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ON THE OSTROWSKI'S INTEGRAL INEQUALITY FOR MAPPINGS WITH BOUNDED VARIATION AND APPLICATIONS

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Abstract. A generalization of Ostrowski's inequality for mappings with bounded variation and applications in Numerical Analysis for Euler's Beta function is given.

1. Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

THEOREM 1.1. *Let* $f : [a,b] \to \mathbf{R}$ *be a differentiable mapping on* (a,b) *whose derivative is bounded on* (a,b) *and denote* $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. *Then for all* $x \in [a,b]$ *we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty .$$

The constant $\frac{1}{4}$ *is sharp in the sense that it can not be replaced by a smaller one.*

In this paper we prove an Ostrowski's type inequality for mappings with bounded variation and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

2. Ostrowski's inequality for mappings with bounded variation

The following inequality for mappings with bounded variation holds:

THEOREM 2.1. *Let* $u : [a,b] \to \mathbf{R}$ *be a mapping with bounded variation on* $[a,b]$. *Then for all* $x \in [a,b]$, *we have the inequality*

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leqslant \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u). \tag{2.1}$$

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where $\bigvee_a^b(u)$ *denotes the total variation of* u.
The constant $\frac{1}{2}$ *is the best possible one.*

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a)du(t) = u(x)(x-a) - \int_a^x u(t)dt$$

and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t \in [a,x) \\ \\ t-b & \text{if} \quad x \in [x,b], \end{cases}$$

for all $x,t \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max\limits_{i \in \{0,\ldots,n-1\}} (x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)}\right]$.
If $p : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbf{R}$ is with bounded variation on $[a,b]$, then

$$\left|\int_a^b p(x)dv(x)\right| = \left|\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)}) \left[v\left(x_{i+1}^{(n)}\right) - v(x_i^{(n)})\right]\right| \tag{2.3}$$

$$\leqslant \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left|p(\xi_i^{(n)})\right| \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right|$$

$$\leqslant \sup_{x\in[a,b]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} \left|v\left(x_{i+1}^{(n)}\right) - v\left(x_i^{(n)}\right)\right| = \sup_{x\in[a,b]} |p(x)| \bigvee_a^b(v).$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x), x \in [a,b]$, we get

$$\left|\int_a^b p(x,t)du(t)\right| \leqslant \sup_{t\in[a,b]} |p(x,t)| \bigvee_a^b(u) \tag{2.4}$$

$$= \max\{x-a, b-x\} \bigvee_a^b (u) = \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u)$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left|\int_a^b u(t)dt - u(x)(b-a)\right| \leqslant \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u). \tag{2.5}$$

for all $x \in [a,b]$.

Consider the mapping $u : [a,b] \to \mathbf{R}$, given by

$$u(x) = \begin{cases} 0 & \text{if} \quad x \in [a,b] \setminus \left\{\frac{a+b}{2}\right\} \\ 1 & \text{if} \quad x = \frac{a+b}{2} \end{cases}$$

in (2.5). Then u is with bounded variation on $[a,b]$, and

$$\bigvee_a^b (u) = 2, \qquad \int_a^b u(t)dt = 0$$

and for $x = \frac{a+b}{2}$, we get in (2.5)

$$1 \leqslant 2C$$

which implies that $C \geqslant \frac{1}{2}$ and the theorem is completely proved.

The following corollary holds:

COROLLARY 2.2. *Let $u : [a,b] \to \mathbf{R}$ be a monotonous mapping on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b u(t)dt - u(x)(b-a)\right| \leqslant \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] |u(b) - u(a)|.$$

The case of lipschitzian mappings is embodied in the following corollary.

COROLLARY 2.3. *Let $u : [a,b] \to \mathbf{R}$ be an L-lipschitzian mapping on $[a,b]$, i.e., we recall*

$$|u(x) - u(y)| \leqslant L|x-y| \quad \text{for all } x, y \in [a,b].$$

Then we have the inequality

$$\left|\int_a^b u(t)dt - u(x)(b-a)\right| \leqslant L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](b-a).$$

The best inequality we can get from (2.1) is that one for which $x = \frac{a+b}{2}$ obtaining

COROLLARY 2.4. *Let* $u : [a,b] \to \mathbf{R}$ *be as above . Then we have the inequality:*

$$\left| \int_a^b u(t)dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \leqslant \frac{1}{2}(b-a)\bigvee_a^b(u). \tag{2.6}$$

Similar inequalities can be found if we assume that u is monotonous or lipschitzian on $[a,b]$. We shall omit the details.

REMARK 2.1. If we assume that u is continuous differentiable on (a,b) and u' is integrable on (a,b), then by (2.1) we get

$$\left| \int_a^b u(t)dx - u\left(\frac{a+b}{2}\right)(b-a) \right| \leqslant \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right|\right] \|u'\|_1$$

which is the inequality obtained by Dragomir and Wang in the recent paper [1].

REMARK 2.2. It is well known that if $f : [a,b] \to \mathbf{R}$ is a convex mapping on $[a,b]$, then Hermite-Hadamard's inequality holds

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset \mathbf{R} \to \mathbf{R}$ is convex on I and $a,b \in Int(I), a < b$; then f'_+ is monotonous nondecreasing on $[a,b]$ and by Corollary 2.4 we get

$$0 \leqslant \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\|f'_+\|_1 \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

Similar results can be obtained if we assume that f is convex and monotonous or convex and lipschitzian on $[a,b]$.

3. A quadrature formula of Riemann type

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,\ldots,n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)\, h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

THEOREM 3.1. *Let* $f : [a,b] \to \mathbf{R}$ *be a mapping with bounded variation on* $[a,b]$ *and* $I_n, \xi_i\ (i = 0,\dots,n-1)$ *be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi) \tag{3.1}$$

where the remainder satisfies the estimation

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leqslant \sup_{i=0,\dots,n} \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \\ &\leqslant \left[\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \leqslant \nu(h) \bigvee_a^b(f) \end{aligned} \tag{3.2}$$

for all $\xi_i (i = 0,\dots,n-1)$ *as above, where* $\nu(h) := \max_{i=0,\dots,n} h_i$.
The constant $\frac{1}{2}$ *is sharp in* (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right| \leqslant \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f). \tag{3.3}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$\begin{aligned} |W_n(f, I_n, \xi)| &\leqslant \sum_{i=0}^{n-1} \left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right| \\ &\leqslant \sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_{x_i}^{x_{i+1}}(f) \\ &\leqslant \sup_{i=0,\dots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(f). \\ &= \sup_{i=0,\dots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b(f) \end{aligned}$$

The second inequality follows by the properties of $\sup(.)$.
Now, as

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leqslant \frac{1}{2}h_i$$

for all $\xi_i \in [x_i, x_{i+1}]\ (i = 0,\dots,n-1)$ the last part of (3.2) is also proved.

COROLLARY 3.2. *Let $f : [a,b] \to \mathbf{R}$ be a monotonous mapping on $[a,b]$ and $I_n, \xi_i\,(i = 0,\ldots,n-1)$ be as above. Then we have the Riemann quadrature formula (3.1) and the remainder satisfies the estimation*

$$|W_n(f,I_n,\xi)| \leqslant \sup_{i=0,\ldots,n}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]|f(b)-f(a)|$$

$$\leqslant \left[\frac{1}{2}\nu(h) + \sup_{i=0,\ldots,n}\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]|f(b)-f(a)| \leqslant \nu(h)\,|f(b)-f(a)|$$

for all $\xi_i\,(i = 0,\ldots,n-1)$ as above.

The case of lipschitzian mappings is embodied into the following corollary.

COROLLARY 3.3. *Let $f : [a,b] \to \mathbf{R}$ be an L-lipschitzian mapping on $[a,b]$ and $I_n, \xi_i\,(i = 0,\ldots,n-1)$ be as above. Then we have the Riemann quadrature formula (3.1) and the remainder satisfies the estimation*

$$|W_n(f,I_n,\xi)| \leqslant L\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]h_i$$

$$\leqslant L\sum_{i=0}^{n-1}h_i^2$$

The proof is obvious by Corollary 2.3 applied on the intervals $[x_i, x_{i+1}]$ and summing the obtained inequalities.

We shall omit the details.

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i+x_{i+1}}{2}$ obtaining the following midpoint formula:

COROLLARY 3.4. *Let f, I_n be as Theorem 3.1. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f,I_n) + S_n(f,I_n)$$

where

$$M_n(f,I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i$$

and the remainder $S_n(f,I_n)$ satisfies the estimation

$$|S_n(f,I_n)| \leqslant \frac{1}{2}\nu(h)\bigvee_a^b(f).$$

Similar results can be obtained from Corollaries 3.2 and 3.3.

REMARK 3.1. If we assume that $f : [a,b] \to \mathbf{R}$ is differentiable on (a,b) and whose derivative f' is integrable on (a,b) we can put instead of $\bigvee_a^b(f)$ the L_1-norm $\|f'\|_1$ obtaining the estimation due to Dragomir-Wang from the paper [1].

4. Applications for Euler's Beta mapping

Consider the mapping Beta for real numbers

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q>0$$

and the mapping $e_{p,q}(t) := t^{p-1}(1-t)^{q-1}, t \in [0,1]$.

We have for $p,q>1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)[p-1-(p+q-2)t]$$

and as

$$|p-1-(p+q-2)t| \leqslant \max\{p-1,q-1\}$$

for all $t \in [0,1]$, then

$$\left\|e'_{p,q}\right\|_1 \leqslant \max\{p-1,q-1\} \left\|e_{p-2,q-2}\right\|_1 \tag{4.1}$$

$$= \max\{p-1,q-1\}B(p-1,q-1); \qquad p,q>1.$$

The following inequality for Beta mapping holds

PROPOSITION 4.1. *Let $p,q>1$ and $x \in [0,1]$. Then we have the inequality*

$$\left|B(p,q) - x^{p-1}(1-x)^{q-1}\right| \tag{4.2}$$

$$\leqslant \max\{p-1,q-1\} B(p-1,q-1) \left[\frac{1}{2} + \left|x - \frac{1}{2}\right|\right].$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\left\|e'_{p,q}\right\|_1$ satisfies the inequality (4.1).

COROLLARY 4.2. *Let $p,q>1$. Then we have the inequality*

$$\left|B(p,q) - \frac{1}{2^{p+q-2}}\right| \leqslant \frac{1}{2} \max\{p-1,q-1\} B(p-1,q-1).$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

PROPOSITION 4.3. *Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,\ldots,n-1)$ a sequence of intermediate points for I_n and $p,q>1$. Then we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1}(1-\xi_i)^{q-1} h_i + T_n(p,q)$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$|T_n(p,q)|$$

$$\leqslant \max\{p-1,q-1\} \left[\frac{1}{2}\nu(h) + \sup_{i=0,\ldots,n} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] B(p-1,q-1)$$

$$\leqslant \max\{p-1,q-1\}\,\nu(h)B(p-1,q-1).$$

Particularly, if we choose above $\xi_i = \frac{x_i+x_{i+1}}{2}$ $(i=0,\dots,n-1)$ *then we get the approximation*

$$B(p,q) = \frac{1}{2^{p+q-2}}\sum_{i=0}^{n-1}(x_i+x_{i+1})^{p-1}(2-x_i-x_{i+1})^{q-1} + V_n(p,q)$$

where

$$|V_n(p,q)| \leqslant \frac{1}{2}\max\{p-1,q-1\}\,\nu(h)B(p-1,q-1).$$

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