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SHORT PROOFS OF THE GALE & RYSER AND FORD & FULKERSON CHARACTERIZATIONS OF THE ROW AND COLUMN SUM VECTORS OF (0,1)–MATRICES

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Abstract. We simultaneously give short, direct proofs of the Gale & Ryser and Ford & Fulkerson characterizations of the row and column sum vectors of (0,1)-matrices.

Let $R := r_1 \geqslant r_2 \geqslant \cdots \geqslant r_m$ and $S := s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$ be two nonincreasing sequences of nonnegative integers with

$$\tau := r_1 + r_2 + \cdots + r_m = s_1 + s_2 + \cdots + s_n.$$

Let $\bar{R} : \bar{r}_1 \geqslant \bar{r}_2 \geqslant \cdots \geqslant \bar{r}_n$ be the *conjugate* of R:

$$\bar{r}_j = |\{i : r_i \geqslant j\}|,$$

and let $T = [t_{kl}]$ be the $m+1$ by $n+1$ *structure matrix* defined by

$$t_{kl} = kl + \sum_{i=k+1}^{m} r_i - \sum_{j=1}^{l} s_j, \quad (0 \leqslant k \leqslant m, 0 \leqslant l \leqslant n).$$

One easily verifies that

$$t_{00} = \tau,\ t_{0n} = 0,\ t_{m0} = 0,\ t_{1n} = n - r_1,\ t_{m1} = m - s_1, \text{ and } t_{mn} = mn - \tau.$$

For vectors $X : x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ and $Y : y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$, define X is *majorized by* Y, denoted $X \preceq Y$, to mean

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i \quad (1 \leqslant k \leqslant n)$$

with equality for $k = n$.

By use of direct combinatorial reasoning (Ryser [5] for (1) below) and the theory of network flows (Gale [3] also for (1), and Ford and Fulkerson [2] for (2)), the following theorem has been proved (see also [1]).

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THEOREM 1. *Each of the following conditions is equivalent to the existence of an m by n $(0,1)$-matrix A with row sums given by R and column sums given by S:*

$$r_1 \leqslant n \text{ and } S \preceq \bar{R} \tag{1}$$

$$\textstyle\sum_{j=1}^{l} s_j - \sum_{i=k+1}^{m} r_k \leqslant kl, \quad (0 \leqslant k \leqslant m,\ 0 \leqslant l \leqslant n) \tag{2}$$

$$(T \text{ is a nonnegative matrix}).$$

Proof. (This part of the proof is standard.) Assume there is a matrix A with row sums R and column sums S. Then $r_1 \leqslant n$ and if one partitions A into

$$\left[\begin{array}{c|c} B_1 & B_2 \end{array}\right] \qquad (B_1 \text{ is } m \text{ by } l),$$

then the number of 1's in B_1 does not exceed the number of 1's in the first l columns of the matrix obtained from A by sliding the 1's in each row as far to the left as possible; that is, $\sum_{i=1}^{l} s_i \leqslant \sum_{i=1}^{l} \bar{r}_i$. Likewise, if one partitions A into

$$\left[\begin{array}{c|c} A_1 & X \\ \hline Y & A_2 \end{array}\right] \qquad (A_1 \text{ is } k \text{ by } l),$$

then t_{kl} equals the number of 0's in A_1 plus the number of 1's in A_2 (either or both of A_1 and A_2 may be vacuous); hence T is a nonnegative matrix.

We prove the converses together by contradiction; a direct proof of the equivalence of (1) and (2) can also be given. Choose a counterexample with $m+n$ minimal, and for that $m+n$, with τ minimal. Clearly $t_{00} = \tau > 0$ (otherwise take A to be the zero matrix O) and $t_{mn} > 0$ (otherwise take A to be the matrix J of all 1's).

Case 1: $\sum_{i=1}^{l} s_i = \sum_{i=1}^{l} \bar{r}_i$ for some l with $0 < l < n$, and $t_{kl} = 0$ for some k and l with $(k,l) \neq (0,n)$ or $(m,0)$, respectively. It is an immediate consequence that each of

$$R_1 := r_1' = \min\{l, r_1\}, \ldots, r_m' = \min\{l, r_m\} \text{ and } S_1 := s_1, \ldots, s_l$$

and

$$R_2 := r_1 - r_1', \ldots r_m - r_m', \text{ and } S_2 := s_{l+1}, \ldots, s_n$$

both satisfy the majorization criteria (1) of Gale and Ryser, and that the structure matrices for $R_3 := r_1 - l, \ldots, r_m - l$, $S_3 := s_{l+1}, \ldots, s_n$ and for $R_4 := r_{k+1}, \ldots, r_m$, $S_4 := s_1 - k, \ldots, s_l - k$ satisfy the nonnegativity criteria (2) of Ford and Fulkerson. By minimality there exists a $(0,1)$-matrices A_1 and A_2 with row sums R_1 and column sums S_1, and row sums R_2 and column sums S_2, respectively. Hence the matrix

$$\left[\begin{array}{cc} A_1 & A_2 \end{array}\right]$$

has row sums R and column sums S, a contradiction. By minimality there also exists a $(0,1)$-matrix A_3 with row sums R_3 and column sums S_3, and a $(0,1)$-matrix A_4 with row sums R_4 and column sums S_4. Hence the matrix

$$\left[\begin{array}{c|c} J & A_3 \\ \hline A_4 & O \end{array}\right] \qquad (J \text{ is the } k \text{ by } l \text{ matrix of all } 1\text{'s})$$

has row sums R and column sums S, also a contradiction.

Case 2: $\sum_{i=1}^{l} s_i > \sum_{i=1}^{l} \bar{r}_i$ for all l with $0 < l < n$, and $t_{kl} > 0$ for all k and l with $(k,l) \neq (0,n)$ or $(m,0)$, respectively. By minimality, $r_m \geqslant 1$ and $s_n \geqslant 1$. Let $R' := r_1, \dots, r_{m-1}, r_m - 1$ and $S' := s_1, \dots, s_{n-1}, s_n - 1$. Then $R' \preceq S'$ and the structure matrix for R' and S' is nonnegative. By minimality again there is a matrix $A' = [a'_{ij}]$ with row sums R' and column sums S'. If $a'_{mn} = 0$, then changing this 0 to a 1 we obtain a matrix with row sums R and column sums S, a contradiction. Suppose that $a'_{mn} = 1$. Since $r_m - 1 \leqslant n - 1$, there is a q such that $a'_{mq} = 0$. Since $s_q \geqslant s_n$, there is a p such that $a'_{pq} = 1$ and $a'_{pn} = 0$. Interchanging 0 with 1 in this 2 by 2 matrix gives a matrix A'' with row sums R' and column sums S', and we get a contradiction as before. $\square$

Simple, 'minimal counterexample' proofs of e.g. Landau's existence theorem for tournaments with a given score vector and König's matching theorem can be found in [4, 6].

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