limit circle case if and only if $\gamma>\max\{-1,2\alpha-3\}$. For the limit point case the results of [1] are complete except for the rays $\{(\alpha,-1):\alpha\geqslant 1\}$ and $\{(\alpha,2\alpha-3):\alpha\leqslant 1\}$. We state two theorems from [1] for comparison purposes with the limit circle results in this paper.

THEOREM 1.1. *Suppose* $I = [c,\infty)$, $\gamma \geqslant \alpha - 2$, *and* L_1, A_1, *and* B_1 *are the maximal operators defined by* L, A, *and* B. *Define* $s(x) = x^{(\alpha-\gamma)/2}$, $N_0 = |\gamma - \alpha| c^{(\alpha-\gamma)/2}/2$,

$$g_{0,\delta}(x) = \frac{1}{s(x)} \int_x^{x+\delta s(x)} u^{-2\gamma} |a(u)|^2 \, du,$$

$$g_{1,\delta}(x) = \frac{1}{s(x)} \int_x^{x+\delta s(x)} u^{-\gamma-\alpha} |b(u)|^2 \, du$$

Then

(i) A_1 *is* L_1*-bounded if and only if* $\sup_{c \leqslant x < \infty} g_{0,\delta}(x) < \infty$ *for some* $\delta \in (0, \frac{1}{2N_0})$,

(ii) A_1 *is* L_1*-compact if and only if* $\lim_{x\to\infty} g_{0,\delta}(x) = 0$ *for some* $\delta \in (0, \frac{1}{2N_0})$,

(iii) B_1 *is* L_1*-bounded if and only if* $\sup_{c \leqslant x < \infty} g_{1,\delta}(x) < \infty$ *for some* $\delta \in (0, \frac{1}{2N_0})$,

(iv) B_1 *is* L_1*-compact if and only if* $\lim_{x\to\infty} g_{1,\delta}(x) = 0$ *for some* $\delta \in (0, \frac{1}{2N_0})$.

THEOREM 1.2. *Suppose* $\gamma \leqslant \alpha - 2$ *and either* $\gamma > -1$ *or* $\gamma > 2\alpha - 3$. *Let* I, L_1, A_1, *and* B_1 *be as in Theorem 1.1. Define*

$$\varepsilon_0 = \begin{cases} (3/2)^{1/(\gamma+1)} - 1, & \text{if } \gamma > -1 \\ (3/2)^{1/(\gamma-2\alpha+3)} - 1, & \text{if } \gamma > 2\alpha - 3 \end{cases},$$

$$h_{0,\varepsilon}(x) = \frac{1}{x} \int_x^{x+\varepsilon x} u^{-2\alpha+4} |a(u)|^2 \, du,$$

$$h_{1,\varepsilon}(x) = \frac{1}{x} \int_x^{x+\varepsilon x} u^{-2\alpha+2} |b(u)|^2 \, du.$$

Then

(i) A_1 *is* L_1*-bounded if and only if* $\sup_{c \leqslant x < \infty} h_{0,\varepsilon}(x) < \infty$ *for some* $\varepsilon \in (0, \varepsilon_0)$,

(ii) A_1 *is* L_1*-compact if and only if* $\lim_{x\to\infty} h_{0,\varepsilon}(x) = 0$ *for some* $\varepsilon \in (0, \varepsilon_0)$,

(iii) B_1 *is* L_1*-bounded if and only if* $\sup_{c \leqslant x < \infty} h_{1,\varepsilon}(x) < \infty$ *for some* $\varepsilon \in (0, \varepsilon_0)$,

(iv) B_1 *is* L_1*-compact if and only if* $\lim_{x\to\infty} h_{1,\varepsilon}(x) = 0$ *for some* $\varepsilon \in (0, \varepsilon_0)$.

Only partial results were given in [1] for the limit circle case. In this paper we give a complete solution of necessary and sufficient criteria for L in the limit circle case for relative boundedness or compactness for both maximal and minimal operators. Unlike the limit point case, the criteria for the limit circle case are different for maximal and minimal operators.

2. Preliminaries

We state in this section some results which are used in later proofs. The first three theorems may be found in [5]. While these theorems hold for differential expressions of arbitrary order, we apply them in the situation where F is L and G is either A or B.

THEOREM 2.1. *Let F and G be formal differential expressions on an interval I where F is symmetric, the order of G is less than the order of F, and the coefficients of F and G are sufficiently smooth so that* $\mathscr{D}(F_0') \subset \mathscr{D}(G_0')$.

(i) If G_0' *is* F_0'*-bounded, then* G_0 *is* F_0*-bounded.*

(ii) If G_0' *is* F_0'*-compact, then* G_0 *is* F_0*-compact.*

THEOREM 2.2. *Let F and G be as in Theorem 2.1 and let* $\mathscr{D}(F_1) \subset \mathscr{D}(G_1)$. *Then*

(i) G_1 *is* F_1*-bounded.*

(ii) G_0 *is* F_0*-bounded.*

(iii) If G_1 *is* F_1*-compact, then* G_0 *is* F_0*-compact.*

THEOREM 2.3. *Let F and G be as in Theorem 2.1,* G_0 *be* F_0*-compact, and* $\mathscr{D}(F_1) \subset \mathscr{D}(G_1)$. *Then* G_1 *is* F_1*-compact.*

The following interpolation theorem from [4] is needed as well as the following lemma, also from [4], which allows a function to be treated as approximately constant on sufficiently short intervals.

THEOREM 2.4. *Let* $I = [a,\infty)$ *and let N, W, and P be positive measurable functions such that* N, W^{-1}, *and* $P^{-1} \in \mathscr{L}_{loc}(I)$. *Suppose there exists an* $\varepsilon_0 > 0$ *and a positive continuous function* $f = f(t)$ *on I such that*

$$S_1(\varepsilon) := \sup_{t\in I}\{f^2\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} P^{-1}]\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} N]\} < \infty$$

and

$$S_2(\varepsilon) := \sup_{t\in I}\{[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} W^{-1}]\,[\frac{1}{\varepsilon f}\int_t^{t+\varepsilon f} N]\} < \infty$$

for all $\varepsilon \in (0,\varepsilon_0)$. *Then there exists a constant* $k > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$ *and* $y \in D$,

$$\int_I N|y|^2 \leqslant k\{S_2(\varepsilon)\int_I W|y|^2 + \varepsilon^2 S_1(\varepsilon)\int_I P|y'|^2\}$$

where

$$\mathscr{D} = \{y : y \in AC_{loc}(I), \int_I W|y|^2 < \infty, \text{ and } \int_I P|y'|^2 < \infty\}.$$

LEMMA 2.5. *Let* $f, g \in AC_{loc}(I)$ *be positive functions on an interval* $I = [a,\infty)$ *satisfying* $|f'(t)| \leqslant N_0$ *and* $|f(t)\,g'(t)| \leqslant M_0\,g(t)$ *a.e. on I for some constants* N_0 *and* M_0. *Then for each* $t \in I$, $0 < \varepsilon < \frac{1}{N_0}$, *and* $t \leqslant u \leqslant t + \varepsilon f(t)$, *we have that*

$$(1-\varepsilon N_0)f(t) \leqslant f(u) \leqslant (1+\varepsilon N_0)f(t)$$

and

$$e^{-\frac{M_0}{N_0}}g(t) \leqslant g(u) \leqslant e^{\frac{M_0}{N_0}}g(t).$$

Finally, we need Hardy's inequality with power weights [7] which is stated for a function in $AC[c,d]$ as

$$\int_c^d x^\beta |y(x)|^2\,dx \leqslant \frac{4}{(\beta+1)^2}\int_c^d x^{\beta+2}|y'(x)|^2\,dx, \tag{3}$$

where $0 < c < d < \infty$, $\beta \neq -1$, and $y(c) = 0 = y(d)$.

Two consequences of Hardy's inequality are the following lemmas.

LEMMA 2.6. *If y has compact support in $[c,d]$, $c>0$, and is absolutely continuous, then for $\gamma \neq 2\alpha-3$,*

$$\int_c^d x^{-\gamma+2\alpha-4}|y(x)|^2\,dx \leqslant \frac{4}{(-\gamma+2\alpha-3)^2}\int_c^d x^{-\gamma+2\alpha-2}|y'(x)|^2\,dx.$$

Proof. Since $y(c)=0=y(d)$ and $\gamma\neq 2\alpha-3$, we apply Hardy's inequality with $\beta=-\gamma-4+2\alpha$ to obtain the result.

LEMMA 2.7. *If y has compact support in $[c,d]$, $c>0$, and is absolutely continuous, then for $\gamma\neq -1$,*

$$\int_c^d x^{-\gamma+2\alpha-2}|y'(x)|^2\,dx \leqslant \frac{4}{(-\gamma-1)^2}\int_c^d x^{-\gamma}|[x^\alpha y'(x)]'|^2\,dx.$$

Proof. Notice that $\int_c^d x^{-\gamma+2\alpha-2}|y'(x)|^2 = \int_c^d x^{-\gamma-2}|x^\alpha y'(x)|^2$. Since $y'(c)=0=y'(d)$ and $\gamma\neq-1$, we apply Hardy's inequality with $\beta=-\gamma-2$ to obtain the result.

3. Perturbations A

In this section, we consider perturbations A of a higher-ordered, symmetric differential operator L in the limit circle case. These operators L and A are defined on an interval I by the equations (2) and (1), with weight $w(x)=x^\gamma$.

THEOREM 3.1. *Let $I=[c,\infty)$ for some $c>0$ and let $\gamma<min\{-1,2\alpha-3\}$. Then*

(i) A_0 is L_0-bounded if and only if

$$\sup_{x\in I}\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}\,|a(u)|^2\,du<\infty, \tag{4}$$

for some $\varepsilon\in(0,\frac{1}{2})$;

(ii) A_0 is L_0-compact if and only if

$$\lim_{x\to\infty}\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}\,|a(u)|^2\,du=0, \tag{5}$$

for some $\varepsilon\in(0,\frac{1}{2})$.

Proof.

(i) Sufficiency We begin by showing that A_0' is L_0'-bounded if inequality (4) holds. Let us consider $y\in\mathscr{D}(L_0')$. Since y has compact support in the interior of I, there exists a $d<\infty$ such that the support of y is contained in $[c,d]$. Then we have

$$\|y\|^2 = \int_c^d x^\gamma|y(x)|^2\,dx, \tag{6}$$

$$\|Ay\|^2 = \int_c^d x^{-\gamma}|a(x)|^2|y(x)|^2\,dx, \text{ and} \tag{7}$$

$$\|Ly\|^2 = \int_c^d x^{-\gamma}\,|[x^\alpha y'(x)]'|^2\,dx. \tag{8}$$

Now, we show that the hypotheses of Theorem 2.4 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma}|a(x)|^2$, $W = x^{-\gamma+2\alpha-4}$, $P = x^{-\gamma+2\alpha-2}$, and $f = x$. By applying Lemma 2.5 with $g \equiv 1$ and $f = x$, we know that positive and negative powers of x are essentially constant on intervals of length εx. Thus, we have for some constants $C_1, C_2 > 0$

$$\begin{aligned} S_1 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2}\left[\int_x^{x+\varepsilon x} u^{\gamma-2\alpha+2}\,du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma}|a(u)|^2\,du\right]\right\} \qquad (9)\\ &\leqslant \frac{C_1}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\} \text{ and}\\ S_2 &= \sup_{x\in I}\left\{\frac{1}{\varepsilon^2 x^2}\left[\int_x^{x+\varepsilon x} u^{\gamma+4-2\alpha}\,du\right]\left[\int_x^{x+\varepsilon x} u^{-\gamma}|a(u)|^2\,du\right]\right\} \qquad (10)\\ &\leqslant \frac{C_2}{\varepsilon}\sup_{x\in I}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du]\right\} \end{aligned}$$

for some $\varepsilon \in (0, \frac{1}{2})$. Inequalities (4), (9), and (10) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$. Therefore (via Theorem 2.4), there exists a constant $C_3 > 0$ such that

$$\int_c^d x^{-\gamma}|a(x)|^2|y(x)|^2\,dx \leqslant C_3\left\{\int_c^d x^{-\gamma+2\alpha-4}|y(x)|^2\,dx + \varepsilon^2\int_c^d x^{-\gamma+2\alpha-2}|y'(x)|^2\,dx\right\}. \qquad (11)$$

Via Lemmas 2.6 and 2.7 each integral on the right-hand side is bounded above by a constant multiple of $\|Ly\|^2$. Therefore, we obtain for some constant $C_4 > 0$

$$\|Ay\|^2 \leqslant C_4\,\|Ly\|^2 \leqslant C_4\,\|y\|^2_{L_0'}. \qquad (12)$$

Thus, $y \in \mathscr{D}(A_0')$. Since y is arbitrary, the inequality above implies that A_0' is L_0'-bounded. Via Theorem 2.1, A_0 is L_0-bounded.

Necessity. Let ϕ be a function in $C_0^\infty(\mathbf{R})$ such that $\phi \equiv 1$ on $[0,1]$ and $\operatorname{supp}(\phi) = [-2,2]$. Fix $\varepsilon \in (0, \frac{1}{2})$. For each $r \geqslant c$ we define $\phi_r(x) = \phi\left(\frac{x-r}{\varepsilon r}\right)$, for $x \geqslant c$. Then $\phi_r \equiv 1$ on $[r, r+\varepsilon r]$ and $\operatorname{supp}(\phi_r) = [r-2\varepsilon r, r+2\varepsilon r]$. Via Lemma 2.5, a change of variables, the continuity of ϕ, and the fact that $\gamma < -1$, there exist constants C_1, C_2, and $C_3 > 0$ such that for each $r \geqslant c$

$$\|\phi_r\|^2 = \int_I x^\gamma \phi_r^2(x)\,dx \leqslant C_1\varepsilon r^{\gamma+1}\int_{-2}^2 \phi^2(u)\,du \leqslant C_2\,r^{\gamma+1} \leqslant C_3. \qquad (13)$$

Notice that for each $r \geqslant c$

$$\begin{aligned} \|L\phi_r\|^2 &= \int_I x^{-\gamma}\left|\frac{d}{dx}\left\{x^\alpha\left[\frac{d}{dx}\phi_r(x)\right]\right\}\right|^2 dx\\ &\leqslant \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma}\left\{\left|x^\alpha\left[\frac{d^2}{dx^2}\phi_r(x)\right]\right| + \left|\alpha x^{\alpha-1}\left[\frac{d}{dx}\phi_r(x)\right]\right|\right\}^2 dx. \end{aligned}$$

Applying Lemma 2.5, a change of variables, and the continuity of ϕ' and ϕ'', we obtain for some constants C_4 and $C_5 > 0$ for each $r \geqslant c$

$$\begin{aligned}\|L\phi_r\|^2 &\leqslant C_4 \left\{ r^{-\gamma+2\alpha} \int_{r-2\varepsilon r}^{r+2\varepsilon r} \left[\frac{d^2}{dx^2}\phi_r(x)\right]^2 dx + r^{-\gamma+2\alpha-2} \int_{r-2\varepsilon r}^{r+2\varepsilon r} \left[\frac{d}{dx}\phi_r(x)\right]^2 dx \right\} \\ &= C_4 r^{-\gamma+2\alpha-3} \left\{ \varepsilon^{-3} \int_{-2}^{2} [\phi''(u)]^2\, du + \varepsilon^{-1} \int_{-2}^{2} [\phi'(u)]^2\, du \right\} \qquad (14) \\ &\leqslant C_5\, r^{-\gamma+2\alpha-3}.\end{aligned}$$

Since $\phi_r \equiv 1$ on $[r, r+\varepsilon r]$ and supp(ϕ_r)$=[r-2\varepsilon r, r+2\varepsilon r]$,

$$\int_r^{r+\varepsilon r} x^{-\gamma}|a(x)|^2\, dx \;\leqslant\; \int_{r-2\varepsilon r}^{r+2\varepsilon r} x^{-\gamma}|a(x)|^2\phi_r^2(x)\, dx \;=\; \|A\phi_r\|^2. \qquad (15)$$

Thus, via the L_0-boundedness of A_0 and inequalities (13) - (15), there exists a constant $C_6 > 0$ such that

$$\int_r^{r+\varepsilon r} x^{-\gamma}|a(x)|^2\, dx \leqslant C_6 \left(C_3 + C_5 r^{-\gamma+2\alpha-3}\right).$$

After multiplying the above inequality by $r^{\gamma-2\alpha+3}$, we apply Lemma 2.5 to the left-hand side and obtain a constant $C_7 > 0$ such that

$$\frac{1}{r}\int_r^{r+\varepsilon r} x^{2(2-\alpha)}|a(x)|^2\, dx \leqslant C_7 \left(r^{\gamma-2\alpha+3} + 1\right).$$

Since $\gamma - 2\alpha + 3 < 0$, the right-hand side of the above inequality is bounded on I. Hence, inequality (4) holds.

(ii) Sufficiency By the previous argument A_0 is L_0-bounded. Thus, $\mathscr{D}(L_0) \subset \mathscr{D}(A_0)$. For every positive integer $N > c$, define A_N on $\mathscr{D}(L_0)$ by

$$A_N y = \begin{cases} Ay & \text{on } [c, N] \\ 0 & \text{on } (N, \infty) \end{cases}$$

Notice that each A_N is L_0-bounded with the same norm as A_0 since $\|A_N y\| \leqslant \|Ay\|$. In order to simplify the proof, we break the argument into two Claims.

CLAIM 3.1.1. *$A_N \to A_0$ in the space of bounded operators on $\mathscr{D}(L_0)$ with the L_0-norm.*

Proof of Claim 3.1.1. By definition L_0 is closed. Therefore, $\mathscr{D}(L_0)$ is complete under the L_0-norm.

Let $y \in \mathscr{D}(L_0)$. Since L_0 is the closure of L_0', for each integer $n \geqslant 1$ there exists a $y_n \in \mathscr{D}(L_0')$ such that

$$\|y - y_n\| + \|Ly - Ly_n\| < \frac{1}{n}. \qquad (16)$$

For each $y_n \in L_0'$ we have

$$\|Ay_n - A_N y_n\|^2 = \int_N^\infty x^{-\gamma}|a(x)|^2|y_n(x)|^2\,dx. \tag{17}$$

Since each y_n has compact support in the interior of I, there exists a $d < \infty$ such that the support of y_n is contained in $[c,d]$. Thus, we can apply the sufficiency argument in part *(i)* with $I_N = [N,\infty)$ to inequality (17).

Via Theorem 2.4 and inequalities (9), (10), (11), and (17), there exists a constant $k_1 > 0$ such that

$$\begin{aligned}\|Ay_n - A_N y_n\|^2 \;&\leqslant\; k_1 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\times\\ &\times\left\{\int_c^d x^{-\gamma+2\alpha-4}|y_n(x)|^2\,dx + \varepsilon^2\int_c^d x^{-\gamma+2\alpha-2}|y_n'(x)|^2\,dx\right\}.\end{aligned}$$

We apply Lemmas 2.6 and 2.7, as before, to obtain a constant $k_2 > 0$ such that

$$\begin{aligned}\|Ay_n - A_N y_n\|^2 \;&\leqslant\; k_2 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\cdot\|Ly_n\|^2 \\ &\leqslant\; k_2 \sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\cdot\|y_n\|^2_{L_0'}.\end{aligned} \tag{18}$$

Therefore, via the triangle inequality, the L_0-boundedness of A_0, and inequalities (16) and (18), we have for each n

$$\begin{aligned}\|Ay - A_N y\| \;&\leqslant\; \|Ay - Ay_n\| + \|Ay_n - A_N y_n\| + \|A_N y_n - A_N y\|\\ &\leqslant\; 2k_3\left(\|y - y_n\| + \|Ly - Ly_n\|\right) + \|Ay_n - A_N y_n\|\\ &<\; \frac{2k_3}{n} + k_2^{\frac12}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\right)^{\frac12}\|y_n\|_{L_0'}\end{aligned}$$

for some constant $k_3 > 0$. By applying the triangle inequality and inequality (16) again, we obtain for each n

$$\begin{aligned}\|Ay - A_N y\| \;&\leqslant\; \frac{2k_3}{n} + k_2^{\frac12}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\right)^{\frac12}\left(\|y_n - y\|_{L_0} + \|y\|_{L_0}\right)\\ &<\; \frac{2k_3}{n} + k_2^{\frac12}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\right)^{\frac12}\left(\frac{1}{n} + \|y\|_{L_0}\right).\end{aligned}$$

We let $n \to \infty$ to obtain for each N

$$\|Ay - A_N y\| \;\leqslant\; k_2^{\frac12}\left(\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\}\right)^{\frac12}\|y\|_{L_0}.$$

Since $I_N = [N,\infty)$, equation (5) implies that $\sup_{x\in I_N}\left\{\frac{1}{x}\int_x^{x+\varepsilon x} u^{2(2-\alpha)}|a(u)|^2\,du\right\} \to 0$ as $N\to\infty$ so that, via the above inequality, we have for $y\neq 0$

$$\|A-A_N\| \to 0 \text{ as } N\to\infty\,. \qquad \square$$

CLAIM 3.1.2. *Each A_N is L_0-compact.*

Proof of Claim 3.1.2. Let $\{y_n\}_{n=1}^{\infty}\subset \mathscr{D}(L_0)$ be an L_0-bounded sequence. By part (ii) of Theorem 2.1, it is sufficient to consider only y_n with compact support. We need to show that for each N the sequence $\{A_Ny_n\}_{n=1}^{\infty}$ has a convergent subsequence. We make use of the Arzela-Ascoli Theorem.

Since the interval $[c,N]$ is compact and $c>0$, powers of x are bounded above and below on $[c,N]$ by positive constants. It is clear then from the boundedness of $\{\|y_n\|_{L_0}\}_{n=1}^{\infty}$ and Lemma 2.7 that the sequence $\left\{\int_c^N |y_n'(x)|^2\,dx\right\}_{n=1}^{\infty}$ is bounded. Via the compact support of each y_n and the Cauchy-Schwarz inequality, we have for each n

$$|y_n(x)| = |y_n(x)-y_n(c)| = |\int_c^x y_n'(u)\,du| \leqslant |N-c|^{\frac12}\left(\int_c^N |y_n'(u)|^2\,du\right)^{\frac12}, \; c\leqslant x\leqslant N.$$

Thus, the sequence $\{y_n(x)\}_{n=1}^{\infty}$ is uniformly bounded on $[c,N]$. We also have that $\{y_n\}_{n=1}^{\infty}$ is equicontinuous on $[c,N]$ since by the Cauchy-Schwarz inequality

$$|y_n(s)-y_n(t)| = |\int_t^s y_n'(u)\,du| \leqslant |s-t|^{\frac12}\left(\int_c^N |y_n'(u)|^2\,du\right)^{\frac12}.$$

Therefore, via the Arzela-Ascoli Theorem we conclude that there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ which converges uniformly on $[c,N]$. WLOG, we assume the sequence $\{y_n\}_{n=1}^{\infty}$ converges uniformly on $[c,N]$.

Now, we have for each N

$$\begin{aligned}\|A_Ny_n - A_Ny_m\|^2 &= \int_c^N u^{-\gamma}|a(u)|^2|y_n(u)-y_m(u)|^2\,du\\ &\leqslant \sup_{c\leqslant x\leqslant N}|y_n(x)-y_m(x)|^2\int_c^N u^{-\gamma}|a(u)|^2\,du.\end{aligned}$$

Since the integral on the right-hand side is finite, there exists a constant $K>0$ such that

$$\|A_Ny_n - A_Ny_m\|^2 \leqslant K\sup_{c\leqslant x\leqslant N}|y_n(x)-y_m(x)|^2.$$

Since $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the uniform norm, the above inequality implies that the sequence $\{A_Ny_n\}_{n=1}^{\infty}$ is Cauchy in $\mathscr{L}^2(x^\gamma,I)$ for each N. Therefore, $\{A_Ny_n\}_{n=1}^{\infty}$ converges for each N as $n\to\infty$ since $\mathscr{L}^2(x^\gamma,I)$ is complete. Hence, each A_N is L_0-compact. $\square$

Since A_0 is the uniform limit of L_0-compact operators, A_0 is L_0-compact.

Necessity. We use a contradiction argument to show that equation (5) must hold. Suppose that for some $\varepsilon \in (0, \frac{1}{2})$ there exists a $\rho > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $r_n \to \infty$ as $n \to \infty$ and for each n

$$\frac{1}{r_n} \int_{r_n}^{r_n + \varepsilon r_n} x^{2(2-\alpha)} |a(x)|^2 \, dx \geqslant \rho. \tag{19}$$

Let $\{\phi_r\}_{r \geqslant c}$ be defined as in part *(i)*. Then via inequalities (13) and (14) there exist constants $C_3, \ C_5 > 0$ such that for each n

$$\|\phi_{r_n}\|_{L_0}^2 \ = \ (\|\phi_{r_n}\| + \|L\phi_{r_n}\|)^2 \ \leqslant \ 2\,(\|\phi_{r_n}\|^2 + \|L\phi_{r_n}\|^2) \ \leqslant \ 2C_3 + 2C_5\, r_n^{-\gamma+2\alpha-3}. \tag{20}$$

For each $r \geqslant c$ define $\widehat{\phi_r}(x) = r^{\frac{1}{2}(\gamma - 2\alpha + 3)} \phi_r(x)$ for $x \geqslant c$. Via inequality (20) we have for each n

$$\|\widehat{\phi}_{r_n}\|_{L_0}^2 \ = \ r_n^{\gamma - 2\alpha + 3}\, \|\phi_{r_n}\|_{L_0}^2 \ \leqslant \ 2C_3\, r_n^{\gamma - 2\alpha + 3} + 2C_5.$$

Since $\gamma - 2\alpha + 3 < 0$, the above inequality implies that $\{\widehat{\phi}_{r_n}\}_{n=1}^{\infty}$ is an L_0-bounded sequence. Via the L_0-compactness of A_0, $\{A\widehat{\phi}_{r_n}\}_{n=1}^{\infty}$ has a convergent subsequence. WLOG, we assume $\{A\widehat{\phi}_{r_n}\}_{n=1}^{\infty}$ converges, say to some y_0. Via inequality (19), properties of ϕ_{r_n}, and Lemma 2.5, there exists a constant $K > 0$ such that for each n

$$\begin{aligned}
\rho \ &\leqslant \ \frac{1}{r_n} \int_{r_n}^{r_n + \varepsilon r_n} x^{2(2-\alpha)} |a(x)|^2 \, dx \\
&\leqslant \ \frac{1}{r_n} \int_{r_n - 2\varepsilon r_n}^{r_n + 2\varepsilon r_n} x^{2(2-\alpha)} |a(x)|^2 \phi_{r_n}^2(x) \, dx \\
&\leqslant \ K\, r_n^{\gamma - 2\alpha + 3} \int_{r_n - 2\varepsilon r_n}^{r_n + 2\varepsilon r_n} x^{-\gamma} |a(x)|^2 \phi_{r_n}^2(x) \, dx \\
&= \ K\, \|A\widehat{\phi}_{r_n}\|^2 .
\end{aligned}$$

Hence, $\|A\widehat{\phi}_{r_n}\| \geqslant \left(\frac{\rho}{K}\right)^{\frac{1}{2}} > 0$ for each n.

Notice that a contradiction is reached if we show that $y_0 = 0$ a.e. in $[c, \infty)$. Let J_0 be a finite subinterval of $[c, \infty)$. Since $r_n \to \infty$ as $n \to \infty$ and supp(ϕ_{r_n}) $= [r_n - 2\varepsilon f(r_n), r_n + 2\varepsilon f(r_n)]$, we conclude that $\phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. Hence, $A\phi_{r_n} \equiv 0$ on J_0 for sufficiently large n. For such n

$$\|y_0\|_{J_0} \ = \ \|y_0 - A\phi_{r_n}\|_{J_0} \ \leqslant \ \|y_0 - A\phi_{r_n}\|.$$

Since $A\phi_{r_n} \to y_0$ as $n \to \infty$ and the left-hand side of the above inequality is independent of n, we have that $\|y_0\|_{J_0} = 0$. Thus, $y_0 = 0$ a.e. in $[c, \infty)$ since the interval J_0 is arbitrary. This contradiction implies that equation (5) holds.

Next, we prove a corollary of Theorem 3.1. We consider the minimal operators L_0 and A_0 associated with the differential expressions (2) and (1), respectively, on the interval $(0, \frac{1}{c}]$. We prove this result by using a unitary transformation to transform the singularity at 0 to a singularity at ∞ and then applying Theorem 3.1 to the new operators. Note that L is limit circle at zero if and only if $\gamma > \max\{-1, 2\alpha - 3\}$.

COROLLARY 3.2. *Let* $I = (0, \frac{1}{c}]$ *for some* $c > 0$ *and let* $\gamma > max\{-1, 2\alpha - 3\}$. *Then*

(i) A_0 *is* L_0*-bounded if and only if*

$$\sup_{x\in I} \frac{1}{x} \int_{x-\varepsilon' x}^{x} u^{2(2-\alpha)} \, |a(u)|^2 \, du < \infty, \tag{21}$$

for some sufficiently small ε' *;*

(ii) A_0 *is* L_0*-compact if and only if*

$$\lim_{x\to 0} \frac{1}{x} \int_{x-\varepsilon' x}^{x} u^{2(2-\alpha)} \, |a(u)|^2 \, du = 0, \tag{22}$$

for some sufficiently small ε'.

Proof. By applying the argument in [1, p.152] (with $Uy = z$ and $z(t) = y(x)$ where $t = \frac{1}{x}$), we transform the operators L and A unitarily into the operators $\widehat{L} = ULU^{-1}$ and $\widehat{A} = UAU^{-1}$, which are defined on $[c, \infty)$ by the equations

$$\widehat{L}[z](t) = -t^{\gamma+2}[t^{2-\alpha} z'(t)]' \text{ and} \tag{23}$$

$$\widehat{A}[z](t) = t^{\gamma+2}[t^{-2} \, a\,(1/t)]z(t). \tag{24}$$

Notice that the operators $\widehat{L}$ and $\widehat{A}$ are of the form (2) and (1), respectively, if we replace $-\gamma$ with $-\gamma' = \gamma + 2$, α with $\alpha' = 2 - \alpha$, and $a(x)$ with $t^{-2}a(1/t)$, where $t = \frac{1}{x}$. Thus, the condition of Theorem 3.1 that $\gamma' < \min\{-1, 2\alpha' - 3\}$ becomes $\gamma > \max\{-1, 2\alpha - 3\}$. We apply Theorem 3.1, with the appropriate replacements, to the minimal operators $\widehat{L_0}$ and $\widehat{A_0}$, associated with the differential expressions (23) and (24), respectively, to obtain the following results:

(i) $\widehat{A_0}$ is $\widehat{L_0}$-bounded if and only if

$$\sup_{t\in[c,\infty)} \frac{1}{t} \int_{t}^{t+\varepsilon t} \tau^{2(\alpha-2)} |a(1/\tau)|^2 \, d\tau < \infty, \tag{25}$$

for some $\varepsilon \in (0, \frac{1}{2})$.

(ii) $\widehat{A_0}$ is $\widehat{L_0}$-compact if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_{t}^{t+\varepsilon t} \tau^{2(\alpha-2)} |a(1/\tau)|^2 \, d\tau = 0, \tag{26}$$

for some $\varepsilon \in (0, \frac{1}{2})$.

Via a change of variables, we can show that (25) and (26) are equivalent to (21) and (22), respectively, with $\varepsilon' = \varepsilon/(1+\varepsilon)$.

In the next result, we consider the maximal operators, L_1 and A_1, and the minimal operators, L_0 and A_0, associated with the differential expressions (2) and (1), respectively, on the interval $I = [c, \infty)$.

THEOREM 3.3. *Let* $I = [c, \infty)$ *for some* $c > 0$ *and let* $\gamma < min\{-1, 2\alpha - 3\}$. *Then the following three statements are equivalent.*

(i) If $\alpha < 1$, *then* $\int_I x^{-\gamma-2\alpha+2} |a(x)|^2 dx < \infty$;

if $\alpha = 1$, *then* $\int_I x^{-\gamma} |a(x)|^2 [\ln(x)]^2 dx < \infty$;

and if $\alpha > 1$, *then* $\int_I x^{-\gamma} |a(x)|^2 dx < \infty$.

(ii) A_1 *is* L_1*-bounded.*

(iii) A_1 *is* L_1*-compact.*

Proof. (i) $\Rightarrow$ *(iii)* First we show that $\int_I x^{3-2\alpha} |a(x)|^2 dx < \infty$. If $\alpha < 1$, then $\gamma < -1$ implies

$$\int_I x^{3-2\alpha} |a(x)|^2 dx \leqslant \int_c^1 x^{3-2\alpha} |a(x)|^2 dx + \int_1^\infty x^{-\gamma-2\alpha+2} |a(x)|^2 dx < \infty.$$

The cases $\alpha > 1, \alpha = 1$ are similar. Hence, *(i)* implies that A_0 is L_0-compact via Theorem 3.1. Therefore, $\mathscr{D}(L_0) \subset \mathscr{D}(A_0)$.

Via the first formula of Von Neumann [8], we can write

$$\mathscr{D}(L_1) = \mathscr{D}(L_0) \oplus S, \tag{27}$$

where S has dimension four since L is regular at c and limit circle at ∞.

CLAIM 3.3.1. $S \subset \mathscr{D}(A_1)$.

Proof of Claim 3.3.1. Set $z_1(x) = \begin{cases} x^{1-\alpha} & \text{if } \alpha \neq 1 \\ \ln x & \text{if } \alpha = 1 \end{cases}$ and $z_2(x) \equiv 1$. Let z_3 and z_4 be functions in $C^{(2)}(\mathbf{R})$ with compact support in [c-1, c+1] such that $z_3(c) = 1$, $z_3'(c) = 0$, $z_4(x) = 0$, and $z_4'(c) = 1$. Notice that these four real-valued functions are linearly independent.

We show that each $z_i \in \mathscr{L}^2(x^\gamma, I)$. Since $\gamma < \min\{-1, 2\alpha - 3\}$,

$$\|z_1\|^2 = \begin{cases} \int_I x^{\gamma-2\alpha+2}\, dx & \text{if } \alpha \neq 1 \\ \int_I x^\gamma [\ln x]^2\, dx & \text{if } \alpha = 1 \end{cases}$$

is finite. We also have that $\|z_2\|^2 = \int_I x^\gamma\, dx < \infty$ since $\gamma < -1$. Lastly, since z_3 and z_4 are continuous on I and have compact support, $\|z_3\|$ and $\|z_4\|$ are finite.

Now, we show that each $Lz_i \in \mathscr{L}^2(x^\gamma, I)$, so that each $z_i \in \mathscr{D}(L_1)$. We have that $\|Lz_1\| = 0 = \|Lz_2\|$ since $Lz_1 = 0 = Lz_2$. Moreover, since z_3 and z_4 have continuous second derivatives and compact support, $\|Lz_3\|$ and $\|Lz_4\|$ are finite.

Define $S =$ span$\{z_1, z_2, z_3, z_4\}$. We now prove that $\mathscr{D}(L_1) = \mathscr{D}(L_0) \oplus S$, i.e., we show that no linear combination of the z_i is in $\mathscr{D}(L_0)$. Suppose to the contrary that there exist constants k_1, k_2, k_3, k_4 (not all zero) such that $z := k_1 z_1 + k_2 z_2 + k_3 z_3 + k_4 z_4 \in \mathscr{D}(L_0)$. Since L is regular at c and $z \in \mathscr{D}(L_1)$, we have that $z \in \mathscr{D}(L_0)$ if and only if $z(c) = 0 = z'(c)$ and $[y, z](x) \to 0$ as $x \to \infty$ for every $y \in \mathscr{D}(L_1)$, where the *Lagrange identity* is defined by $[y, \hat{y}](x) = y(x)[x^\alpha \hat{y}'(x)] - [x^\alpha y'(x)]\hat{y}(x)$ for real-valued functions y and $\hat{y}$ [8, Theorem 3.12].

For $\alpha \neq 1$ we have that $\lim_{x\to\infty}[z_2, z_1](x) = \lim_{x\to\infty}[x^{\alpha}(x^{1-\alpha})'] = (1-\alpha)$; and for $\alpha = 1$ we have that $\lim_{x\to\infty}[z_2, z_1](x) = \lim_{x\to\infty}[x(\ln x)'] = 1$. Since $\lim_{x\to\infty}[z_2, z_1](x) \neq 0$, we must take $k_1 = 0$. Note that $[z_1, z_2](x) = -[z_2, z_1](x)$, so we must also take $k_2 = 0$. Thus, $z = k_3 z_3 + k_4 z_4$. Since $z_3(c)$ and $z_4'(c)$ are not zero, we must take $k_3 = 0$ and $k_4 = 0$. Hence, no linear combination of the z_i is in $\mathscr{D}(L_0)$.

If $\alpha \neq 1$, then $\|Az_1\|^2 = \int_I x^{-\gamma}|a(x)|^2|z_1(x)|^2\,dx = \int_I x^{-\gamma-2\alpha+2}|a(x)|^2\,dx$; and if $\alpha = 1$, then $\|Az_1\|^2 = \int_I x^{-\gamma}|a(x)|^2|z_1(x)|^2\,dx = \int_I x^{-\gamma}|a(x)|^2[\ln x]^2\,dx$. Thus, $\|Az_1\|$ is finite via the inequalities in *(i)*. Moreover, $\|Az_2\|^2 = \int_I x^{-\gamma}|a(x)|^2|z_2(x)|^2\,dx = \int_I x^{-\gamma}|a(x)|^2\,dx$, which is finite via the inequalities in *(i)*. Since z_3 and z_4 are continuous and have compact support, $\|Az_3\|$ and $\|Az_4\|$ are finite. Therefore, each $z_i \in \mathscr{D}(A_1)$. □

Equation (27) and Claim 3.3.1 imply that $\mathscr{D}(L_1) \subset \mathscr{D}(A_1)$. Via Theorem 2.3 A_1 is L_1-compact.

(iii) ⇒ *(ii)* Since A_1 is L_1-compact, $\mathscr{D}(L_1) \subset \mathscr{D}(A_1)$. Via Theorem 2.2 A_1 is L_1-bounded.

(ii) ⇒ *(i)* Since A_1 is L_1-bounded, $\mathscr{D}(L_1) \subset \mathscr{D}(A_1)$. Via equation (27) $S \subset \mathscr{D}(A_1)$. Therefore, $\|Az_i\| < \infty$ for each i, i.e., the inequalities in *(i)* hold.

We obtain a similar result for the maximal operators, L_1 and A_1, associated with the differential expressions (2) and (1), respectively, on the interval $I = (0, \frac{1}{c}]$.

COROLLARY 3.4. *Let $I = (0, \frac{1}{c}]$ for some $c > 0$ and let $\gamma > max\{-1, 2\alpha - 3\}$. Then the following three statements are equivalent.*

(i) If $\alpha > 1$, then $\displaystyle\int_I x^{-\gamma-2\alpha+2}|a(x)|^2 dx < \infty$;

if $\alpha = 1$, then $\displaystyle\int_I x^{-\gamma}|a(x)|^2[\ln(x)]^2 dx < \infty$;

and if $\alpha < 1$, then $\displaystyle\int_I x^{-\gamma}|a(x)|^2 dx < \infty$.

(ii) A_1 is L_1-bounded.

(iii) A_1 is L_1-compact.

Proof. As in the proof of Corollary 3.2, we unitarily transform the singularity at 0 to one at ∞. We apply Theorem 3.3, with the appropriate replacements, to the maximal operators $\widehat{L_1}$ and $\widehat{A_1}$ associated with the transformed differential expressions (23) and (24), respectively, to obtain the equivalence of the following three statements:

(i') If $\alpha > 1$, then $\displaystyle\int_c^{\infty} t^{\gamma+2\alpha-4}|a(1/t)|^2 dt < \infty$;

if $\alpha = 1$, then $\displaystyle\int_c^{\infty} t^{\gamma-2}|a(1/t)|^2[\ln(t)]^2 dt < \infty$;

and if $\alpha < 1$, then $\displaystyle\int_c^{\infty} t^{\gamma-2}|a(1/t)|^2 dt < \infty$.

(ii') $\widehat{A_1}$ is $\widehat{L_1}$-bounded.

(iii') $\widehat{A_1}$ is $\widehat{L_1}$-compact.

Via a change of variables and the unitary transformation, we obtain the desired result.

4. Perturbations B

In this section, we consider perturbations B of L in the limit circle case, where the operators are defined on an interval I by equation (1), with weight $w(x) = x^\gamma$, and equation (2).

THEOREM 4.1. *Let $I = [c, \infty)$ for some $c > 0$ and let $\gamma < min\{-1, 2\alpha - 3\}$. Then*

(i) B_0 is L_0-bounded if and only if

$$\sup_{x \in I} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2(1-\alpha)} |b(u)|^2 \, du < \infty, \tag{28}$$

for some $\varepsilon \in (0, \frac{1}{2})$;

(ii) B_0 is L_0-compact if and only if

$$\lim_{x \to \infty} \frac{1}{x} \int_x^{x+\varepsilon x} u^{2(1-\alpha)} |b(u)|^2 \, du = 0, \tag{29}$$

for some $\varepsilon \in (0, \frac{1}{2})$.

Proof. (i) Sufficiency We begin by showing that B_0' is L_0'-bounded if inequality (28) holds. Let us consider $y \in \mathscr{D}(L_0')$. Since y has compact support in the interior of I, there exists a $d < \infty$ such that the support of y is contained in $[c, d]$. Then we have

$$\|y\|^2 = \int_c^d x^\gamma |y(x)|^2 \, dx, \tag{30}$$

$$\|By\|^2 = \int_c^d x^{-\gamma} |b(x)|^2 |y'(x)|^2 \, dx, \text{ and} \tag{31}$$

$$\|Ly\|^2 = \int_c^d x^{-\gamma} \, |[x^\alpha y'(x)]'|^2 \, dx. \tag{32}$$

We show that the hypotheses of Theorem 2.4 hold for some $\varepsilon \in (0, \frac{1}{2})$ with $N = x^{-\gamma-2\alpha}|b(x)|^2$, $W = x^{-\gamma-2}$, $P = x^{-\gamma}$, and $f = x$. By applying Lemma 2.5 with $g \equiv 1$ and $f = x$, we know that positive and negative powers of x are essentially constant on intervals of length εx. Thus, we have for some constants $C_1, C_2 > 0$

$$\begin{aligned} S_1 &= \sup_{x \in I} \left\{ \frac{1}{\varepsilon^2} \left[\int_x^{x+\varepsilon x} u^\gamma \, du \right] \left[\int_x^{x+\varepsilon x} u^{-\gamma-2\alpha} |b(u)|^2 \, du \right] \right\} \\ &\leqslant \frac{C_1}{\varepsilon} \sup_{x \in I} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2(1-\alpha)} |b(u)|^2 \, du \right\} \text{ and} \end{aligned} \tag{33}$$

$$\begin{aligned} S_2 &= \sup_{x \in I} \left\{ \frac{1}{\varepsilon^2 x^2} \left[\int_x^{x+\varepsilon x} u^{\gamma+2} \, du \right] \left[\int_x^{x+\varepsilon x} u^{-\gamma-2\alpha} |b(u)|^2 \, du \right] \right\} \\ &\leqslant \frac{C_2}{\varepsilon} \sup_{x \in I} \left\{ \frac{1}{x} \int_x^{x+\varepsilon x} u^{2(1-\alpha)} |b(u)|^2 \, du] \right\} \end{aligned} \tag{34}$$

for some $\varepsilon \in (0, \frac{1}{2})$. Inequalities (28), (33), and (34) give us $S_1, S_2 < \infty$ for some $\varepsilon \in (0, \frac{1}{2})$. Therefore (via Theorem 2.4), there exists a constant $C_3 > 0$ such that

$$\begin{aligned} &\int_c^d x^{-\gamma-2\alpha}|b(x)|^2|x^\alpha y'(x)|^2\,dx \\ &\leqslant C_3\left\{\int_c^d x^{-\gamma-2}|x^\alpha y'(x)|^2\,dx + \varepsilon^2\int_c^d x^{-\gamma}|[x^\alpha y'(x)]'|^2\,dx\right\} \\ &= C_3\left\{\int_c^d x^{-\gamma+2\alpha-2}|y'(x)|^2\,dx + \varepsilon^2\|Ly\|^2\right\} \end{aligned} \tag{35}$$

Via Lemma 2.7 the integral on the right-hand side is bounded above by a constant multiple of $\|Ly\|^2$. Therefore, we obtain for some constant $C_4 > 0$

$$\|By\|^2 \leqslant C_4\,\|Ly\|^2 \leqslant C_4\,(\|y\| + \|Ly\|)^2. \tag{36}$$

Thus, $y \in \mathscr{D}(B_0')$. Since y is arbitrary, the inequality above implies that B_0' is L_0'-bounded. Via Theorem 2.1, B_0 is L_0-bounded.

We omit the remaining proofs of Theorem 4.1. The proofs are modifications of the proofs of the corresponding parts of Theorem 3.1, but have the same structure. In particular, for the necessity proofs, the same function ϕ is used, but with $\phi' \equiv 1$ on $[0,1]$.

Next, we prove a corollary of Theorem 4.1. We consider the minimal operators L_0 and B_0 associated with the differential expressions (2) and (1), respectively, on the interval $(0, \frac{1}{c}]$. We prove this result by using a unitary transformation to transform the singularity at 0 to a singularity at ∞ and then applying Theorem 4.1 to the new operators.

COROLLARY 4.2. *Let $I = (0, \frac{1}{c}]$ for some $c > 0$ and let $\gamma > max\{-1, 2\alpha - 3\}$. Then*

(i) B_0 is L_0-bounded if and only if

$$\sup_{x\in I}\ \frac{1}{x}\int_{x-\varepsilon' x}^{x} u^{2(1-\alpha)}\ |b(u)|^2\,du < \infty, \tag{37}$$

for some sufficiently small ε';

(ii) B_0 is L_0-compact if and only if

$$\lim_{x\to 0}\ \frac{1}{x}\int_{x-\varepsilon' x}^{x} u^{2(1-\alpha)}\ |b(u)|^2\,du = 0, \tag{38}$$

for some sufficiently small ε'.

Proof. By applying the same unitary transformation as in Corollary 3.2, we transform the operators L and B into the operators $\widehat{L} = ULU^{-1}$ and $\widehat{B} = UBU^{-1}$, which are defined on $[c, \infty)$ by (23) and

$$\widehat{B}[z](t) = -t^{\gamma+2}\,b(1/t)z'(t) \tag{39}$$

Notice that the operators $\widehat{L}$ and $\widehat{B}$ are of the form (2) and (1), respectively, if we replace $-\gamma$ with $-\gamma' = \gamma + 2$, α with $\alpha' = 2 - \alpha$, and $b(x)$ with $-b(1/t)$, where $t = \frac{1}{x}$. Thus, the condition of Theorem 4.1 that $\gamma' < \min\{-1, 2\alpha' - 3\}$ becomes $\gamma > \max\{-1, 2\alpha - 3\}$. We apply Theorem 4.1, with the appropriate replacements, to the minimal operators $\widehat{L}_0$ and $\widehat{B}_0$, associated with the differential expressions (23) and (39), respectively, to obtain the following results:

(i) $\widehat{B}_0$ is $\widehat{L}_0$-bounded if and only if

$$\sup_{t\in[c,\infty)} \frac{1}{t} \int_t^{t+\varepsilon t} \tau^{2(\alpha-1)} |b(1/\tau)|^2 \, d\tau < \infty, \tag{40}$$

for some $\varepsilon \in (0, \frac{1}{2})$.

(ii) $\widehat{B}_0$ is $\widehat{L}_0$-compact if and only if

$$\lim_{t\to\infty} \frac{1}{t} \int_t^{t+\varepsilon t} \tau^{2(\alpha-1)} |b(1/\tau)|^2 \, d\tau = 0, \tag{41}$$

for some $\varepsilon \in (0, \frac{1}{2})$.

Via a change of variables, we can show that (40) and (41) are equivalent to (37) and (38), respectively, with $\varepsilon' = \varepsilon/(1+\varepsilon)$.

In the next result, we consider the maximal operators, L_1 and B_1, and the minimal operators, L_0 and B_0, associated with the differential expressions (2) and (1), respectively, on the interval $I = [c, \infty)$.

THEOREM 4.3. *Let $I = [c, \infty)$ for some $c > 0$ and let $\gamma < min\{-1, 2\alpha - 3\}$. Then the following three statements are equivalent.*

(i) $\int_I x^{-\gamma-2\alpha} |b(x)|^2 dx < \infty$.

(ii) B_1 is L_1-bounded.

(iii) B_1 is L_1-compact.

Proof. (i) $\Rightarrow$ (iii) We have that $\int_I x^{1-2\alpha} |b(x)|^2 dx \leqslant \int_c^1 x^{1-2\alpha} |b(x)|^2 dx + \int_1^\infty x^{-\gamma-2\alpha} |b(x)|^2 dx$ since $\gamma < -1$. Hence, *(i)* implies that B_0 is L_0-compact via Theorem 4.1. Therefore, $\mathscr{D}(L_0) \subset \mathscr{D}(B_0)$.

Via the first formula of Von Neumann, we can write

$$\mathscr{D}(L_1) = \mathscr{D}(L_0) \oplus S, \tag{42}$$

where S has dimension four since L is regular at c and limit circle at ∞.

CLAIM 4.3.1. $S \subset \mathscr{D}(B_1)$.

Proof of Claim 4.3.1. Define z_1, z_2, z_3, and z_4 as in the proof of Claim 3.3.1. As before, each of these linearly independent functions is contained in $\mathscr{D}(L_1)$.

Define $S = \operatorname{span}\{z_1, z_2, z_3, z_4\}$. Since the proof of Claim 3.3.1 shows that no linear combination of the z_i is in $\mathscr{D}(L_0)$, we have that $\mathscr{D}(L_1) = \mathscr{D}(L_0) \oplus S$. Now, we show that each of the z_i is in $\mathscr{D}(B_1)$.

We have that $\|Bz_1\|^2 = \int_I x^{-\gamma}\,|b(x)|^2|z_1'(x)|^2\,dx = k\int_I x^{-\gamma-2\alpha}\,|b(x)|^2\,dx$, where the constant $k = (1-\alpha)$ if $\alpha \neq 1$ and $k = 1$ if $\alpha = 1$. Thus, $\|Bz_1\|$ is finite via inequality *(i)*. Since $z_2'(x) \equiv 0$, $\|Bz_2\|^2 = 0$. Moreover, $\|Bz_3\|$ and $\|Bz_4\|$ are finite since z_3 and z_4 are continuous and have compact support. Therefore, each $z_i \in \mathscr{D}(B_1)$. □

Equation (42) and Claim 4.3.1 imply that $\mathscr{D}(L_1) \subset \mathscr{D}(B_1)$. Via Theorem 2.3 B_1 is L_1-compact.

(iii) $\Rightarrow$ *(ii)* Since B_1 is L_1-compact, $\mathscr{D}(L_1) \subset \mathscr{D}(B_1)$. Via Theorem 2.2 B_1 is L_1-bounded.

(ii) $\Rightarrow$ *(i)* Since B_1 is L_1-bounded, $\mathscr{D}(L_1) \subset \mathscr{D}(B_1)$. Via equation (42) $S \subset \mathscr{D}(B_1)$. Therefore, $\|Bz_i\| < \infty$ for each i, i.e.,

$$\int_I x^{-\gamma-2\alpha}\,|b(x)|^2\,dx < \infty.$$

Next, we obtain a similar result for the maximal operators, L_1 and B_1, associated with the differential expressions (2) and (1), respectively, on the interval $I = (0, \frac{1}{c}]$.

COROLLARY 4.4. *Let $I = (0, \frac{1}{c}]$ for some $c > 0$ and let $\gamma > max\{-1, 2\alpha - 3\}$. Then the following three statements are equivalent.*

(i) $\int_I x^{-\gamma-2\alpha}\,|b(x)|^2 dx < \infty$.

(ii) B_1 is L_1-bounded.

(iii) B_1 is L_1-compact.

Proof. As in the proof of Corollary 3.2, we unitarily transform the singularity at 0 to one at ∞. We apply Theorem 4.3, with the appropriate replacements, to the maximal operators $\widehat{L}_1$ and $\widehat{B}_1$ associated with the transformed differential expressions (23) and (39), respectively, to obtain the equivalence of the following three statements:

(*i′*) $\int_c^{\infty} t^{\gamma+2\alpha-2}|b(1/t)|^2\,dt < \infty$.

(*ii′*) $\widehat{B}_1$ is $\widehat{L}_1$-bounded.

(*iii′*) $\widehat{B}_1$ is $\widehat{L}_1$-compact.

Via a change of variables and the unitary transformation, we obtain the desired result.

We close with an application of Corollary 3.4 to the energy operator of the hydrogen atom,

$$M[y](x) = -y''(x) + \left[\frac{\ell(\ell+1)}{x^2} + V(x)\right] y(x), \quad 0 < x \leqslant 1,\ 0 \leqslant \ell,$$

(M acts in $\mathscr{L}^2(1, I)$) in the limit circle case, i.e., $0 \leqslant \ell < 1/2$. In [2] it was proved that $V(x)\,y$ is a relatively compact perturbation of the maximal operator N_1 for

$$N[y](x) = -y''(x) + \left[\frac{\ell(\ell+1)}{x^2}\right] y(x),$$

in the limit point case $\ell > 1/2$ if and only if

$$\lim_{x \to 0} \frac{1}{x} \int_{x-\varepsilon x}^{x} u^4 |V(u)|^2 \, du = 0$$

for some sufficiently small ε. To apply Corollary 3.4, we transform M as in [2] by the unitary transformation $Uy = z$ where $z(t) = x^{\ell} y(x)$, $t = x^{1-2\ell}/(1-2\ell)$. Then $K = UMU^{-1}$ is given by

$$K[z](t) = -[P(t)\dot{z}(t)]^{\cdot} + Q(t)z(t), \quad {}^{\cdot} = \frac{d}{dt}, \quad P(t) = x^{-4\ell}, \quad Q(t) = V(x).$$

Applying Corollary 3.4 to K and transforming back to M yields the result that the multiplication operator $V(x)\,y$ is a relatively compact perturbation of N_1 in the limit circle case if and only if

$$\int_0^{1/(1-2\ell)} t^{-2\alpha+2} |Q(t)|^2 \, dt < \infty \iff \int_0^1 x^{2+2\ell} |V(x)|^2 \, dx < \infty,$$

where $\alpha = -4\ell/(1-2\ell)$. As in the limit point case, a Coulomb type potential $V(x) = c/x$ is a relatively compact perturbation of N_1.

REFERENCES

[1] T. G. ANDERSON, R. C. BROWN, AND D. B. HINTON, *Perturbation Theory for a One-Term Weighted Differential Operator, Spectral Theory and Computational Methods of Sturm-Liouville Problems*, Marcel Dekker, Inc., New York, 1997, 149–170.

[2] T. G. ANDERSON AND D. B. HINTON, *Relative Boundedness and Compactness Theory for Second Order Differential Operators*, J. of Ineq. & Appl. **1** (1997), 375–400.

[3] F. V. ATKINSON, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[4] R. C. BROWN, AND D. B. HINTON, *Sufficient Conditions for Weighted Inequalities of Sum Form*, J. Math. Anal. Appl. **112** (1985), 563–578.

[5] S. C. MELESCUE, *A Theory on Perturbations of the Dirac Operator*, J. of Ineq. & Appl., **4** (1999), 17–56

[6] M. A. NAIMARK, *Linear Differential Operators, Part II*, Ungar, New York, 1968.

[7] B. OPIC AND A. KUFNER, *Hardy-type Inequalities*, Longman Scientific and Technical Harlow, Essex, UK, 1990.

[8] J. WEIDMANN, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, Vol 1258, Springer-Verlag, Berlin, 1987.

(Received June 4, 1999)

Don B. Hinton
Mathematics Department
University of Tennessee
Knoxville, TN 37996, USA
e-mail: hinton@math.utk.edu

Suzanne Collier Melescue
Department of Computer Science and Mathematics
Arkansas State University
State University, AR 72467, USA
e-mail: melescue@csm.astate.edu