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A NOTE ON SOME CLASSES OF REAL SEQUENCES

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Dedicated to Professor A. Meir, on his 70th birthday

(*communicated by R. N. Mohapatra*)

Abstract. It is shown that the class of real sequences defined by $\check{Z}$. Tomovski recently is identical with the Fomin's class, furthermore one new class of sequences is defined and compared with some known classes of sequences.

1. Introduction

The literature studying the cosine and sine series is very plentiful. E.g. we can refer to the excellent monograph by R. P. Boas, Jr. [1] or the interesting paper by Č. V. Stanojević [8] and the references given there.

The first results pertaining to the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{1.2}$$

considered the case of monotone coefficients. Later the authors investigated the series (1.1) and (1.2) with *quasi-monotone coefficients* ($a_{n+1} \leqslant a_n(1+\alpha/n)$, $n \geqslant n_0$, $\alpha > 0$).

Several papers deal with *convex* or *quasi-convex null-sequences* ($\Delta^2 a_n \geqslant 0$ or $\sum \mathrm{n}|\Delta^2 \mathrm{a_n}| < \infty$), furthermore with *null-sequences of bounded variation* ($\sum |\Delta a_n| < \infty$).

S. A. Telyakovskiĭ [10] introduced a very effective idea, defined a new class of coefficients. He denoted this class by S; the letter S refers to an esteemed result of S. Sidon [6].

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A null-sequence $\mathbf{a} := \{a_n\}$ belongs to *the class* S, or briefly $\mathbf{a} \in S$, if there exists a monotonically decreasing sequence $\mathbf{A} := \{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leqslant A_n$ hold for all n.

It is easy to verify that this class is a generalization of the class of quasi-convex null-sequences $(A_n := \sum_{k=n}^{\infty} |\Delta^2 a_k|)$.

Utilizing the notion of the class S, Telyakovskiĭ, among others, extended the classical result of A. N. Kolmogorov [3] concerning the L^1-convergence of the series (1.1) with quasi-convex null-sequences.

It is quite natural that several mathematicians have utilized this very applicable notion and proved interesting results in connection with cosine and sine series. It is also factual that some authors have extended the class S.

E.g. G. A. Fomin [2] gave the following definition.

A null-sequence **a** belongs to *the class* $F_p, p > 1$, if

$$\sum_{n=1}^{\infty} \left(n^{-1} \sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} < \infty.$$

It can be also shown that

$$S \subseteq F_p \subseteq BV, \tag{1.3}$$

where BV *denotes the class of sequences having bounded variation.*

Later N. Singh and K. M. Sharma [7] gave a further extension of the class S as follows:

A null-sequence **a** belongs to *the class* S' if there exists a quasi-monotone sequence **A** such that $\sum_{n=1}^{\infty} A_n < \infty$ and $|\Delta a_n| \leqslant A_n$ for all n.

S. Zahid Ali Zenei [12], using the concept of the δ-quasi-monotone sequences, introduced the class $S(\delta)$.

A null-sequence **a** is said to be δ-*quasi-monotone* if $a_n > 0$ and $\Delta a_n \geqslant -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers.

The definition of *the class* $S(\delta)$ is the following: A null-sequence **a** belongs to the class $S(\delta)$ if there exists a δ-quasi-monotone sequence **A** such that $\sum_{n=1}^{\infty} A_n < \infty$, $|\Delta a_n| \leqslant A_n$ for all n, furthermore $\sum_{n=1}^{\infty} n\, \delta_n < \infty$.

In [9] Č. V. Stanojević and V. B. Stanojević defined a new class of sequences as follows:

A null-sequence **a** belongs to *the class* S_p if there exists a positive monotonically decreasing sequence **A** such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(1). \tag{1.4}$$

In [11] $\check{Z}$ Tomovski defined a further class of sequences in the following manner.

A null-sequence **a** belongs to *the class* $S_p(\delta)$ if there exists a δ-quasi-monotone sequence **A** satisfying the assumptions (1.4) and $\sum_{n=1}^{\infty} n\delta_n < \infty$.

It is clear that

$$S \subseteq S' \subseteq S(\delta) \subseteq S_p(\delta).$$

Very recently we ([4]) also defined a class S_p^* of sequences. Our *class* S_p^* contains the null-sequences **a** having the property

$$\sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} < \infty$$

for a fixed $p > 1$.

The real concrete aim of these extensions was always the generalization of some theorems, proved on sine and cosine series, to a wider class. These plans have been successful in general.

In my view the kernel of these succesful enterprises has been based on the following facts proved very rencently.

In [4] we verified that if $p > 1$ then

$$F_p \subseteq S_p \subseteq S_p^* \subseteq F_p \tag{1.5}$$

holds and in [5] we showed that

$$S \subseteq S' \subseteq S(\delta) \subseteq S(A) \subseteq S \tag{1.6}$$

is also valid.

The class $S(A)$ has not been mentioned so far, because it appeared only in the proof of $S(\delta) \subset S$ given in [5]. Namely analyzing the proof in question it looked like that the class $S(A)$ is a wider class than $S(\delta)$, but finally it has turned out that the classes $S(A)$ and $S(\delta)$ are the same.

However, in my view, the class $S(A)$ is worth for recealling its definition and explaining that it is also identical with the class S.

A null-sequence **a** belongs to *the class* $S(A)$ if there exists a null-sequence **A** such that

$$|\Delta a_n| \leqslant A_n \tag{1.7}$$

and

$$\sum_{n=1}^{\infty} n|\Delta A_n| < \infty. \tag{1.8}$$

As we have already mentioned, see (1.3),

$$S \subseteq F_p \tag{1.9}$$

holds, but an example given in [4] shows that this embedding is strict, that is, there exists a null-sequence **a** such that

$$\mathbf{a} \in F_p \qquad \text{but} \qquad \mathbf{a} \notin S. \tag{1.10}$$

Collecting the partial results of (1.5), (1.6), (1.9) and (1.10) we get the following embedding relations:

$$S \equiv S' \equiv S(\delta) \equiv S(A) \underset{\neq}{\subset} F_p \equiv S_p \equiv S_p^* \tag{1.11}$$

The goal of the present note is to clear up the relation of the class $S_p(\delta)$ to the others, furthermore we define the analogue of the class $S(A)$ for $p>1$, to be denoted by $S_p(A)$, and compare it with the classes appearing in (1.11) and also with $S_p(\delta)$.

A null-sequence $\mathbf{a}$ belongs to the class $S_p(A)$ if there exists a null-sequence $\mathbf{A}$ such that (1.8) and

$$\frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1) \tag{1.12}$$

hold.

We shall show that if $p>1$ then

$$S_p\subseteq S_p(\delta)\subseteq S_p(A)\subseteq S_p^* \tag{1.13}$$

2. Result

The embedding relations (1.5) and (1.13) will lead to the following result.

THEOREM. *If $p>1$ then the following identity*

$$F_p\equiv S_p\equiv S_p^*\equiv S_p(\delta)\equiv S_p(A) \tag{2.1}$$

holds.

By (1.11) and (2.1) we have the following

COROLLARY. *If $p>1$ then the following embedding relations*

$$S\equiv S'\equiv S(\delta)\equiv S(A)\underset{\neq}{\subset}F_p\equiv S_p\equiv S_p^*\equiv S_p(\delta)\equiv S_p(A)$$

are valid.

3. Lemma

The following result can be found in [7].

LEMMA. *Let $\{c_n\}$ be a δ-quasi-monotone sequence with*

$$\sum_{n=1}^{\infty}n\,\delta_n<\infty.$$

If

$$\sum_{n=1}^{\infty}c_n$$

converges, then

$$\sum_{n=1}^{\infty}(n+1)|\Delta c_n|<\infty.$$

4. Proof

The embedding relation $S_p \subseteq S_p(\delta)$ evidently follows from the definitions. It is enough to choose $\delta_k := k^{-3}$.

Next we show that if $\mathbf{a} \in S_p(\delta)$ then $\mathbf{a} \in S_p(A)$ also holds.

Since $\mathbf{a} \in S_p(\delta)$ there exists a δ-quasi-monotone sequence $\mathbf{A}$ with $\sum_{n=1}^{\infty} n\,\delta_n < \infty$. Applying the Lemma with A_n we get that the condition (1.8) holds.

On the other hand the estimation (1.12) is automatically satisfied by the assumption $\mathbf{a} \in S_p(\delta)$, see the conditions (1.4).

Thus the embedding statement

$$S_p(\delta) \subseteq S_p(A)$$

is verified.

Finally we prove the embedding relation

$$S_p(A) \subseteq S_p^*.$$

Setting

$$D_m := \sum_{n=2^m}^{2^{m+1}} |\Delta A_n|,$$

by (1.8) we obtain that

$$\sum_{m=0}^{\infty} 2^m D_m < \infty. \tag{4.1}$$

Since $A_n \to 0$ thus

$$A_{2^m} = \sum_{n=2^m}^{\infty} \Delta A_n \leqslant \sum_{n=m}^{\infty} D_n.$$

Utilizing the last inequality and (4.1) we get that

$$\sum_{m=1}^{\infty} 2^m A_{2^m} \leqslant \sum_{m=1}^{\infty} 2^m \sum_{n=m}^{\infty} D_n = \sum_{n=1}^{\infty} D_n \sum_{m=1}^{n} 2^m \leqslant 2 \sum_{n=1}^{\infty} 2^n D_n < \infty. \tag{4.2}$$

Now we define one more sequence $\{C_m\}$ as follows:

$$C_m := A_{2^m} + D_m \qquad \text{for all} \qquad \text{m} \geqslant 1.$$

If $2^m < k \leqslant 2^{m+1}$ then

$$A_k = A_{2^m} - \sum_{n=2^m}^{k-1} \Delta A_n \leqslant A_{2^m} + \sum_{n=2^m}^{k-1} |\Delta A_n| \leqslant A_{2^m} + D_m = C_m.$$

Using this estimation we obviously have the following inequality

$$\sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} \leqslant \sum_{m=1}^{\infty} 2^m C_m \left\{ 2^{-m} \sum_{n=2^m+1}^{2^{m+1}} \frac{|\Delta a_n|^p}{A_n^p} \right\}^{1/p}. \tag{4.3}$$

Here the sum in the bracket is $O(1)$ by (1.12), thus if we can show that

$$\sum_{m=1}^{\infty} 2^m C_m < \infty, \tag{4.4}$$

then (4.3) implies that $\mathbf{a} \in S_p^*$. But (4.4) clearly stays by (4.1) and (4.2).

Herewith the embedding relation

$$S_p(A) \subseteq S_p^*$$

is also proved.

Summing up our partial results we obtain the assertion (2.1) of the Theorem.

The proof is complete.

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