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COMPACTNESS OF THE EMBEDDING OPERATORS FOR ROUGH DOMAINS

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Abstract. New classes of non-smooth bounded domains D, for which the embedding operator from $H^1(D)$ into $L^2(D)$ is compact, are introduced. These classes include, in particular, the domains whose boundary locally are graphs of $C-$ functions, but also contain much larger classes of domains. Examples of non-smooth domains for which the above embedding is compact are given. Applications to scattering by rough obstacles are mentioned.

1. Introduction

In this paper we prove some results about compactness of the embedding operator $H^1(\Omega) \to L^2(\Omega)$ for rough bounded domains, that is, for domains with non-smooth boundaries which do not satisfy the usual for the embedding theorems conditions, such as cone condition, Lipschitz domains, and extension domains (Ext-domains). First, we prove compactness of the embedding operators for "elementary" domains which can be approximated by Lipschitz domains in the sense described below (see the paragraph above Lemma 1.2). This class ET of "elementary" domains is larger then the known classes of domains used in embedding theorems. Let us give some bibliographical discussion. In [12] a necessary and sufficient condition for compactness of the embedding operator is given in an abstract setting. A version of this result is presented in the Appendix. Compactness of the embedding operator for bounded domains with "segment property" is proved in [1]. In [9] it was shown that the class of domains with "segment property" coincides with the class C of domains whose boundaries are locally graphs of continuous functions. Compactness of the embeddings for the class C was proved in [2]. The reader can found an interesting discussion of these results in [11]. The class ET is much larger than the class C and includes (in the two-dimensional case) bounded domains whose boundaries are locally graphs of piecewise-continuous functions with "jump"-type discontinuity at a finite number of points. Boundaries of the domains of class ET can have singularities more complicated than the "jump"-type singularities (see example 3.4).

Using lemma 3.10 for the union of "elementary" domains of the class ET we extend this result to domains of the class T which are finite unions of the "elementary"

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domains. Simple examples demonstrate that boundary of a bounded domain of class T can have countably many connected components (see example 3.12). This is impossible for the classes of domains used in embedding theorems earlier (compare, for example, classes C and E in [9], [11], [3] with our class T).

Our construction can be generalized. First, we construct a class of "elementary" domains with the compactness property for the embedding operator. Secondly, we extend this compactness property to finite unions of "elementary" domains. This scheme is used for quasiisometrical (the class L) and $2-$quasiconformal (the class Q) cases. Our class L includes the Fraenkel class E. Let us explain this. Note that E in [9] is not the class of extension domains. According to [9], p. 411, any domain Ω of class E is locally C^1-diffeomorphic at any boundary point to a domain of class C and $\partial\Omega = \partial\overline{\Omega}$, where $\overline{\Omega}$ is the closure of Ω. Any domain of our class L is a finite union of domains that are locally quasiisometrically equivalent at any boundary point to domains of class ET . The condition $\partial\Omega = \partial\overline{\Omega}$ is not necessary for the domains in this class. For example, if Ω is a disc with an extracted radius, then Ω is a domain of the class L, but $\partial\Omega \neq \partial\overline{\Omega}$.

Our class Q is much larger then the class L and includes domains with some "anisotropic" behavior of their boundaries (see section 4.3 for a detailed explanation).

Our results allow one to use the results in [13] and [14] on the existence and uniqueness of the solutions to the scattering problem in the exterior of rough obstacles and consider larger class of rough obstacles in scattering theory than it was done earlier.

2. Abstract result

In this section we prove some results which give conditions for the compactness of an embedding operator, and use these results in a study of compactness of the embeddings of Sobolev spaces. An abstract necessary and sufficient condition for the compactness of an embedding operator is proved in [12]. Let H_1 and H_2 be Hilbert spaces and $H_1 \subset H_2$. Here the embeddings are understood as the set-theoretical inclusions and the inequalities $\|u\|_1 \geqslant \|u\|_2$ are assumed, where $\|u\|_j := \|u\|_{H_j}$. Suppose that T_s , $s \in (0,1)$, is a family of closed subspaces of H_2 and $T_\sigma \subset T_s$ for $s < \sigma$. Having in mind applications, we assume also that the closure of the union of T_s for $s > 0$ equals H_2.

In our applications $T_s = L^2(D_s)$, where $D_s \subset D$, $D_s \subset D_\sigma$ for $s > \sigma$. We assume below (see lemma 1.2) that a domain D, for which we study the compactness of the embedding operator from $H^1(D)$ into $L^2(D)$, contains a Lipschitz subdomain G_s, $D_s \subset G_s \subset D$. Let P_s be the orthogonal projection onto T_s in H_2, $i : H_1 \to H_2$ be the embedding operator, and $i_s := P_s i$. Let us state two results. The above assumptions and notations are not repeated.

PROPOSITION 2.1. *If the operator $i : H_1 \to H_2$ is compact, then the operator i_s is compact for any $s \in (0,1)$.*

Proof. If i: $H_1 \to H_2$ is compact, then i_s is a composition of a bounded linear operator P_s and a compact operator i, so i_s is compact. □

The following proposition is used in the proof of proposition 2.4 below.

PROPOSITION 2.2. *If the following conditions hold:*
1) i_s *is compact for all* $s \in (0,1)$, $\lim_{s\to 0}\|P_s u - u\|_2 = 0$, *and*
2) $\|u\|_2 \leqslant a(s)\|u\|_1 + b\|P_s u\|_2,\ a(s) > 0,\ \lim_{s\to 0} a(s) = 0,\ b \geqslant 1$ *for any* $u \in H_1$, *where b is independent of s,*
then the embedding $i : H_1 \to H_2$ *is compact.*

Proof. Choose a sequence $\{s_m\}$ such that $a(s_m) < \frac{1}{m}$. Denote by P_m the projection P_{s_m}. Let u_n be an arbitrary normalized sequence of elements of H_1. If i_s is compact then $\|u_n\|_1 = 1$ implies $\|P_s u_n\|_2 \leqslant 1$ and for any m there exists a subsequence $u_{n,m}$ and a number $n(m)$ such that $\|P_m(u_{n,m} - u_{n_1,m})\|_2 < \frac{1}{m}$ for any $n, n_1 \geqslant n(m)$. Without loss of generality assume that the sequence u_{n,m_1} is a subsequence of $u_{n,m}$ and $n(m) < n(m_1)$ for $m < m_1$. Therefore

$$\|P_m(u_{n(m),m} - u_{n(m_1),m})\|_2 < \frac{1}{m}$$

for any m and for any $m_1 > m$.

For the subsequence $u_m := u_{n(m),m}$ condition 2) implies $\|u_m - u_{m_1}\|_2 \leqslant a(s_m)\|u_m - u_{m_1}\|_1 + b\|P_m(u_m - u_{m_1})\|_2$. By the choice of the subsequence $\{u_m\}$ this implies $\|u_m - u_{m_1}\|_2 < (1+b)\frac{1}{m}$ for any m . We have proved the convergence of $\{u_m\}$ in H_2. Because the original sequence $\|u_n\|_1 = 1$ was arbitrary, compactness of the operator i is proved. □

We apply this abstract result to Sobolev spaces. Below we assume that $D \subset R^n$ is a bounded domain and $\{D_s\}$, $0 < s < 1$, is a family of subdomains such that $D_s \subset D_\sigma$ for any $s > \sigma$ and for any s there exists a Lipschitz domain G_s such that $D_s \subset G_s \subset D$. A bounded domain is a Lipschitz domain if its boundary is locally graph of a Lipschitz function. Let $H^1_s(D_s)$ denote the set of functions which are restrictions of the $H^1(D)$-functions to D_s.

LEMMA 2.3. *Suppose that* $\{u_n\}$ *is a bounded sequence in* $H^1(D)$. *Then there exists a subsequence* $\{u_{n_k}\}$ *of the sequence* $\{u_n\}$ *which converges in* $L^2(D_s)$, *i.e.* $i_s : H^1_s(D_s) \to L^2(D_s)$ *is compact for all* $s \in (0,1)$.

Proof. One takes a Lipschitz domain G_s such that $D_s \subset G_s \subset D$. By the known embedding theorem for Lipschitz domains the embedding $H^1(G_s) \to L^2(G_s)$ is compact. Since $D_s \subset G_s \subset D$, one obtains the conclusion of the lemma. □

PROPOSITION 2.4. *If the following condition holds*

$$\|u\|_{L^2(D)} \leqslant a(s)\|u\|_{H^1(D)} + b\|u\|_{L^2(D_s)}, \quad a(s) > 0,\ \lim_{s\to 0} a(s) = 0, \quad b \geqslant 1,$$

for any $u \in H^1(D)$, *then the operator* $i:\ H^1(D) \to L^2(D)$ *is compact*

By Lemma 2.3 $i_s : H^1_s(D_s) \to L^2(D_s)$ is compact for all $s \in (0,1)$. Hence the claim follows from Proposition 2.2.

In section 3-4 we describe classes of domains for which the conditions of proposition 2.4 are satisfied.

3. Domains of class T

Below we denote a domain by Ω. The main purpose of this section is to prove compactness of the embedding operators $H^1(\Omega) \to L^2(\Omega)$ for domains of the class T which we describe below. Domains of the class T are finite unions of elementary domains of the class ET whose boundaries are locally graphs of "good" functions: these domains can be approximated by Lipschitz subdomains in such a way, that conditions of Proposition 2.4 hold. For example, in the two-dimensional case the function is "good" if it is piecewise-continuous with discontinuity points of "finite jump" type.

In the first part of this section we describe exactly classes T and ET. In the second part we derive an auxiliary one-dimensional inequality. This inequality is not new, but its proof is. It is a version of Agmon's inequality [1] adopted for our purposes. In the final part of this section we prove compactness of the embedding operator for domains of class T using the results of section 2.

3.1. Preliminaries.

Let $x \in R^n$, $x = (x_1, x_2, ..., x_n)$ and $Q_n = [0,1]^n$ be the standard closed cube in R^n. Denote $x' := (x_1, x_2, ..., x_{n-1})$.

A bounded function $f : Q_{n-1} \to R$ is an admissible function if f is continuous on a set $C\{f\} \subset Q_{n-1}$ such that $\mu(Q_{n-1} \setminus C(f)) = 0$, where μ is $n-1$-dimensional Lebesgue measure. Denote by $IntA$ the set of all interior points of a set A. By $Q_n + (0, 0, ..., f(x'))$ we denote the set $0 \leqslant x_n \leqslant 1 + f(x'), x' \in Q_{n-1}$, and assume $1 + \inf f > 0$.

DEFINITION 3.1. *We call* $U := Int\{Q_n + (0, ..., 0, f(x'))\}$ *a standard elementary domain if the function* f *is admissible.*

Let $Q_{n,h} = [h, 1-h]^n$ be a subcube of the standard cube Q_n. If $U := Int\{Q_n + (0, ..., 0, f(x'))\}$ is an elementary domain then denote
$U_h = Int\{Q_{n,h} + (0, ..., 0, f(x'))\}$.

DEFINITION 3.2. *We call a standard elementary domain* U *a standard elementary domain of class* ET *if for any* $0 < h < 1/3$ *there exists a Lipschitz domain* V_h *such that* $U_h \subset V_h \subset U$. *We call* U *an elementary domain of class* ET *if it is an image of a standard elementary domain of class* ET *under an affine invertible mapping of* R^n *onto* R^n.

EXAMPLE 3.3. Suppose that $f : [0,1] \to R$ is a piecewise-continuous bounded function with a finite number of discontinuity points $x_1, x_2, .., x_k$ and at any discontinuity point the function f has right and left limits (i.e. discontinuity points are the "jump" points). The domain $U := Int\{Q_2 + (0, f(x))\}$ is a standard elementary domain of the class ET.

Proof. It is obvious that U is a standard elementary domain. Fix $0 < h < 1/3$. The open set $W_h = Int(U \setminus U_h)$ is a finite union of domains $U_i = (x_{i-1}, x_i) \times (1 - h + f(x), 1 + f(x))$ and domains $V_i = (x_{i-1}, x_i) \times (f(x), h + f(x))$, $i = 1, ..., k+1$,

$x_0 = a$, $x_{k+1} = b$. Join any two points $(x_{i-1}, 1 - h/2 + \lim_{x\to x_{i-1}^+} f(x))$, $(x_i, 1 - h/2 + \lim_{x\to x_i^-} f(x))$ by a smooth curve α_i and any pair $(x_{i-1}, h/2 + \lim_{x\to x_{i-1}^+} f(x))$, $(x_i, h/2 + \lim_{x\to x_i^-} f(x))$ by a smooth curve β_i . The set $\partial U \setminus \partial W_h \cup (\cup_{i=1}^k \alpha_i) \cup (\cup_{i=1}^k \beta_i)$ is a closed Lipschitz curve that is the boundary of a Lipschitz domain V_h. By construction $U_h \subset V_h \subset U$. Therefore U is a standard elementary domain of the class ET. □

EXAMPLE 3.4. Suppose that $f : [0,1] \to R$ is a piecewise-continuous bounded function with countably many isolated discontinuity points $x_1, x_2, .., x_k, ...$ and at any discontinuity point the function f has right and left limits (i.e. any discontinuity points are "jump" points). Suppose also that the sequence $\{x_k\}$ converges to x_0. The domain $U := Int\{Q_2 + (0, f(x))\}$ is a standard elementary domain of class ET.

Proof. Because f is continuous in x_0 for any $0 < h < 1/3$ the open set $W_h = Int U \setminus U_h$ is a finite union of domains of the same type as in example 3.3. Therefore the domain $U := Int\{Q_2 + (0, f(x))\}$ is a standard elementary domain of the class ET. □

3.2. One-dimensional inequality.

LEMMA 3.5. *If $u \in H^1((-h,h))$, then*

$$|\|u\|_{L^2((0,h))} - \|u\|_{L^2((-h,0))}| \leqslant \sqrt{2}h\|\frac{du}{dt}\|_{L^2((-h,h))}.$$

Proof. Since smooth functions are dense in $H^1((-h,h))$ it is sufficient to prove the desired estimate only for smooth functions $u \in H^1((-h,h))$. Integrating the inequality $|u(t+h) - u(t)|^2 \leqslant (\int_t^{t+h} |\frac{du}{ds}(s)|ds)^2 \leqslant (\int_{-h}^h |\frac{du}{ds}(s)|ds)^2$ with respect to t over the segment $[-h,0]$ and using the Hölder inequality we obtain

$$\int_{-h}^0 |u(t+h) - u(t)|^2 dt \leqslant h(\int_{-h}^h |\frac{du}{dt}(t)|dt)^2 \leqslant 2h^2 \int_{-h}^h |\frac{du}{dt}(t)|^2 dt.$$

For any normed space X and any $x, y \in X$ the following inequality holds

$$|\|x\| - \|y\|| \leqslant \|x - y\|.$$

Combining this inequality with the previous one, we obtain

$$|(\int_{-h}^0 |u(t+h)|^2 dt)^{1/2} - (\int_{-h}^0 |u(t)|^2 dt)^{1/2}| \leqslant (\int_{-h}^0 |u(t+h) - u(t)|^2 dt)^{1/2}$$

$$\leqslant \sqrt{2}h(\int_{-h}^h |\frac{du}{dt}(t)|^2 dt)^{1/2}.$$

Because $\int_{-h}^0 |u(t+h)|dt = \int_0^h |u(t)|dt$ we have finally

$$|\|u\|_{L^2((0,h))} - \|u\|_{L^2((-h,0))}| \leqslant \sqrt{2}h\|\frac{du}{dt}\|_{L^2((-h,h))}.$$

□

COROLLARY 3.6. *If* $u \in H^1((-h,h))$, *then*

$$\int_0^h |u(t)|^2 dt \leqslant 2\int_{-h}^0 |u(t)|^2 dt + 4h^2 \int_{-h}^h |\frac{du}{dt}(t)|^2 dt,$$

and

$$\int_{-h}^0 |u(t)|^2 dt \leqslant 2\int_0^h |u(t)|^2 dt + 4h^2 \int_{-h}^h |\frac{du}{dt}(t)|^2 dt.$$

Proof. Using Lemma 3.5, one gets:

$$\begin{aligned}\int_0^h |u(t)|^2 dt &\leqslant [(\int_{-h}^0 |u(t)|^2 dt)^{1/2} + \sqrt{2}h(\int_{-h}^h |\frac{du}{dt}(t)|^2 dt)^{1/2}]^2 \\ &\leqslant 2\int_{-h}^0 |u(t)|^2 dt + 4h^2 \int_{-h}^h |\frac{du}{dt}(t)|^2 dt.\end{aligned}$$

□

PROPOSITION 3.7. *If* $u \in H^1((a,b))$, *then*

$$\int_a^b |u(t)|^2 dt \leqslant 3\int_{a+h}^{b-h} |u(t)|^2 dt + 4h^2 \int_a^b |\frac{du(t)}{dt}|^2 dt.$$

for any $h < \frac{b-a}{4}$.

Proof. By the previous corollary

$$\begin{aligned}\int_a^b |u(t)|^2 dt &\leqslant \int_{b-h}^b |u(t)|^2 dt + \int_{a+h}^{b-h} |u(t)|^2 + \int_a^{a+h} |u(t)|^2 dt \\ &\leqslant 2\int_{b-2h}^{b-h} |u(t)|^2 dt + \int_{a+h}^{b-h} |u(t)|^2 + 2\int_{a+h}^{a+2h} |u(t)|^2 dt \\ &\qquad + 4h^2 \int_a^{a+h} |\frac{du}{dt}(t)|^2 dt + 4h^2 \int_{b-h}^b |\frac{du}{dt}(t)|^2 dt \\ &\leqslant 3\int_{a+h}^{b-h} |u(t)|^2 dt + 4h^2 \int_a^b |\frac{du}{dt}(t)|^2 dt.\end{aligned}$$

□

3.3. Compactness for elementary domains of class ET.

PROPOSITION 3.8. *If* U *is an elementary domain of the class* ET, *then the embedding operator* $i : H^1(U) \Rightarrow L^2(U)$ *is compact.*

Proof. It is sufficient to prove this proposition for a standard elementary domain of class ET.

Fix $h < \frac{1}{3}$ and choose a sequence $\{u_n\} \subset H^1(U)$, $\|u_n\|_{H^1(U)} \leqslant 1$ for all n.

Using Proposition 3.7 for almost all x' in the domain of definition Q_{n-1} of an admissible function f we get

$$\int_0^{1+f(x')} |u_n(x',t)|^2 dt \leqslant 3 \int_h^{f(x')+1-h} |u_n(x',t)|^2 dt + 4h^2 \int_0^{1+f(x')} |\frac{du_n}{dt}(x',t)|^2 dt.$$

Integrating this inequality over Q_{n-1} we obtain

$$\int_U |u_n(x)|^2 dx \leqslant 3 \int_{U_h} |u_n(x)|^2 dx + 4h^2 \int_U |\nabla u_n|^2 dx.$$

The role of s in Proposition 2.4 is played by the parameter h, and by this Proposition the embedding operator i is compact. □

3.4. Compactness for domains of class T.

DEFINITION 3.9. *A domain Ω belongs to class T if it is a finite union of elementary domains of class ET.*

First, we prove

LEMMA 3.10. *Let Ω_1 and Ω_2 be such domains that embedding operators $H^1(\Omega_1) \to L^2(\Omega_1)$ and $H^1(\Omega_2) \to L^2(\Omega_2)$ are compact, then the embedding operator $H^1(\Omega_1 \cup \Omega_2) \to L^2(\Omega_1 \cup \Omega_2)$ is also compact.*

Proof. Choose a sequence $\{w_n\} \subset H^1(\Omega_1 \cup \Omega_2)$, $\|w_n\|_{H^1(\Omega_1 \cup \Omega_2)} \leqslant 1$ for all n. Let $u_n := w_n|\Omega_1$ and $v_n := w_n|\Omega_2$. Then $u_n \in H^1(\Omega_1)$, $v_n \in H^1(\Omega_2)$, $\|u_n\|_{H^1(\Omega_1)} \leqslant 1$, $\|v_n\|_{H^1(\Omega_2)} \leqslant 1$.

Because the embedding operator $H^1(\Omega_1) \to L^2(\Omega_1)$ is compact we can choose a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ which converges in $L^2(\Omega_1)$ to a function $u_0 \in L^2(\Omega_1)$. Because the second embedding operator $H^1(\Omega_2) \Rightarrow L^2(\Omega_2)$ is also compact we can choose a subsequence $\{v_{n_{k_m}}\}$ of the sequence $\{v_{n_k}\}$ which converges in $L^2(\Omega_2)$ to a function $v_0 \in L^2(\Omega_2)$. It is evident that $u_0 = v_0$ almost everywhere in $\Omega_1 \cap \Omega_2$ and the function $w_0(x)$ which is defined as $w_0(x) := u_0(x)$ on Ω_1 and $w_0(x) := v_0(x)$ on Ω_2 belongs to $L^2(\Omega_1 \cup \Omega_2)$.

Hence

$$\|w_{n_{k_m}} - w_0\|_{L^2(\Omega_1 \cup \Omega_2)} \leqslant \|u_{n_{k_m}} - u_0\|_{L^2(\Omega_1)} + \|v_{n_{k_m}} - v_0\|_{L^2(\Omega_2)}.$$

Therefore $\|w_{n_{k_m}} - w_0\|_{L^2(\Omega_1 \cup \Omega_2)} \to 0$ for $m \to \infty$. □

From Proposition 3.8 and Lemma 3.10 the main result of this section follows immediately:

THEOREM 3.11. *If a domain Ω belongs to class T then the embedding operator $H^1(\Omega) \to L^2(\Omega)$ is compact.*

The example below demonstrates the difference between class T and the class of bounded domains whose boundaries are locally graphs of continuous functions (C-domains). The boundary of a domain of class T can have countably many connected components, while this is not possible for C-domains.

EXAMPLE 3.12. Take: $U := \{(x_1,x_2) : 0 < x_1 < 1/\pi, x_1 \sin\frac{1}{x_1} < x_2 < x_1 \sin\frac{1}{x_1} + 4\}$; $V = (0,1/\pi) \times (-2,0)$, $\Omega = U \cup V$.

Domains U and V are elementary domains of class ET. Therefore Ω is a domain of class T . By Theorem 3.11 the embedding operator $H^1(\Omega) \Rightarrow L^2(\Omega)$ is compact.

Let us discuss the structure of $\partial\Omega$. The boundary ∂U is connected and contains the graph $\Gamma_f = \{(x_1,x_2) : x_2 = x_1 \sin\frac{1}{x_1}$ of the function $f : [0,\frac{1}{\pi}] \to R$, $f(x_1) = x_1 \sin\frac{1}{x_1}$. The graph Γ_f can be divided on two parts: the "nonnegative" part $\Gamma_f^+ := \{(x_1,x_2) \subset \Gamma_f : x_2 \geqslant 0\}$ and "negative part $\Gamma_f^- := \{(x_1,x_2) \subset \Gamma_f : x_2 < 0\}$. The "negative" part $\Gamma_f^- \subset V$. Therefore the boundary $\partial\Omega$ of the plane domain Ω does not contain Γ_f^- and consists of the countably many connected components: $S_1 = ([0,1/\pi] \times \{-2\}) \cup (\{0\} \times (-2,4)) \cup (\{\frac{1}{\pi}\} \times (-2,4)) \cup \Gamma_g$, where Γ_g is the graph of the function $g : [0,\frac{1}{\pi}] \to R$, $g(x_1) = x_1 \sin\frac{1}{x_1} + 4$; $S_i = ([\frac{1}{(2i-1)\pi}, \frac{1}{2(i-1)\pi}] \times \{0\}) \cup \Gamma_i$ $i = 2, \dots$, $\Gamma_i \subset \Gamma_f^+$ is the graph of the restriction of the function $f(x_1) = x_1 \sin\frac{1}{x_1}$ to the segment $[\frac{1}{(2i-1)\pi}, \frac{1}{2(i-1)\pi}]$; and $\widetilde{S} = \{0,0\}$ is also a point of the boundary $\partial\Omega$.

Notice that any neighboorhood of the point $\{0,0\}$ the boundary $\partial\Omega$ has countably many connected components and therefore can not be presented as a graph of any continuous function which is a connected set.

Higher-dimensional examples can be constructed using the rotation of two-dimensional domain Ω around x_1-axis.

The following corollary is practically convenient for using the main theorem.

COROLLARY 3.13. *If a bounded domain U is an extension domain, a domain V belongs to class T and $\Omega := U \cup V$, then the embedding operator $H^1(\Omega) \to L^2(\Omega)$ is compact.*

This corollary follows from Theorem 3.11 and Lemma 3.10.

EXAMPLE 3.14. Let $U := U(f,g,x_0',r) := \{(x',x_n) : g(x') < x_n < f(x')\}$ where a continuous real-valued functions f,g defined on the closed ball $\overline{B} := \overline{B_{n-1}}(x_0',r) \subset R^{n-1}$ and $H := \max_{x' \in \overline{B}}(f(x') - g(x')) > 0$. Then the embedding operator $H^1(U) \to L^2(U)$ is compact.

The above claim follows from corollary 3.13. We need only to represent U as a union of domains of class C and an extension domain (in our case a domain with Lipschitz boundary).

REMARK. Extension domains can have very rough boundary. In the plane a bounded domain U has an extension property if and only if it is an image of the unit disc under quasiconformal homeomorphism $\phi : R^2 \to R^2$ (see [5],[6]). For example the Hausdorff dimension of an image ∂U of a unit circle under quasiconformal homeomorphism $\phi : R^2 \to R^2$ can be any number $1 \leqslant \alpha < 2$ [4].

4. Quasiisometrical homeomorphisms and compact embeddings

A large class of bounded domains in R^n does not belong to class T but still have "good" properties like compactness of the embedding $H^1(\Omega) \Rightarrow L^2(\Omega)$. To study these domains we will introduce a larger and more flexible class of "elementary" domains, i.e. quasiisometrical images of elementary domains of class ET. Then we extend the main theorem to the finite unions of quasiisometrical elementary domains. Our proof is based on the well-known fact that a quasiisometrical homeomorphism $\varphi : U \to V$ induces a bounded composition operator $\varphi^* : H^1(V) \to H^1(U)$ by the rule $\varphi^*(u) = u \circ \varphi$ (see, for example [6] or [15]).

Recall the definition of a quasiisometrical homeomorphism.

DEFINITION 4.1. *Let U and V be two domains in R^n. A homeomorphism φ : $U \to V$ is $Q-$quasiisometrical (or simply quasiisometrical) if for any point $x \in U$ there exists such a ball $B(x,r) \subset U$ that*

$$Q^{-1}|y-z| < |\varphi(y) - \varphi(z)| < Q|y-z| \tag{1}$$

for any $y,z \in B(x,r)$. Here the constant $Q > 0$ does not depend on the choice of $x \in U$.

Obviously the inverse homeomorphism $\varphi^{-1} : V \to U$ is also $Q-$ quasiisometrical. Domains U and V are quasiisometrically equivalent if there exists a quasiisometrical homeomorphism $\varphi : U \to V$.

Any quasiisometrical homeomorphism is a locally bi-Lipschitz, weakly differentiable and differentiable almost everywhere.

Any diffeomorphism $\varphi : U \to V$ is quasiisometrical on a subdomain $U_1 \subset U$ if the closure $\overline{U_1}$ of U_1 belongs to U .

Let us demonstrate a practical way to construct a new quasiisometrical homeomorphism using a given one. Suppose that $S_k(x) = kx$ is a similarity transformation (which is called below a similarity) of R^n with the similarity coefficient $k > 0$, $S_{k_1}(x) = k_1x$ is another similarity and $\varphi : U \to V$ is a $Q-$ quasiisometrical homeomorphism. Then a composition $\psi := S_k \circ \varphi \circ S_{k_1}$ is a k_1kQ -quasiisometrical homeomorphism.

It is easy to check this claim. Because $\varphi : U \to V$ is $Q-$ quasiisometrical for any point $x \in U$ there exists such a ball $B(x,r) \subset U$ that the inequality 2.2 holds.

Therefore

$$|\psi(y) - \psi(z)| = k|\varphi(k_1y) - \varphi(k_1z)| < kQ|k_1y - k_1z| < k_1kQ|y-z|$$

for any $y,z \in k_1^{-1}B(k_1^{-1}x, k_1^{-1}r)$. By the same reasons

$$|\psi(y) - \psi(z)| > (k_1kQ)^{-1}|y-z|.$$

If $k_1 = k^{-1}$ then the homeomorphism ψ is $Q-$ quasiisometrical.

This remark will be used in example 2.2 of a domain with "spiral" boundary which is quasiisometrically equivalent to a cube. We start with a two-dimensional example.

EXAMPLE 4.2. We will construct a domain with "spiral" boundary with the help of a quasiisometrical homeomorphism. We can start with the triangle $T := \{(s,t)$:

$0 < s < 1, s < t < 2s\}$ because T is quasiisometrically equivalent to the unit square $Q_2 = (0,1) \times (0,1)$. Hence we need to construct only a quasiisometrical homeomorphism φ_0 from T into R^2.

Let (ρ, θ) be polar coordinates in the plane. Define first a mapping $\varphi : R^2_+ \to R^2$ as follows: $\varphi(s,t) = (\rho(s,t), \theta(s,t))$, $\rho(s,t) = s$, $\theta(s,t) = 2\pi \ln \frac{t}{s^2}$. Here $R^2_+ := \{(s,t) | 0 < s < \infty, 0 < t < \infty\}$. An inverse mapping can be calculated easily: $\varphi^{-1}(\rho, \theta) = (s(\rho,\theta), t(\rho,\theta))$, $s(\rho,\theta) = \rho$, $t(\rho,\theta)) = \rho^2 e^{\frac{\theta}{2\pi}}$. Therefore φ and $\varphi_0 = \varphi | T$ are diffeomorphisms.

The image of the ray $t = ks,\ s > 0, k > 0$ is the logarithmic spiral $\rho = k \exp(-\frac{\theta}{2\pi})$. Hence the image $S := \varphi(T) = \varphi_0(T)$ is an "elementary spiral" plane domain, because ∂T is a union of two logarithmic spirals $\rho = \exp(-\frac{\theta}{2\pi})$, $\rho = 2\exp(-\frac{\theta}{2\pi})$ and the segment of the circle $\rho = 1$.

The domain T is a union of countably many subdomains $T_n := \{(s,t) : e^{-(n+1)} < s < e^{-(n-1)}, s < t < 2s\}$, $n = 1, 2, \ldots$. On the first domain T_1 the diffeomorphism $\varphi_1 := \varphi | T_1$ is $Q-$quasiisometrical, because φ_1 is the restriction on T_1 of a diffeomorphism φ defined in R^2_+ and $\overline{T_1} \subset R^2_+$. We do not calculate the number Q.

If we will prove that any diffeomorphism $\varphi_n := \varphi | T_n$ is the composition $\varphi_n = S_{e^{-(n-1)}} \circ \varphi_1 \circ S_{e^{n-1}}$ of similarities $S_{e^{-(n-1)}}$, $S_{e^{n-1}}$ and the $Q-$quasiisometrical diffeomorphism φ_1, then any diffeomorphism φ_n is $Q-$quasiisometrical, the diffeomorphism φ_0 is also $Q-$quasiisometrical, and the "elementary spiral" domain $U = \varphi_0(T)$ is quasiisometrically equivalent to the unit square.

Let us prove the representation $\varphi_n = S_{e^{-(n-1)}} \circ \varphi_1 \circ S_{e^{n-1}}$.

By construction the domain T_1 is the image of T_n under the similarity transformation $S_{e^{n-1}}(s,t) = e^{n-1}(s,t)$. Therefore we need to prove only the representation $\varphi = S_{e^{-(n-1)}} \circ \varphi \circ S_{e^{n-1}}$. This representation follows from a direct calculation:

$$(S_{e^{-(n-1)}} \circ \varphi \circ S_{e^{n-1}})(s,t) = S_{e^{-(n-1)}}(\rho(e^{n-1}s, e^{n-1}t), \theta(e^{n-1}s, e^{n-1}t))$$

$$= (e^{-(n-1)}\rho(e^{n-1}s, e^{n-1}t), \theta(e^{n-1}s, e^{n-1}t)) = (s, 2\pi \ln \frac{t}{s^2} - 2\pi(n-1))$$

$$= (\rho(s,t), \theta(s,t)) = \varphi(s,t)$$

REMARK. By a rotation we can construct corresponding higher-dimensional examples of domains with "spiral" type singularities.

4.1. Domains of class L.

DEFINITION 4.3. *A domain U is an elementary domain of class L if it is a quasiisometrical image of an elementary domain of class ET.*

A domain U is a domain of class L if it is a finite union of elementary domains of class L.

PROPOSITION 4.4. *(see for example [6] or [15]) Let U and V be domains in R^n. A quasiisometrical homeomorphism $\varphi : U \to V$ induces a bounded composition operator $\varphi^* : H^1(V) \Rightarrow H^1(U)$ by the rule $\varphi^*(u) = u \circ \varphi$.*

Combining this result with Theorem 3.11 and Lemma 3.10 we obtain:

THEOREM 4.5. *If a domain* Ω *belongs to class* L *then the embedding operator* $H^1(\Omega) \to L^2(\Omega)$ *is compact.*

Proof. Let U be an elementary domain of class L. Then there exists an elementary domain V of class ET and a quasiisometrical homeomorphism $\varphi : V \to U$. By the previous theorem operators $\varphi^* : H^1(V) \Rightarrow H^1(U)$ and $(\varphi^{-1})^* : H^1(U) \to H^1(V)$ are bounded. By Theorem 3.8 the embedding operator $I_V : H^1(V) \to L^2(V)$ is compact. The embedding operator $I_U : H^1(U) \to L^2(U)$ is the composition $(\varphi^{-1})^* \circ I_V \circ \varphi^*$. Therefore the embedding operator $I_U : H^1(U) \to L^2(U)$ is compact.

Because any domain Ω of class L is a finite union of elementary domains of class L the result follows from Lemma 3.10. □

5. Domains with nonlocal singularities of the boundaries

The previous section focuses on domains which are locally quasiisometrical images of domains of class T. For the proof of the main result we used the compactness of embedding operators for domains of class T and the boundedness of composition operators induced by quasiisometrical homeomorphisms.

In this section we use similar arguments for the largest class of homeomorphisms that induce bounded composition operators of the Sobolev spaces H^1.

We recall the main idea for a study of the embedding operators proposed in [8]. Let Ω be a domain with "good" boundary, for example, domain of class L, and U be a domain with "bad" boundary. Suppose that there exists a homeomorphism $\phi : \Omega \to U$ such that ϕ induces a bounded composition operator $\phi^* : H^1(U) \to H^1(\Omega)$ by the rule $\phi^*(u) = u \circ \varphi$ and ϕ^{-1} induces a bounded composition operator $(\phi^{-1})^* : L^2(\Omega) \to L^2(U)$. If the embedding operator $I_\Omega : H^1(\Omega) \to L^2(\Omega)$ is compact, then the embedding operator $I_U = (\phi^{-1})^* I_\Omega \phi^* : H^1(\Omega) \Rightarrow L^2(\Omega)$ is also compact.

This method was used in [8] for a study of the embedding operators in domains with "nonlocal" singularities.

5.1. 2-quasi-conformal homeomorphisms.

Composition operators for Sobolev spaces with first generalized derivatives were studied in detail in [7]. We restrict ourselves to the practically important class of locally bi-Lipschitz homeomorphisms.

DEFINITION 5.1. *A locally bi-Lipschitz homeomorphism* $\phi : \Omega \to U$ *is 2-quasi-conformal if there exists a constant* K *such that*

$$\|\phi'(x)\|^2 \leqslant K |\det \phi'(x)|$$

for almost all $x \in \Omega$ *. The 2-quasi-conformal dilatation* $K(\phi)$ *is a minimal number* K *for which the previous inequality holds.*

Here $\phi'(x) = (\frac{\partial \varphi_i}{\partial x_j}(x)), i,j = 1,2,..,n,$ is the Jacobi matrix of the mapping φ at the point x and $\|\phi'(x)\| := \sqrt{\sum_{i,j=1}^{n} |\frac{\partial \varphi_i}{\partial x_j}(x)|^2}$ is the norm of the Jacobi matrix.

Obviously any quasiisometrical homeomorphism is 2-quasi-conformal. Composition of 2-quasi-conformal homeomorphisms is 2-quasi-conformal [8].

Choose two bounded domains Ω, U in R^n , $n > 2$.

THEOREM 5.2. *(see [8]) A locally bi-Lipschitz homeomorphism* $\phi : \Omega \to U$ *induces a bounded composition operator* $\phi^* : H^1(U) \to H^1(\Omega)$ *if and only if* ϕ *is 2-quasi-conformal.*

This result was used in the following version of the so-called "relative" embedding theorem.

THEOREM 5.3. *(see [7]) Suppose that a homeomorphism* $\phi : \Omega \to U$ *is 2-quasi-conformal and* $\| \det \phi'(x)\|_{L^\infty(\Omega)} < \infty$. *If the embedding operator* $I_\Omega : H^1(\Omega) \to L^2(\Omega)$ *is compact then the embedding operator* $I_U : H^1(U) \to L^2(U)$ *is also compact.*

The following corollary helps to use this result practically:

COROLLARY 5.4. *Suppose that* Ω *is domain of class* L *and there exists a 2-quasi-conformal homeomorphism* $\phi : \Omega \to U$. *If* $\| \det \phi'(x)\|_{L^\infty(\Omega)} < \infty$, *then the embedding operator* $I_U : H^1(U) \to L^2(U)$ *is compact.*

This corollary follows immediately from the previous theorem and the embedding theorem for $T-$ domains.

It allows one to use the method of Section 2 for 2-quasi-conformal case.

5.2. Domains of class Q.

DEFINITION 5.5. *A domain* U *is an elementary domain of class* Q *if there exist an elementary domain* V *of class* L *and a 2-quasi-conformal homeomorphism* $\varphi : U \to V$ *such that* $\| \det \phi'(x)\|_{L^\infty(\Omega)} < \infty$.

A domain U *is a domain of class* Q *if it is a finite union of elementary domains of class* Q.

Combining Corollary 5.4 with the Theorem 4.5 and Lemma 3.10 we obtain

THEOREM 5.6. *If a domain* Ω *belongs to class* Q *then the embedding operator* $H^1(\Omega) \Rightarrow L^2(\Omega)$ *is compact.*

Proof. Let U be an elementary domain of class Q. Then there exists an elementary domain V of class L and a 2-quasiisometrical homeomorphism $\varphi : V \to U$ such that $\| \det \phi'(x)\|_{L^\infty(\Omega)} < \infty$. By Theorem 5.2 operators $\varphi^* : H^1(V) \to H^1(U)$ and $(\varphi^{-1})^* : H^1(U) \Rightarrow H^1(V)$ are bounded. By Corollary 5.4 the embedding operator $I_V : H^1(V) \to L^2(V)$ is compact. The embedding operator $I_U : H^1(U) \to L^2(U)$ is equal to the composition $(\varphi^{-1})^* \circ I_V \circ \varphi^*$. Therefore the embedding operator $I_U : H^1(U) \to L^2(U)$ is compact.

Because any domain Ω of class Q is a finite union of elementary domains of class Q, the result follows from Lemma 3.10. □

Let us demonstrate a simple example of an elementary domain of class Q with "non local" singularity near the point $\{0\}$.

EXAMPLE 5.7. Let $\Omega \in R^2$ be the union of rectangles $T_k = \{x \in R^2 : |x_1 - 2^{-\alpha k}| \leqslant 2^{-\alpha(k+2)}; 0 \leqslant x_2 < 2^{-\alpha(k+2)}\}, 0 < \alpha$ and the square $Q = (0,1) \times (-1,0)$. It is easy to check that the homeomorphism $\varphi(x_1,x_2) = (x_1|x|^{\frac{1}{\alpha}-1}, x_2|x|^{\frac{1}{\alpha}-1}\}$ is 2-quasi-conformal and $\Omega_1 = \varphi(\Omega)$ is the union of rectangles $P_k = \{x \in R^2 : |x_1 - 2^{-k}| \leqslant 2^{-(k+2)}; 0 \leqslant x_2 < 2^{-(k+2)}\}, 0 < \alpha$ and the square $Q = (0,1) \times (-1,0)$. In [10] a quasiisometrical homeomorphism ψ from Ω_1 to the unit square is constructed. Hence the composition $\phi = \psi \circ \varphi$ is a 2-quasi-conformal homeomorphism and by direct calculation we can check that $\| \det \phi'(x)\|_{L^\infty(\Omega)} < \infty$. Therefore the domain Ω is an elementary domain of class Q.

A projection of $B(0,r) \cap \partial\Omega$ onto an arbitrary straight line $L \in R^2$ is not a one-to-one correspondence for any r and L . Therefore the domain Ω is not an elementary domain of class C.

Higher-dimensional examples can be constructed using rotations.

5.3. Discussion of 2-quasiconformal homeomorphisms and 2-quasi-conformal domains.

Let us give first a geometrical interpretation of 2-quasi-conformality.

Suppose that $\phi : R^n \to R^n$ is a linear homeomorphism, φ' is its matrix and $(\phi')^T$ its adjoint matrix. Denote by $\lambda_1^2 \leqslant \lambda_2^2 \leqslant ... \leqslant \lambda_n^2$ eigenvalues of $(\phi')^T\phi'$. There exist two orthogonal bases $e_1, e_2, ..., e_n$ and $g_1, g_2, ..., g_n$ such that $\phi(e_i) = \lambda_i g_i$ for every $i = 1,2,...,n$. Geometrically λ_i is length of $i-$ th semi-axis of the ellipsoid $\phi(B(0,1))$. The 2-quasi-conformal dilatation $K(\phi) = \frac{\lambda_n}{\lambda_1\lambda_2...\lambda_{n-1}}$.

If $\varphi : \Omega \to U$ is a diffeomorphism then the numbers $\lambda_1(x) \leqslant \lambda_2(x) \leqslant ... \leqslant \lambda_n(x)$ correspond to the linear homeomorphism $d\phi$ and

$$K(\phi) = \sup_{x\in\Omega}[\frac{\lambda_n(x)}{\lambda_1(x)\lambda_2(x)...\lambda_{n-1}(x)}].$$

If $\varphi : \Omega \to U$ is only locally Lipschitz then

$$K(\phi) = \operatorname*{ess\,sup}_{x\in\Omega}[\frac{\lambda_n(x)}{\lambda_1(x)\lambda_2(x)...\lambda_{n-1}(x)}].$$

The relations of 2-quasi-conformal homeomorphisms with the traditional classes can be described as follows:

1) In the two-dimensional case 2-quasi-conformal homeomorphisms are quasi-conformal. A homeomorphism inverse to a quasi-conformal homeomorphism is also quasi-conformal. Therefore a homeomorphism inverse to 2-quasi-conformal homeomorphism is 2-quasi-conformal (for plane domains). Unfortunately, this property does not hold in the higher-dimensional cases. In [7] an example of 2-quasi-conformal

homeomorphism with non-2-quasi-conformal inverse homeomorphism is constructed. Composition of 2-quasi-conformal homeomorphisms is a 2-quasi-conformal homeomorphism.

2) Two-dimensional conformal mappings are 2-quasi-conformal homeomorphisms with $K(\phi) = 1$.

3) Any quasiisometrical homeomorphism is 2-quasi-conformal.

Appendix

In this section an abstract necessary and sufficient condition for the embedding operator to be compact is given. In our presentation the work [12] is used.

Let H_j, $j = 1,2,3$, be Hilbert spaces, $H_1 \subset H_2 \subset H_3$, the embeddings mean set-theoretical inclusions and the inequalities $\|u\|_1 \geqslant \|u\|_2 \geqslant \|u\|_3$, where $\|u\|_j := \|u\|_{H_j}$. This implies the compatibility of the norms:

if $\|u_n\|_3 \to 0$ and $\|u_n - u\|_2 \to 0$ then $u = 0$

Denote by i the embedding operator from H_1 into H_2 and by j the embedding operator from H_1 into H_3.

PROPOSITION 5.8. *The operator* $i : H_1 \to H_2$ *is compact if and only if the following conditions hold:*

1) j *is compact,*

and

2) $\|u\|_2 \leqslant \varepsilon\|u\|_1 + c(\varepsilon)\|u\|_3$ *for all* $\varepsilon \in (0,\varepsilon_0)$, $c(\varepsilon) = const > 0$, *for all* $u \in H_1$.

Proof. Necessity: condition 1) is clearly necessary: if $i\colon H_1 \to H_2$ is compact, and $H_2 \subset H_3$, $\|u\|_2 \geqslant \|u\|_3$, then $j\colon H_1 \to H_3$ is compact.

To prove 2), assume the contrary: there exists $u_n \in H_1$, $\|u_n\|_1 = 1$, and $\varepsilon \in (0,\varepsilon_0)$ such that

$$\|u_n\|_2 > \varepsilon\|u_n\|_1 + n\|u_n\|_3$$

for all $n = 1,2,\dots$.

Since $\|u_n\|_1 = 1 \geqslant \|u_n\|_2$, one concludes from previous inequality that $\|u_n\|_3 \to 0$ as $n \to \infty$ and $u_n \rightharpoonup u$ in H_1, $\rightharpoonup$ stands for weak convergence. Since $i\colon H_1 \to H_2$ is compact, it follows that $\|u_n - u\|_2 \to 0$. Since $\|u_n\|_3 \to 0$ it follows that $u = 0$ and $\|u_n\|_2 \to 0$. This is a contradiction: by condition 2) the inequality $\|u_n\|_2 \geqslant \varepsilon > 0$ holds. The necessity of the conditions 1) and 2) is established.

Sufficiency: if j is compact then $\|u_n\|_1 = 1$ implies that a subsequence u_n (denoted again u_n) converges in H_3, that is $\|u_n - u_m\|_3 \to 0$ as $n,m \to \infty$. Condition 2) implies $\|u_n - u_m\|_2 \leqslant \varepsilon\|u_n - u_m\|_1 + c(\varepsilon)\|u_n - u_m\|_3$.

Fix an arbitrary small $\delta > 0$. Note that $\|u_n - u_m\|_1 \leqslant 2$. Choose $\varepsilon = \delta/4$ and fix it. Then choose n,m so large that $c(\varepsilon)\|u_n - u_m\|_3 < \delta/2$. Then $\|u_n - u_m\|_2 < \delta$. This implies convergence of u_n in H_2. The sufficiency is proved. □

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