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BEST POSSIBLE BOUNDS FOR ORDERED POSITIVE NUMBERS USING THEIR SUM AND PRODUCT

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Abstract. Best possible bounds for real numbers $\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0$ with prescribed sum $a = \lambda_1 + \cdots + \lambda_n$ and product $d = \lambda_1 \cdots \lambda_n$ are presented. These bounds can be expressed algebraically only in certain special cases. In the general case, explicit bounds are found by using extra bounds. The results are applied to eigenvalue estimation, when the λ_k 's are regarded as eigenvalues of an n by n matrix A, $a = \operatorname{tr} A$, and $d = \det A$. The case when the eigenvalues are real but not necessarily positive is also discussed. The bounds are compared with bounds using a and $b = \lambda_1^2 + \cdots + \lambda_n^2$; i.e, with eigenvalue bounds using $\operatorname{tr} A$ and $\operatorname{tr} A^2$.

1. Introduction

Throughout this paper, we consider real numbers $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ $(n \geqslant 2)$ with prescribed sum, sum of squares, and product,

$$a = \lambda_1 + \cdots + \lambda_n, \qquad b = \lambda_1^2 + \cdots + \lambda_n^2, \qquad d = \lambda_1 \cdots \lambda_n.$$

Then $nb \geqslant a^2$. If $\lambda_n > 0$, we have $a > 0$ and $0 < n^n d \leqslant a^n$.

The problem to find the best possible bounds for n ordered real numbers with prescribed arithmetic mean and standard deviation is widely studied by many solvers. Jensen and Styan [3, 4] credit priorities to Laguerre, Samuelson, Brunk, Boyd, and Hawkins. An equivalent question is to find bounds for an individual λ_k, using only a and b (and, of course, n and k). Wolkowicz and Styan [13] applied these results to eigenvalue estimation, regarding the λ_k 's as eigenvalues of an n by n matrix A with

$$a = \operatorname{tr} A, \qquad b = \operatorname{tr} A^2.$$

We call these bounds Wolkowicz–Styan bounds (shortly WS bounds) rather than "Laguerre–Samuelson–Brunk–Boyd–Hawkins–etc. bounds".

More generally, the best possible bounds for

$$s_{kl} = \lambda_k + \cdots + \lambda_l \qquad (1 \leqslant k \leqslant l \leqslant n),$$

using only a and b, were presented in [13]. Under an assumption to guarantee $\lambda_l > 0$, the best possible bounds for λ_k / λ_l were found in [6].

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The problem to find bounds for n ordered positive numbers with prescribed arithmetic mean and *geometric mean* has received less attention.

Regarding the λ_k's as eigenvalues with $\lambda_n > 0$, the problem to find an upper bound for λ_1/λ_n with prescribed

$$a = \operatorname{tr} A, \qquad d = \det A$$

was discussed by Guggenheimer, Edelman, and Johnson [2]. The best possible bound was found in [8], [7]. Bounds for s_{kl} and

$$p_{kl} = \lambda_k \cdots \lambda_l \qquad (1 \leqslant k \leqslant l \leqslant n),$$

involving only a and d, have also been found ([9], [10]), but these bounds are not the best possible.

We pursue this topic further. We will first (Sec. 2.) find the best possible bounds for λ_k (and, more generally, for s_{kl} and p_{kl}), using only a and d (and, of course, n, k, and l). We will then meet a pair of equations (Lemma 1), which cannot be solved algebraically in general, but can be if n is even and $k = n/2$ (Sec. 3.). Next (Sec. 4.) we will consider other cases. Thereafter (Sec. 5.), we will compare our bounds with the WS bounds. Further (Sec. 6.), we will show that the WS eigenvalue bounds of A are obtained by applying our bounds to $A + tI$, subtracting t, and letting $t \to \infty$. Finally (Sec. 7.), we will study eigenvalue estimation with examples.

A more general approach would be possible. Instead of s_{kl} and p_{kl}, we may consider any elementary symmetric function of $\lambda_k, \ldots, \lambda_l$. Instead of prescribing a and d, we may fix the values of any two (or more) elementary symmetric functions of $\lambda_1, \ldots, \lambda_n$. Instead of prescribing a and b, we may fix the values of any two (or more) power sums, thus generalizing [13]. Further, we may generalize power sums to $g(\lambda_1) + \cdots + g(\lambda_n)$ where g is a given strictly convex or concave function. See [11].

2. Bounds as solutions of equations

LEMMA 1. *Let $1 \leqslant k < n$. The function*

$$f_k(x) = x^k \left(\frac{a - kx}{n - k} \right)^{n-k}$$

is strictly decreasing on the interval $[a/n, a/k]$. Moreover, the system

$$\begin{aligned} &kx + (n-k)y = a, \\ &x^k y^{n-k} = d, \\ &x \geqslant y > 0 \end{aligned} \tag{S}$$

has a unique solution x, y.

Proof. Denoting

$$g(x) = (n-k)^{n-k} f_k(x) = x^k (a - kx)^{n-k},$$

we have, for $a/n < x < a/k$,

$$g'(x) = kx^{k-1}(a-kx)^{n-k-1}(a-nx) < 0.$$

Therefore g is strictly decreasing, and so is also f_k.

Eliminating y from (S), we have

$$f_k(x) = x^k \left(\frac{a-kx}{n-k}\right)^{n-k} = d,$$

where

$$\frac{a-kx}{n-k} \leqslant x < \frac{a}{k}; \quad \text{i.e., } \frac{a}{n} \leqslant x < \frac{a}{k}.$$

Since

$$f_k\left(\frac{a}{n}\right) = \left(\frac{a}{n}\right)^n \geqslant d,\ f_k\left(\frac{a}{k}\right) = 0 < d,$$

a solution exists, and, since f is strictly decreasing, the solution is unique.

We denote the solution (x,y) of (S) by $(\xi_k(n,a,d), \eta_k(n,a,d))$ or shortly by (ξ_k, η_k).

THEOREM 1. *Assume $\lambda_n > 0$. The best possible bounds for the λ_i's, using only a and d, are*

$$\begin{aligned}(\xi_{n-1}, \eta_1, \eta_2, \ldots, \eta_{n-1}) &\leqslant \\ (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) &\leqslant \\ (\xi_1, \xi_2, \ldots, \xi_{n-1}, \eta_1).\end{aligned}$$

Here $\leqslant$ denotes elementwise ordering.

Proof. To show $\lambda_i \leqslant \xi_i$ $(1 \leqslant i < n)$, suppose $\lambda_i > \xi_i$. Then

$$\begin{gathered}(\lambda_1 \cdots \lambda_i)(\lambda_{i+1} \cdots \lambda_n) \leqslant \\ \left(\frac{\lambda_1 + \cdots + \lambda_i}{i}\right)^i \left(\frac{\lambda_{i+1} + \cdots + \lambda_n}{n-i}\right)^{n-i} = \\ f_i\left(\frac{\lambda_1 + \cdots + \lambda_i}{i}\right) < f_i(\xi_i) = d,\end{gathered}$$

since f_i is strictly decreasing by Lemma 1, and

$$\frac{\lambda_1 + \cdots + \lambda_i}{i} \geqslant \lambda_i > \xi_i.$$

But this contradicts $\lambda_1\lambda_2\cdots\lambda_n = d$. Therefore $\lambda_i \leqslant \xi_i$ and, as above,

$$\frac{\lambda_1 + \cdots + \lambda_i}{i} \leqslant \xi_i.$$

To show $\lambda_i \geqslant \eta_{i-1}$ $(1 < i \leqslant n)$, we now get

$$\lambda_1 + \cdots + \lambda_{i-1} \leqslant (i-1)\xi_{i-1}.$$

Hence

$$\lambda_i \geqslant \frac{\lambda_i + \cdots + \lambda_n}{n-i+1} \geqslant \frac{a-(i-1)\xi_{i-1}}{n-i+1} = \eta_{i-1}.$$

To show $\lambda_1 \geqslant \xi_{n-1}$, suppose $\lambda_1 < \xi_{n-1}$. If $x \geqslant y \geqslant t > 0$, then clearly $(x+t)(y-t) < xy$. By applying this fact $n-1$ times, we get

$$\lambda_1\lambda_2\cdots\lambda_n >$$

$$\xi_{n-1}\lambda_2\cdots\lambda_{n-1}(\lambda_n - (\xi_{n-1} - \lambda_1)) > \cdots >$$

$$\xi_{n-1}^{n-1}(\lambda_n - (\xi_{n-1} - \lambda_1) - (\xi_{n-1} - \lambda_2) - \cdots - (\xi_{n-1} - \lambda_{n-1})) =$$

$$\xi_{n-1}^{n-1}(a - (n-1)\xi_{n-1}) =$$

$$\xi_{n-1}^{n-1}\eta_{n-1} = d,$$

which is a contradiction.

To show $\lambda_n \leqslant \eta_1$, suppose $\lambda_n > \eta_1$. Then

$$\lambda_1\lambda_2\cdots\lambda_n >$$

$$(\lambda_1 + (\lambda_n - \eta_1))\lambda_2\cdots\lambda_{n-1}\eta_1 > \cdots >$$

$$(\lambda_1 + (\lambda_n - \eta_1) + (\lambda_{n-1} - \eta_1) + \cdots + (\lambda_2 - \eta_1))\eta_1^{n-1} =$$

$$(a - (n-1)\eta_1)\eta_1^{n-1} = \xi_1\eta_1^{n-1} = d,$$

which is a contradiction. This completes the proof of the inequalities.

To show that the bounds are the best possible, we note that, for given a and d (with $0 < d \leqslant (a/n)^n$) and i $(1 \leqslant i < n)$, there exist by Lemma 1 $\lambda_1 = \cdots = \lambda_i = \xi_i$, $\lambda_{i+1} = \cdots = \lambda_n = \eta_i$ satisfying $\lambda_1 + \cdots + \lambda_n = a$, $\lambda_1 \cdots \lambda_n = d$.

The proof of Theorem 1 implies also the following:

THEOREM 2. *Assume $\lambda_n > 0$. Then*

$\lambda_1 = \xi_{n-1}$ $(\lambda_n = \eta_{n-1})$ *if and only if* $\lambda_1 = \cdots = \lambda_{n-1} = \xi_{n-1}$ *and* $\lambda_n = \eta_{n-1}$,

$\lambda_1 = \xi_1$ $(\lambda_n = \eta_1)$ *if and only if* $\lambda_1 = \xi_1$ *and* $\lambda_2 = \cdots = \lambda_n = \eta_1$

and, for $1 < k < n$,

$\lambda_k = \xi_k$ *if and only if* $\lambda_1 = \cdots = \lambda_k = \xi_k$ *and* $\lambda_{k+1} = \cdots = \lambda_n = \eta_k$,

$\lambda_k = \eta_{k-1}$ *if and only if* $\lambda_1 = \cdots = \lambda_{k-1} = \xi_{k-1}$ *and* $\lambda_k = \cdots = \lambda_n = \eta_{k-1}$.

The best possible bounds for $\lambda_1 + \lambda_n$ and $\lambda_1\lambda_n$, using only a, d, are obviously

$$\xi_{n-1} + \eta_{n-1} \leqslant \lambda_1 + \lambda_n \leqslant \xi_1 + \eta_1,$$

$$\xi_{n-1}\eta_{n-1} \leqslant \lambda_1\lambda_n \leqslant \xi_1\eta_1.$$

Next, we study s_{kl} and p_{kl} $(1 \leqslant k \leqslant l < n)$. We have

$$s_{kl} = \lambda_k + \cdots + \lambda_l \leqslant$$

$$(l-k+1)\frac{\lambda_1 + \cdots + \lambda_l}{l} \leqslant$$

$$(l-k+1)\xi_l.$$

Similarly,

$$p_{kl} = \lambda_k \cdots \lambda_l \leqslant$$

$$\left(\frac{\lambda_k + \cdots + \lambda_l}{l-k+1}\right)^{l-k+1}$$

$$\leqslant \xi_l^{l-k+1}.$$

These are the best possible upper bounds for s_{kl} and p_{kl}. In a similar way we see that if $1 < k \leqslant l \leqslant n$, then the the best possible lower bound for s_{kl} is $(l-k+1)\eta_{k-1}$. Since

$$\lambda_k \cdots \lambda_l \geqslant$$

$$(\lambda_k \cdots \lambda_n)^{\frac{l-k+1}{n-k+1}} =$$

$$\left(\frac{d}{\lambda_1 \cdots \lambda_{k-1}}\right)^{\frac{l-k+1}{n-k+1}} \geqslant$$

$$\left(\frac{d}{\xi_{k-1}^{k-1}}\right)^{\frac{l-k+1}{n-k+1}} =$$

$$\eta_{k-1}^{l-k+1},$$

the best possible lower bound for p_{kl} is η_{k-1}^{l-k+1}. The problems of finding upper bounds in the case $1 < k \leqslant l = n$ and lower bounds in the case $1 = k \leqslant l < n$ are more complicated, see [11].

3. Special case

If n is even and $k = n/2$, we can solve algebraically (S), which in this case is

$$\frac{n}{2}x + \frac{n}{2}y = a,$$

$$x^{\frac{n}{2}}y^{\frac{n}{2}} = d,$$

$$x \geqslant y > 0;$$

i.e.,

$$x + y = \frac{2a}{n},$$

$$xy = d^{\frac{2}{n}},$$

$$x \geqslant y > 0.$$

The solution is

$$x = U(n,a,d) = \frac{a}{n} + \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}},$$

$$y = L(n,a,d) = \frac{a}{n} - \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}}.$$

Hence, by Theorem 1, we obtain

THEOREM 3. *Assume n even, $\lambda_n > 0$. Then*

$$\frac{a}{n} - \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}} \leqslant \lambda_{\frac{n}{2}+1} \leqslant \lambda_{\frac{n}{2}} \leqslant \frac{a}{n} + \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}}.$$

These bounds are the best possible, using only a and d.

Consider the ordered multiset $\{\mu_1, \ldots, \mu_{2n}\} = \{\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n\}$. Then $\mu_{2i-1} = \mu_{2i} = \lambda_i$ $(1 \leqslant i \leqslant n)$, $\mu_1 + \cdots + \mu_{2n} = 2a$, $\mu_1 \cdots \mu_{2n} = d^2$. Moreover, $L(2n, 2a, d^2) = L(n,a,d)$ and $U(2n, 2a, d^2) = U(n,a,d)$. Applying Theorem 3 to the μ_i's, we have

THEOREM 4. *Assume $\lambda_n > 0$. Then*

$$\frac{a}{n} - \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}} \leqslant \lambda_{\lceil \frac{n+1}{2} \rceil} \leqslant \lambda_{\lfloor \frac{n+1}{2} \rfloor} \leqslant \frac{a}{n} + \sqrt{\left(\frac{a}{n}\right)^2 - d^{\frac{2}{n}}}.$$

Here $\lceil x \rceil$ denotes the smallest integer $\geqslant x$ and $\lfloor x \rfloor$ the largest integer $\leqslant x$. If n is odd, these bounds are not the best possible.

4. General case

Let $\alpha > 0$. Define $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n+r}$ as

$$\{\mu_1, \mu_2, \cdots, \mu_{n+r}\} = \{\lambda_1, \ldots, \lambda_n, \alpha, \ldots, \alpha\},$$

where the multiset on the right is not neccessarily ordered. By Theorem 4,

$$L(n+r, a+r\alpha, d\alpha^r) \leqslant \mu_{\lceil \frac{n+r+1}{2} \rceil} \leqslant \mu_{\lfloor \frac{n+r+1}{2} \rfloor} \leqslant U(n+r, a+r\alpha, d\alpha^r). \qquad (*)$$

If α is some bound for λ_k, say a lower bound, we can get an upper bound and a new lower bound for λ_k by choosing r suitably and applying $(*)$. The seemingly most interesting applications are presented in the following two theorems.

THEOREM 5. *Assume $\lambda_n > 0$. Let $1 \leqslant k \leqslant n/2$. If $\lambda_k \leqslant \beta_k$, then*

$$\lambda_k \leqslant \frac{a+(n-2k)\beta_k}{2(n-k)} + \sqrt{\left(\frac{a+(n-2k)\beta_k}{2(n-k)}\right)^2 - (d\beta_k^{n-2k})^{\frac{1}{n-k}}}.$$

Proof. Choose $r = n-2k$, $\alpha = \beta_k$. Then

$$\{\mu_{n-k}, \ldots, \mu_{2(n-k)}\} = \{\lambda_k, \ldots, \lambda_n\}$$

in this order. Hence

$$\mu_{\lfloor \frac{n+r+1}{2} \rfloor} = \mu_{n-k} = \lambda_k$$

and the theorem follows from $(*)$.

In a similar way, by choosing $r = 2k-2-n$, $\alpha = \alpha_k$, we get

THEOREM 6. *Assume $\lambda_n > 0$. Let $1+n/2 \leqslant k \leqslant n$. If $\alpha_k \leqslant \lambda_k$, then*

$$\frac{a+(2k-2-n)\alpha_k}{2(k-1)} - \sqrt{\left(\frac{a+(2k-2-n)\alpha_k}{2(k-1)}\right)^2 - (d\alpha_k^{2k-2-n})^{\frac{1}{k-1}}} \leqslant \lambda_k.$$

We study, whether the bounds of Theorem 5 and Theorem 6 are better than the original bounds β_k and α_k.

Since trivially $0 < \lambda_k \leqslant a/k$, we assume that $\alpha_k > 0$ and $\beta_k \leqslant a/k$. Let $k \leqslant n/2$ and $t > 0$. Denote

$$x(t) = U(2(n-k), a+(n-2k)t, dt^{n-2k}),$$

$$y(t) = L(2(n-k), a+(n-2k)t, dt^{n-2k}).$$

Then

$$(n-k)x(t) + (n-k)y(t) = a+(n-2k)t,$$

$$x(t)^{n-k}y(t)^{n-k} = dt^{n-2k},$$

$$x(t) \geqslant y(t) > 0.$$

Thus $x(t) = t$ if and only if

$$kt + (n-k)y(t) = a,$$

$$t^k y(t)^{n-k} = d,$$

$$t \geqslant y(t) > 0,$$

i.e., if and only if $t = \xi_k$ (and $y(t) = \eta_k$), see Lemma 1.

Now,

$$x\left(\frac{a}{n}\right) = U\left(2(n-k), a + (n-2k)\frac{a}{n}, d\left(\frac{a}{n}\right)^{n-2k}\right) =$$

$$\frac{a}{n} + \sqrt{\left(\frac{a}{n}\right)^2 - \left(d\left(\frac{a}{n}\right)^{n-2k}\right)^{\frac{1}{n-k}}} \geqslant \frac{a}{n}.$$

(Equality holds if and only if $d = (a/n)^n$. Then $\xi_k = a/n$.) Moreover,

$$x\left(\frac{a}{k}\right) = \frac{a}{2k} + \sqrt{\left(\frac{a}{2k}\right)^2 - \left(d\left(\frac{a}{k}\right)^{n-2k}\right)^{\frac{1}{n-k}}} < \frac{a}{k}.$$

Since $x(t)$ is continuous, we have

$$\begin{aligned} x(t) > t \quad & \text{if } a/n \leqslant t < \xi_k, \\ x(t) = t \quad & \text{if } t = \xi_k, \\ x(t) < t \quad & \text{if } t > \xi_k. \end{aligned}$$

If $\beta_k < a/n$, then

$$U(2(n-k), a + (n-2k)\beta_k, d\beta_k^{n-2k}) \geqslant$$

$$\frac{a + (n-2k)\beta_k}{2(n-k)} =$$

$$\frac{a}{2(n-k)} + \beta_k - \frac{n\beta_k}{2(n-k)} > \beta_k.$$

We have proved

THEOREM 7. *Assume $\lambda_n > 0$. Let $1 \leqslant k \leqslant n/2$. If $\lambda_k \leqslant \beta_k \leqslant a/k$, then*

$$\frac{a + (n-2k)\beta_k}{2(n-k)} + \sqrt{\left(\frac{a + (n-2k)\beta_k}{2(n-k)}\right)^2 - (d\beta_k^{n-2k})^{\frac{1}{n-k}}} \lesseqgtr \beta_k$$

if and only if

$$\xi_k \lesseqgtr \beta_k.$$

Thus $U(2(n-k), a+(n-2k)\beta_k, d\beta_k^{n-2k})$ is a better upper bound for λ_k than β_k if and only if β_k is a worse upper bound for λ_k than ξ_k.

Next, let $1+n/2 \leqslant k \leqslant n$. Denoting

$$x(t) = U(2(k-1), a+(2k-2-n)t, dt^{2k-2-n}),$$

$$y(t) = L(2(k-1), a+(2k-2-n)t, dt^{2k-2-n}),$$

we analogously see that $y(t) = t$ if and only if $t = \eta_{k-1}$. Moreover,

$$y\left(\frac{a}{n}\right) \leqslant \frac{a+(2k-2-n)a/n}{2(k-1)} = \frac{a}{n}.$$

A straightforward computation yields that if $t \leqslant a/n$, then

$$y(t) = \frac{a+(2k-2-n)t}{2(k-1)} - \sqrt{\left(\frac{a+(2k-2-n)t}{2(k-1)}\right)^2 - (dt^{2k-2-n})^{\frac{1}{k-1}}} > t$$

if and only if

$$\left(\frac{a-nt}{2(k-1)}\right)^2 > \left(\frac{a+(2k-2-n)t}{2(k-1)}\right)^2 - (dt^{2k-2-n})^{\frac{1}{k-1}}$$

if and only if

$$d^{\frac{1}{k-1}} > t^{1-\frac{2k-2-n}{k-1}} \frac{a+(k-1-n)t}{k-1},$$

which clearly holds if t is small enough. (Note that $0 \leqslant (2k-2-n)/(k-1) < 1$.) Since

$$L(2(k-1), a+(2k-2-n)\alpha_k, d\alpha_k^{2k-2-n}) \leqslant \frac{a+(2k-2-n)\alpha_k}{2(k-1)} = \alpha_k + \frac{a-n\alpha_k}{2(k-1)},$$

$\alpha_k > a/n$ implies

$$L(2(k-1), a+(2k-2-n)\alpha_k, d\alpha_k^{2k-2-n}) < \alpha_k.$$

We have now proved

THEOREM 8. *Assume $\lambda_n > 0$. Let $1+n/2 \leqslant k \leqslant n$. If $0 < \alpha_k \leqslant \lambda_k$, then*

$$\frac{a+(2k-2-n)\alpha_k}{2(k-1)} - \sqrt{\left(\frac{a+(2k-2-n)\alpha_k}{2(k-1)}\right)^2 - (d\alpha_k^{2k-2-n})^{\frac{1}{k-1}}} \lesseqgtr \alpha_k$$

if and only if

$$\eta_{k-1} \lesseqgtr \alpha_k.$$

Thus $L(2(k-1), a+(2k-2-n)\alpha_k, d\alpha_k^{2k-2-n})$ is a better lower bound for λ_k than α_k if and only if α_k is a worse lower bound for λ_k than η_{k-1}.

Using Theorem 5, we find upper bounds for $\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor n/2 \rfloor}$, and using Theorem 6, we find lower bounds for $\lambda_{\lceil n/2 \rceil+1}, \lambda_{\lceil n/2 \rceil+2}, \ldots, \lambda_n$. We complete our list of bounds as follows:

THEOREM 9. *Assume $\lambda_n > 0$. Let $1 < k \leqslant n$ and $1 \leqslant l < n$. If $\alpha_k \leqslant \eta_{k-1}$ ($\leqslant \lambda_k$), then*

$$\lambda_{k-1} \leqslant \frac{a-(n-k+1)\alpha_k}{k-1}$$

and if ($\lambda_l \leqslant$) $\xi_l \leqslant \beta_l$, then

$$\lambda_{l+1} \geqslant \frac{a-l\beta_l}{n-l}.$$

Proof. By Theorem 1,

$$\lambda_{k-1} \leqslant \xi_{k-1} = \frac{a-(n-k+1)\eta_{k-1}}{k-1} \leqslant \frac{a-(n-k+1)\alpha_k}{k-1}$$

and

$$\lambda_{l+1} \geqslant \eta_l = \frac{a-l\xi_l}{n-l} \geqslant \frac{a-l\beta_l}{n-l}.$$

5. Comparison with the WS bounds

Denote

$$\gamma_k = \gamma_k(n,a,b) = \frac{a}{n} + \sqrt{\frac{n-k}{k}\frac{1}{n}\left(b-\frac{a^2}{n}\right)},$$

$$\delta_k = \delta_k(n,a,b) = \frac{a}{n} - \sqrt{\frac{k}{n-k}\frac{1}{n}\left(b-\frac{a^2}{n}\right)}.$$

Then (not requiring positivity of λ_i's)

$$\begin{aligned}(\gamma_{n-1},\delta_1,\delta_2,\ldots,\delta_{n-1}) &\leqslant\\ (\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n) &\leqslant\\ (\gamma_1,\gamma_2,\ldots,\gamma_{n-1},\delta_1),&\end{aligned}$$

see [13]. These WS bounds are the best possible bounds for λ_i's, using only a and b. It is easy to see that $(x,y) = (\gamma_k,\delta_k)$ is the solution of

$$\begin{aligned}&kx+(n-k)y = a,\\ &kx^2+(n-k)y^2 = b,\\ &x \geqslant y,\end{aligned}$$

and that

$$\delta_{n-1} \leqslant \delta_{n-2} \leqslant \cdots \leqslant \delta_1 \leqslant \gamma_{n-1} \leqslant \cdots \leqslant \gamma_2 \leqslant \gamma_1.$$

We prove that the WS bounds for λ_1 are always at least as good as ours and that our bounds for λ_n are at least as good as the WS bounds:

THEOREM 10. *Assume* $\lambda_n > 0$. *Then*

$$\xi_{n-1} \leqslant \gamma_{n-1} \ (\leqslant \lambda_1) \leqslant \gamma_1 \leqslant \xi_1$$

and

$$\delta_{n-1} \leqslant \eta_{n-1} \ (\leqslant \lambda_n) \leqslant \eta_1 \leqslant \delta_1.$$

This also means that the WS lower bound for λ_2 is at least as good as ours and our upper bound for λ_{n-1} is at least as good as the WS upper bound. To prove the theorem, we need a theorem on d with prescribed a and b, due to Grone, Johnson, de Sá, and Wolkowicz [1, Theorem 3.1].

THEOREM 11. *Assume* $\lambda_n \geqslant 0$. *Then*

$$\max(0, \gamma_{n-1}^{n-1}\delta_{n-1}) \leqslant d \leqslant \gamma_1 \delta_1^{n-1}.$$

These bounds are the best possible, using only a *and* b. *Further,* $d = \gamma_1\delta_1^{n-1}$ *if and only if* $\lambda_1 = \gamma_1$ *and* $\lambda_2 = \cdots = \lambda_n = \delta_1$. *If* $\delta_{n-1} \geqslant 0$, *then* $d = \gamma_{n-1}^{n-1}\delta_{n-1}$ *if and only if* $\lambda_1 = \cdots = \lambda_{n-1} = \gamma_{n-1}$ *and* $\lambda_n = \delta_{n-1}$.

We prove Theorem 10 by contradiction. If $\xi_1 < \gamma_1$, we see as in the proof of Theorem 1 that

$$\xi_1\eta_1^{n-1} >$$

$$\left(\xi_1 + \frac{\gamma_1 - \xi_1}{n-1}\right)\eta_1^{n-2}\left(\eta_1 - \frac{\gamma_1 - \xi_1}{n-1}\right) > \cdots >$$

$$\left(\xi_1 + (n-1)\frac{\gamma_1 - \xi_1}{n-1}\right)\left(\eta_1 - \frac{\gamma_1 - \xi_1}{n-1}\right)^{n-1}.$$

Since $\xi_1 + (n-1)\eta_1 = \gamma_1 + (n-1)\delta_1$, i.e.,

$$\delta_1 = \eta_1 - \frac{\gamma_1 - \xi_1}{n-1},$$

we obtain $\xi_1\eta_1^{n-1} > \gamma_1\delta_1^{n-1}$. But Theorem 11 implies

$$d = \xi_1\eta_1^{n-1} \leqslant \gamma_1\delta_1^{n-1}.$$

Hence $\xi_1 \geqslant \gamma_1$ and, consequently, $\eta_1 \leqslant \delta_1$.

Since, by Theorem 11,

$$\gamma_{n-1}^{n-1}\delta_{n-1} \leqslant \xi_{n-1}^{n-1}\eta_{n-1} = d,$$

we get similarly $\xi_{n-1} \leqslant \gamma_{n-1}$ and $\delta_{n-1} \leqslant \eta_{n-1}$. Thus we have proved Theorem 10.

From the equality conditions of Theorem 11 it follows easily, that, for example, $\delta_{n-1} = \eta_{n-1}$ if and only if $\lambda_n = \delta_{n-1} = \eta_{n-1}$ (and $\lambda_1 = \cdots = \lambda_{n-1} = \gamma_{n-1} = \xi_{n-1}$). Hence Theorems 8 and 10 imply that if $\delta_{n-1} > 0$, then

$$\frac{a + (n-2)\delta_{n-1}}{2n-2} - \sqrt{\left(\frac{a + (n-2)\delta_{n-1}}{2n-2}\right)^2 - (d\,\delta_{n-1}^{n-2})^{\frac{1}{n-1}}}$$

is a better lower bound for λ_n than δ_{n-1} unless they both are equal to η_{n-1} (and to λ_n).

6. Eigenvalue bounds applied to $A+tI$

Let us regard the λ_i's as eigenvalues of an n by n matrix A. If some of them are not positive, and we know only $a = \operatorname{tr} A$ and $d = \det A$, we cannot use our bounds. But if they are real and we know also $b = \operatorname{tr} A^2$, we can apply our bounds to A^2 and get bounds for squares and so also for the absolute values of eigenvalues of A.

If we know A explicitly and an upper bound for λ_n, we can choose such t that the eigenvalues of $A+tI$ are positive, and apply our bounds to $A+tI$.

Let $B = A+tI$, where $t > -\lambda_n$. Then eigenvalues of B are $\lambda_i + t$ $(i = 1,2,\ldots,n)$, which are positive. By Theorem 1,

$$\begin{array}{rcccl}
\xi_{n-1}(n, a+nt, \det\,(A+tI)) - t & \leqslant & \lambda_1 & \leqslant & \xi_1(n, a+nt, \det\,(A+tI)) - t, \\
\eta_1(n, a+nt, \det\,(A+tI)) - t & \leqslant & \lambda_2 & \leqslant & \xi_2(n, a+nt, \det\,(A+tI)) - t, \\
\vdots & & \vdots & & \vdots \\
\eta_{n-2}(n, a+nt, \det\,(A+tI)) - t & \leqslant & \lambda_{n-1} & \leqslant & \xi_{n-1}(n, a+nt, \det\,(A+tI)) - t, \\
\eta_{n-1}(n, a+nt, \det\,(A+tI)) - t & \leqslant & \lambda_n & \leqslant & \eta_1(n, a+nt, \det\,(A+tI)) - t.
\end{array}$$

Note that the WS bounds do not depend on t.

We will show that as t tends to infinity, the above bounds tend to the WS bounds. We denote by $S_k(x_1,\ldots,x_n)$ the kth elementary symmetric function of $x_1,\ldots,x_n$. We also denote

$$(x^{(k)}, y^{(n-k)}) = (\underbrace{x,\ldots,x}_{k \text{ times}}, \underbrace{y,\ldots,y}_{n-k \text{ times}}).$$

LEMMA 2. *Let* $1 \leqslant k < n$ *and* $0 < a/n \leqslant c < a/k$. *Consider a function* $x(t)$, *which, for sufficiently large values of* t, *is defined, continuous, and satisfies* $a/n \leqslant x(t) < a/k$. *If*

$$\lim_{t\to\infty} S_2\left(x(t)^{(k)}, \left(\frac{a-k\,x(t)}{n-k}\right)^{(n-k)}\right) = S_2\left(c^{(k)}, \left(\frac{a-k\,c}{n-k}\right)^{(n-k)}\right),$$

then

$$\lim_{t\to\infty} x(t) = c.$$

Proof. Since

$$S_2\left(c^{(k)}, \left(\frac{a-kc}{n-k}\right)^{(n-k)}\right) =$$

$$\binom{k}{2}c^2 + k(n-k)c\frac{a-kc}{n-k} + \binom{n-k}{2}\left(\frac{a-kc}{n-k}\right)^2 =$$

$$\frac{-kn}{2(n-k)}c^2 + \frac{ak}{n-k}c + \frac{n-k-1}{n-k}\frac{a^2}{2}$$

and

$$S_2\left(x(t)^{(k)}, \left(\frac{a-kx(t)}{n-k}\right)^{(n-k)}\right) =$$

$$\frac{-kn}{2(n-k)}x(t)^2 + \frac{ak}{n-k}x(t) + \frac{n-k-1}{n-k}\frac{a^2}{2},$$

we have

$$\lim_{t\to\infty}\left(\frac{-kn}{2(n-k)}x(t)^2 + \frac{ak}{n-k}x(t)\right) = \frac{-kn}{2(n-k)}c^2 + \frac{ak}{n-k}c.$$

It follows that

$$\lim_{t\to\infty}(c-x(t))\left(\frac{2a}{n} - c - x(t)\right) = 0,$$

and further, since $x(t)$ is continuous,

$$\lim_{t\to\infty} x(t) = c \text{ or } \lim_{t\to\infty} x(t) = \frac{2a}{n} - c.$$

Since $2a/n - c \leqslant a/n$ and $a/n \leqslant x(t)$, we have $\lim_{t\to\infty} x(t) = c$.

THEOREM 12. *Let* $1 \leqslant k < n$. *Then*

$$\lim_{t\to\infty}(\xi_k(n, a+nt, \det\,(A+tI)) - t) = \gamma_k \; (= \gamma_k(n,a,b))$$

and

$$\lim_{t\to\infty}(\eta_k(n, a+nt, \det\,(A+tI)) - t) = \delta_k \; (= \delta_k(n,a,b)).$$

Proof. Denoting $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$, we have

$$\det\,(A+tI) = \sum_{k=0}^{n} S_k(\underline{\lambda})t^{n-k}.$$

If $t > -\lambda_n$, then the solution of

$$kx + (n-k)y = a + nt,$$

$$x^k y^{n-k} = \sum_{k=0}^{n} S_k(\underline{\lambda})t^{n-k},$$

$$x \geqslant y > 0$$

is $(x,y) = (\xi_k(n, a+nt, \det\,(A+tI)), \eta_k(n, a+nt, \det\,(A+tI)))$. Hence an upper bound for λ_k is $x_k(t) = \xi_k(n, a+nt, \det\,(A+tI)) - t$, and a lower bound for λ_{k+1} is

$y_k(t) = \eta_k(n, a+nt, \det(A+tI)) - t$. Moreover,

$$kx_k(t) + (n-k)y_k(t) = a,$$

$$(x_k(t)+t)^k (y_k(t)+t)^{n-k} = \sum_{k=0}^{n} S_k(\underline{\lambda}) t^{n-k},$$

$$x_k(t) \geqslant y_k(t) > 0.$$

Denote $\underline{z}_k(t) = (x_k(t)^{(k)}, y_k(t)^{(n-k)})$. Since $S_0 = 1$, $S_1(\underline{z}_k(t)) = a = S_1(\underline{\lambda})$, and

$$\sum_{k=0}^{n} S_k(\underline{z}_k(t)) t^{n-k} = \sum_{k=0}^{n} S_k(\underline{\lambda}) t^{n-k},$$

we have

$$S_2(\underline{z}_k(t)) + \sum_{k=3}^{n} S_k(\underline{z}_k(t)) \frac{1}{t^{k-2}} = S_2(\underline{\lambda}) + \sum_{k=3}^{n} S_k(\underline{\lambda}) \frac{1}{t^{k-2}}.$$

Clearly the set $\{S_k(\underline{z}_k(t)) \mid t \in \mathbf{R}\}$ is bounded. Hence, $S_2(\underline{z}_k(t)) \to S_2(\underline{\lambda})$ as $t \to \infty$.

On the other hand,

$$k\gamma_k + (n-k)\delta_k = a = \sum \lambda_i$$

and

$$k\gamma_k^2 + (n-k)\delta_k^2 = b = \sum \lambda_i^2,$$

and so

$$S_2(\underline{\varepsilon}_k) = \frac{a^2 - b}{2} = S_2(\underline{\lambda}),$$

where $\underline{\varepsilon}_k = (\gamma_k^{(k)}, \delta_k^{(n-k)})$.

Consequently,

$$kx_k(t) + (n-k)y_k(t) = k\gamma_k + (n-k)\delta_k, \text{ for all } t > -\lambda_n,$$

$$S_2(\underline{z}_k(t)) \to S_2(\underline{\varepsilon}_k) \text{ as } t \to \infty.$$

Since the zeros of a polynomial depend continuously on the coefficients, $x(t)$ is continuous. By Lemma 2, $x_k(t) \to \gamma_k$ as $t \to \infty$, which implies that also $y_k(t) \to \delta_k$.

7. Examples

EXAMPLE 1. ([13], Example 4)

$$A = \begin{pmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{pmatrix}, \qquad \begin{array}{l} a = 22,\ b = 154,\ d = 410, \\ (\lambda_1, \ldots, \lambda_4) = (9.376, 6.423, 4.775, 1.426). \end{array}$$

First we accept only the bounds which can be expressed algebraically.

The WS bounds (best possible using a and b) are

$$\begin{aligned}(\gamma_3,\delta_1,\delta_2,\delta_3) &= (7.158,3.842,2.628,0.525)\\ &\leqslant (\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\ &\leqslant (\gamma_1,\gamma_2,\gamma_3,\delta_1)\\ &= (10.475,8.372,7.158,3.842).\end{aligned}$$

We can improve $\lambda_4 \geqslant \delta_3$ by applying Theorem 6 with $\alpha_4 = \delta_3$. We get $\lambda_4 \geqslant 0.692 = \delta_3'$. We can continue similarly. Putting $\alpha_4 = \delta_3'$ we obtain a still better bound $\lambda_4 \geqslant 0.835$ etc. It appears that $\lambda_3 \geqslant \delta_2$ cannot be improved in this way and that neither $\lambda_1 \leqslant \gamma_1$ nor $\lambda_2 \leqslant \gamma_2$ can be improved by applying Theorem 5.

Since $\delta_3' > \delta_3$ $(= \alpha_4)$ we have $\eta_3 > \delta_3$ $(= \alpha_4)$ by Theorem 8, and so we can apply Theorem 9 with $\alpha_4 = \delta_3$. We obtain $\lambda_3 \leqslant 7.158$. Similarly, putting $\alpha_4 = \delta_3'$, we get $\lambda_3 \leqslant 7.103$, etc.

The bounds of Theorem 3 (best possible using a and d) are

$$\eta_2 = 2.337 \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \xi_2 = 8.663.$$

They do not improve the WS bounds. In conclusion, we have

$$\begin{aligned}(7.158,3.842,2.628,0.835) &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\ &\leqslant (10.475,8.372,7.103,3.842).\end{aligned}$$

Next, we accept also the non-algebraic bounds of Theorem 1 (best possible using a and d). They can be computed iteratively in the view of Theorems 7 and 8.

$$\begin{aligned}(\xi_3,\eta_1,\eta_2,\eta_3) &= (6.921,3.213,2.337,1.237)\\ &\leqslant (\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\ &= (\xi_1,\xi_2,\xi_3,\eta_1)\\ &= (12.361,8.663,6.921,3.213),\end{aligned}$$

The inequalities $\eta_3 > \delta_3$ and $\eta_2 < \delta_2$ confirm, by Theorem 8, that Theorem 6 with $\alpha_4 = \delta_3$ indeed improves $\lambda_4 \geqslant \delta_3$, but does not improve $\lambda_3 \geqslant \delta_2$ with $\alpha_3 = \delta_2$. The inequalities $\xi_1 > \gamma_1$ and $\xi_2 > \gamma_2$ confirm, by Theorem 7, that Theorem 5 with $\beta_1 = \gamma_1$ or $\beta_2 = \gamma_2$ does not indeed improve $\lambda_1 \leqslant \gamma_1$ or $\lambda_2 \leqslant \gamma_2$.

In conclusion of all the above, we have

$$(7.158,3.842,2.628,1.237) \leqslant (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \leqslant (10.475,8.372,6.921,3.213).$$

EXAMPLE 2. ([13], Example 5)

$$A = \begin{pmatrix} 4 & 1 & 1 & 2 & 2\\ 1 & 5 & 1 & 1 & 1\\ 1 & 1 & 6 & 1 & 1\\ 2 & 1 & 1 & 7 & 1\\ 2 & 1 & 1 & 1 & 8 \end{pmatrix}, \qquad \begin{array}{l} a = 30,\ b = 222,\ d = 4377,\\ \\ (\lambda_1,\ldots,\lambda_5) = (11.171,6.527,5.435,4.296,2.571). \end{array}$$

The WS bounds are

$$\begin{aligned}(7.449, 4.551, 3.634, 2.450, 0.203) &\leqslant (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\\ &\leqslant (11.797, 9.550, 8.366, 7.449, 4.551).\end{aligned}$$

(The lower bound for λ_3 is in four decimals 3.6336.) Using the tricks described in Example 1, we find for λ_3, λ_4 and λ_5 better lower bounds $\lambda_3 \geqslant 3.6341$, $\lambda_4 \geqslant 2.725$, $\lambda_5 \geqslant 0.337$ and better upper bounds for λ_2, λ_3 and λ_4: $\lambda_2 \leqslant 9.549$, $\lambda_3 \leqslant 8.183$, $\lambda_4 \leqslant 7.416$. The best possible bounds using only a and d are

$$\begin{aligned}(7.059, 4.300, 3.634, 2.881, 1.762) &\leqslant (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\\ &\leqslant (12.799, 9.548, 8.079, 7.059, 4.300).\end{aligned}$$

(The lower bound for λ_3 is in four decimals 3.6345.)

EXAMPLE 3. In [9] and [10] we found several explicite bounds involving a and d for eigenvalues. We considered $A = \operatorname{diag}(1, 2, 3, \ldots, 8)$ with $a = 36$, $b = 204$ and $d = 40320$. The WS bounds are

$$\begin{aligned}&(5.366, 3.634, 3.177, 2.725, 2.209, 1.542, 0.531, -1.562)\\ &\qquad\leqslant (8, 7, 6, 5, 4, 3, 2, 1)\\ &\qquad\leqslant (10.562, 8.469, 7.458, 6.791, 6.275, 5.823, 5.366, 3.634).\end{aligned}$$

The WS bounds are better than any of our bounds of [9] and [10] except lower bounds for λ_7 and λ_8 and upper bounds for λ_6 and λ_7.

The best possible bounds obtained by using only a and d are

$$\begin{aligned}&(5.077, 3.114, 2.738, 2.392, 2.034, 1.629, 1.129, 0.464)\\ &\qquad\leqslant (8, 7, 6, 5, 4, 3, 2, 1)\\ &\qquad\leqslant (14.203, 9.787, 8.013, 6.966, 6.223, 5.624, 5.077, 3.114).\end{aligned}$$

Now our lower bounds for λ_6, λ_7 and λ_8 and our upper bounds for λ_5, λ_6, λ_7 and λ_8 are better than the WS bounds.

EXAMPLE 4. In [12], a technique to improve WS bounds using certain extra bounds was presented. For

$$A = \begin{pmatrix} 2 & 4 & 1 & 1 & 3 & 1\\ 4 & -1 & 2 & 3 & 1 & 2\\ 1 & 2 & -3 & 2 & 1 & -1\\ 1 & 3 & 2 & 3 & 1 & 1\\ 3 & 1 & 1 & 1 & 1 & -2\\ 1 & 2 & -1 & 1 & -2 & -2 \end{pmatrix}, \qquad \begin{aligned} &a = 0,\ b = 144,\ d = -1851,\\ &(\lambda_1, \ldots, \lambda_6) = (8.82, 2.74, 1.12, -2.91, -4.16, -5.62 \end{aligned}$$

the WS bounds yield

$$\begin{aligned}(2.19, -2.19, -3.46, -4.90, -6.93, -10.95) &\leqslant (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\\ &\leqslant (10.95, 6.93, 4.90, 3.46, 2.19, -2.19).\end{aligned}$$

Since A is symmetric, its eigenvalues majorize the diagonal elements. Using these extra bounds, we get

$$\begin{gathered} 3 \leqslant \lambda_1 \leqslant 10.92, \\ -1.97 \leqslant \lambda_2, \\ \lambda_5 \leqslant 1.97, \\ -10.92 \leqslant \lambda_6 \leqslant -3 \end{gathered}$$

The eigenvalues of $B = A + 10.92I$ are positive. Applying our bounds to B, we get

$$\begin{aligned} (1.57, -2.66, -3.63, -4.64, -5.90, -7.85) &\leqslant (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \\ &\leqslant (13.28, 7.27, 4.64, 2.95, 1.57, -2.66). \end{aligned}$$

Now our lower bounds for λ_4, λ_5 and λ_6 and upper bounds for λ_3, λ_4 and λ_5 are better than the WS bounds and the improved bounds of [12]. Our upper bound for λ_6 is better than the WS bound but worse than the improved bound.

In [5], bounds for singular values, involving $\operatorname{tr} A$, $\operatorname{tr} A^2$ and $\operatorname{tr} A^H A$, were presented. Some of these results are analogous to the WS bounds. If we know $\operatorname{tr} A^H A$ and $\det A$ (or $\det A^H A = |\det A|^2$), we can apply our bounds for finding bounds for singular values. Our last example illustrates this.

EXAMPLE 5. ([5], Example 1)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The best bounds obtained in [5] for singular values $(\sigma_1, \sigma_2, \sigma_3) = (2.2470, 0.802, 0.555)$ are

$$\begin{gathered} 2.236 \leqslant \sigma_1 \leqslant 2.2483, \\ 0.687 \leqslant \sigma_2 \leqslant 1, \\ (0 \leqslant)\ \sigma_3 \leqslant 0.687. \end{gathered}$$

The eigenvalues of $A^H A$ are σ_1^2, σ_2^2 and σ_3^2, $\operatorname{tr} A^H A = 6$ and $\det A^H A = 1$. Our bounds yield now bounds for σ_1^2, σ_2^2 and σ_3^2. By taking square roots, we get

$$\begin{gathered} 1.715 \leqslant \sigma_1 \leqslant 2.262, \\ 0.665 \leqslant \sigma_2 \leqslant 1.715, \\ 0.340 \leqslant \sigma_3 \leqslant 0.665, \end{gathered}$$

improving the bounds for σ_3.

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REFERENCES

[1] B. GRONE, C. JOHNSON, E. M. DE SÁ, AND H. WOLKOWICZ, *Improving Hadamard's Inequality*, Linear and Multilienear Algebra **16**(1984), 305–322.

[2] H. W. GUGGENHEIMER, A. S. EDELMAN, AND C. R. JOHNSON, *A simple estimate of the condition number of a linear system*, College Math. J., **26**(1995), 2–5.

[3] S. T. JENSEN, *The Laguerre–Samuelson inequality with extensions and applications in statistics and matrix theory, Master's thesis*, Department of Mathematics and Statistics, McGill University, 1999.

[4] S. T. JENSEN AND G. P. H. STYAN, *Some comments and a bibliography on the Laguerre–Samuelson inequality with extensions and applications in statistics and matrix theory*, in H. M. Srivastva and T. M. Rassias (eds.) *Analytic and Geometric Inequalities and Their Applications*, Kluwer, 2000.

[5] J. K. MERIKOSKI, H. SARRIA, AND P. TARAZAGA, *Bounds for singular values using traces*, Linear Algebra Appl., **210**(1994), 227–254.

[6] J. K. MERIKOSKI, G. P. H. STYAN, AND H. WOLKOWICZ, *Bounds for ratios of eigenvalues using traces*, Linear Algebra Appl. **55**(1983), 105–124.

[7] J. K. MERIKOSKI, U. URPALA, AND A. VIRTANEN, *Upper bounds for the ratio of the largest and smallest eigenvalues*, Department of Mathematical Sciences, University of Tampere, Report A 307, 1996.

[8] J. K. MERIKOSKI, U. URPALA, A. VIRTANEN, T.-Y. TAM, AND F. UHLIG, *A best upper bound for the 2-condition number of a matrix*, Linear Algebra Appl. **254**(1997), 355–365.

[9] J. K. MERIKOSKI AND A. VIRTANEN, *Bounds for eigenvalues using the trace and determinant*, Department of Mathematical Sciences, University of Tampere, Report A 308, 1996.

[10] J. K. MERIKOSKI AND A. VIRTANEN, *Bounds for eigenvalues using the trace and determinant*, Linear Algebra Appl., **264**(1997), 101–108.

[11] J. K. MERIKOSKI, A. VIRTANEN, AND O. PIKHURKO, *Majorization bounds for vectors and bounds for sums using a sum constraint and a convex constraint*, Manuscript.

[12] J. K. MERIKOSKI AND H. WOLKOWICZ, *Improving eigenvalue bounds using extra bounds*, Linear Algebra Appl., **68**(1985), 93–113.

[13] H. WOLKOWICZ AND G. P. H. STYAN, *Bounds for eigenvalues using traces*, Linear Algebra Appl., **29**(1980), 471–506.

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