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EQUIVALENCE OF THE HÖLDER–ROGERS AND MINKOWSKI INEQUALITIES

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Abstract. It is well-known that the Hölder-Rogers inequality implies the Minkowski inequality. Infantozzi [6] observed implicitely and Royden [15] proved explicitely that the reverse implication is also true. In this note we discuss and give a new proof of this perhaps surprising fact.

The proofs as well as the extensions, inverses and applications of the well-known Hölder-Rogers and Minkowski inequalities can be found in many books about real functions, analysis, functional analysis, L_p-spaces or inequalities (cf. [1], [2], [3], [4], [5], [7], [10], [11], [12], [15]). In this note we discuss and give a new proof of the perhaps surprising fact that these two fundamental inequalities are equivalent.

The classical *Hölder-Rogers inequality* states: Let $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in L_p(\mu)$ and $y \in L_q(\mu)$, then $xy \in L_1(\mu)$ and

$$\|xy\|_1 \leqslant \|x\|_p \|y\|_q. \tag{1}$$

The classical *Minkowski inequality* states: Let $1 \leqslant p < \infty$. If $x, y \in L_p(\mu)$, then $x + y \in L_p(\mu)$ and

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p. \tag{2}$$

Most of mathematicians will say that (1) is just the Hölder inequality but both Rogers and Hölder had an equivalent form of (1). Rogers proved it in 1888 and Hölder one year later refering to Rogers (see [8] for the complete history; cf. also [5], p. 25, [4], p. 202, [3], p. 117). F. Riesz [13] was the first who obtained and used inequalities (1) and (2) precisely in such a form, and in the context of L_p-spaces, therefore sometimes these inequalities are called Hölder-Riesz and Minkowski-Riesz inequalities (cf. for example [2], pp. 193–194 or [3], p. 119). I will put the alphabetical order in the name *Hölder-Rogers inequality* rather than the proper one Rogers-Hölder inequality just not disturbing to much with the priority.

For $p = 1$ Hölder-Rogers and Minkowski inequalities follow immediately from the properties of the integral and the essential supremum. Therefore we can assume that $1 < p < \infty$.

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The most natural deduction of the Minkowski inequality from the Hölder-Rogers inequality is due to F. Riesz [14, pp. 45–46 for sums; for inegrals Riesz showed it in 1928]. We have

$$\begin{aligned}\|x+y\|_p^p &= \int_\Omega |x(s)+y(s)|^p d\mu(s)\\ &\leqslant \int_\Omega |x(s)+y(s)|^{p-1}(|x(s)|+|y(s)|)d\mu(s)\\ &= \int_\Omega |x(s)+y(s)|^{p-1}|x(s)|d\mu(s) + \int_\Omega |x(s)+y(s)|^{p-1}|y(s)|d\mu(s).\end{aligned}$$

Applying the Hölder-Rogers inequality (1) to each integral, and observing that $(p-1)q = p$, we obtain

$$\begin{aligned}\|x+y\|_p^p &\leqslant \|x\|_p \Big(\int_\Omega |x(s)+y(s)|^{(p-1)q}d\mu(s)\Big)^{1/q}\\ &\quad + \|y\|_p \Big(\int_\Omega |x(s)+y(s)|^{(p-1)q}d\mu(s)\Big)^{1/q}\\ &= (\|x\|_p + \|y\|_p)(\|x+y\|_p^p)^{1/q},\end{aligned}$$

and so

$$(\|x+y\|_p^p)^{1-1/q} \leqslant \|x\|_p + \|y\|_p,$$

which, by the fact that $p(1-1/q) = 1$, gives the Minkowski inequality (2).

There are known equivalences of AM-GM inequality and Lyapunov inequality to the Hölder-Rogers inequality (cf. [10, pp. 457–462]) but it is interesting that the Minkowski inequality is also equivalent to the Hölder-Rogers inequality. This observation follows implicitely from the announcement by Infantozzi [**6**, pp. 121-122] (cf. also [12, p. 200]) but there were no proofs of his several equivalences. The proof that Hölder-Rogers inequality can be obtained from the Minkowski inequality has been given, for example, by Royden [**15**, pp. 121–122]. He was using the Bernoulli inequality and the formula for the derivative of a power function.

Let $1 < p < \infty$. The Bernoulli inequality $(1+u)^p \geqslant 1 + pu$ with $u = tb/a$ gives

$$(a+tb)^p \geqslant a^p + pta^{p-1}b$$

for all a, b, $t \geqslant 0$, which implies that

$$pt|x(s)||y(s)| \leqslant (|y(s)|^{1/(p-1)} + t|x(s)|)^p - |y(s)|^{p/(p-1)},$$

for $s \in \Omega$. Integrating over Ω we obtain

$$pt\int_\Omega |x(s)||y(s)|d\mu(s) \leqslant \||y|^{1/(p-1)} + t|x|\|_p^p - \||y|^{1/(p-1)}\|_p^p,$$

which, by the Minkowski inequality, can be estimated by

$$\leqslant (\||y|^{1/(p-1)}\|_p + t\|x\|_p)^p - \||y|^{1/(p-1)}\|_p^p.$$

Hence, by the definition of the derivative,

$$p \int_\Omega |x(s)||y(s)|d\mu(s) \leqslant \liminf_{t\to 0^+} \Big[\Big(\||y|^{1/(p-1)}\|_p + t\|x\|_p\Big)^p - \||y|^{1/(p-1)}\|_p^p\Big]/t$$
$$= p\||y|^{1/(p-1)}\|_p^{p-1}\|x\|_p = p\|x\|_p\|y\|_q,$$

and the Hölder-Rogers inequality (1) is proved.

We will now give another proof of the above implication (which implies that the Hölder-Rogers inequality is equivalent to the Minkowski inequality). The following lemma will be the main step in this proof. This lemma is also in a sense known since it is saying that the power means goes to the geometric mean when the exponent goes to zero (cf. [5, p. 15] or [1, p. 133]).

LEMMA. *For $0 < \theta < 1$ and any $a, b \geqslant 0$, we have*

$$\lim_{p\to\infty} [\theta a^{1/p} + (1-\theta)b^{1/p}]^p = a^\theta b^{1-\theta}. \tag{3}$$

For the sake of completeness we include an elementary proof which only depends on the definition of the derivative.

Proof. If either $a = 0$ or $b = 0$, then equality (3) obviously holds. Therefore we can assume that $a, b > 0$. Let, for a fixed $\theta \in (0,1)$ and $a, b > 0$, the function f be defined by

$$f(x) = ln[\theta a^x + (1-\theta)b^x].$$

Then, using only the definition of the derivative, we have when $p \to \infty$

$$[\theta a^{1/p} + (1-\theta)b^{1/p}]^p = \exp\{p[\theta a^{1/p} + (1-\theta)b^{1/p}]\}$$
$$= \exp\frac{f(1/p) - f(0)}{1/p} \to \exp f'(0)$$
$$= \exp[\theta \ln a + (1-\theta)\ln b] = a^\theta b^{1-\theta}.$$

REMARK 1. The above proof shows also that for any $a_k > 0$ $(k = 1, 2, \ldots, n)$ we have

$$\lim_{p\to\infty} (\theta_1 a_1^{1/p} + \theta_2 a_2^{1/p} + \ldots + \theta_n a_n^{1/p})^p = a_1^{\theta_1} a_2^{\theta_2} \ldots a_n^{\theta_n}, \tag{4}$$

where $\theta_k > 0$ and $\sum_{k=1}^n \theta_k = 1$.

Another proof that the Minkowski inequality (2) implies the Hölder-Rogers inequality (1).

Let $x \in L_p(\mu)$ and $y \in L_q(\mu)$. Denote $u(s) = |x(s)|^p$ and $v(s) = |y(s)|^q$. Then $u^{1/p} \in L_p(\mu)$ and $v^{1/p} \in L_p(\mu)$. For any $0 < \theta < 1$ we have from the Minkowski inequality and the homogeneity of the L_p-norm

$$\|\theta u^{1/p} + (1-\theta)v^{1/p}\|_p \leqslant \|\theta u^{1/p}\|_p + \|(1-\theta)v^{1/p}\|_p$$
$$= \theta\|u^{1/p}\|_p + (1-\theta)\|v^{1/p}\|_p,$$

which gives

$$\int_\Omega [\theta u(s)^{1/p} + (1-\theta)v(s)^{1/p}]^p d\mu(s) \leqslant \Big[\theta\Big(\int_\Omega u(s)d\mu(s)\Big)^{1/p} + (1-\theta)\Big(\int_\Omega v(s)d\mu(s)\Big)^{1/p}\Big]^p.$$

Taking $p \to \infty$ we get, according to our Lemma and the Lebesgue theorem

$$\int_\Omega u(s)^\theta v(s)^{1-\theta} d\mu(s) \leqslant \Big(\int_\Omega u(s)d\mu(s)\Big)^\theta \Big(\int_\Omega v(s)d\mu(s)\Big)^{1-\theta},$$

or

$$\int_\Omega |x(s)|^{p\theta}|y(s)|^{q(1-\theta)} d\mu(s) \leqslant \Big(\int_\Omega |x(s)|^p d\mu(s)\Big)^\theta \Big(\int_\Omega |y(s)|^q d\mu(s)\Big)^{1-\theta},$$

Putting now $\theta = 1/p$ in the last inequality we obtain the Hölder-Rogers inequality (1).

REMARK 2. Our proof of equivalence is still working for a general Banach function spaces $X(\mu)$ instead of the $L_1(\mu)$-space. Let $1 < p < \infty$. The generalized version of the Hölder-Rogers inequality reads: If $|x|^p \in X(\mu)$ and $|y|^q \in X(\mu), \frac{1}{p} + \frac{1}{q} = 1$, then $xy \in X(\mu)$ and

$$\|xy\|_X \leqslant \||x|^p\|_X^{1/p} \||y|^q\|_X^{1/q}. \qquad (1_X)$$

The generalized version of the Minkowski inequality reads: if $|x|^p \in X(\mu)$ and $|y|^p \in X(\mu)$, then $|x+y|^p \in X(\mu)$ and

$$\||x+y|^p\|_X^{1/p} \leqslant \||x|^p\|_X^{1/p} + \||y|^p\|_X^{1/p}. \qquad (2_X)$$

These two inequalities are equivalent when the space $X(\mu)$ has the so called Fatou property (cf. [7, p. 30]). The implication $(1_X) \Longrightarrow (2_X)$ we can get from the above Riesz method (cf. [9, p. 326]). The reverse implication $(2_X) \Longrightarrow (1_X)$ can be proved in a similar way we did under the assumption that space $X(\mu)$ has the Fatou property.

REMARK 3. The fundamental inequalities (1) and (2) are reversed for $p < 1$, $p \neq 0$. The proofs presented here also shows that the reverse Hölder-Rogers and the reverse Minkowski inequalities are equivalent for positive functions x and y.

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