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AN EXTENSION OF THE SIDON–FOMIN TYPE INEQUALITY AND ITS APPLICATIONS

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Abstract. An extension of the Sidon-Fomin type inequality [5] is made by considering the r-th derivate of the Dirichlet's kernel $D_k^{(r)}$ instead of D_k. Namely, two different proofs of the following inequality

$$\int_0^{\pi}\left|\sum_{k=0}^{n}\alpha_k D_k^{(r)}(x)\right| dx = O\left((n+1)^{r+1}\right), \quad |\alpha_k| \leqslant 1 \quad \text{for all} \quad k \qquad (*)$$

are given. Applying the inequality $(*)$ it's shown that the new class S_r is a subclass of $BV \cap C_r$, $r = 0, 1, 2, \ldots$ where C_r is the extension of the Garret-Stanojević class [7] and BV is the class of null sequences of bounded variation. Also, in this paper an extension of the theorem for convergence and integrability for cosine series of a [6] is made by considering the class S_r, $r = 0, 1, 2, \ldots$ instead of S.

1. Introduction and preliminaries

Sidon [5] proved inequality named after him in 1939 year. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications for instance in L^1-convergence problems and summation methods with respects to trigonometric series, newer and newer improvements of the original inequality has been proved by several authors.

Fomin [1] applying the linear method for summing of Fourier series has given another proof of this inequality. Thus the inequality is called as Sidon-Fomin's inequality. Also, S. A. Telyakovskii in [6] has given an elegant proof of Sidon-Fomin's inequality.

LEMMA 1. (Sidon-Fomin). *Let $\{\alpha_k\}_{k=0}^{n}$ be a sequence of real number such that $|\alpha_k| \leqslant 1$ for all k. Then there exists a positive constant M such that for any $n \geqslant 0$,*

$$\left\|\sum_{k=0}^{n}\alpha_k D_k(x)\right\|_1 \leqslant M(n+1),$$

where $||\cdot||$ is the L^1-norm.

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Let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx \tag{C}$$

be the cosine series. Several authors have studied the problem of L^1-convergence of the series (C).

In [6] Telyakovskii defined the following class of L^1-convergence of Fourier series. A sequence $\{a_k\}$ belongs to the class S, or $\{a_k\} \in S$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=0}^{\infty} A_k < \infty$ and $|\Delta a_k| \leqslant A_k$ for all k. The importance of Telyakovskii's contributions are twofold. Firstly, he expressed Sidon's conditions [5] in a succinct equivalent form and secondly, he showed that the class S is also a class of L^1-convergence. Thus the class S is usualy called as Sidon-Telyakovskii class.

In the same paper, Telyakovskii proved the following theorem.

THEOREM 1. [6] *Let the coefficients of the series (C) satisfy the conditions S. Then the series is a Fourier series of some $f \in L^1(0,\pi)$ and the following inequality holds:*

$$\int_0^{\pi} |f(x)|\, dx \leqslant M \sum_{n=0}^{\infty} A_n\,, \quad M>0\,.$$

Next we define a new class S_r, $r=0,1,2,\ldots$ of sequences as follows: $\{a_k\} \in S_r$ if $a_k \to 0$ as $k \to \infty$ and there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \leqslant A_k$ for all k. When $r=0$, we denote $S_r = S$.

We note that by $A_k \downarrow 0$ and

$$\sum_{k=1}^{\infty} k^r A_k < \infty, \quad \text{follows} \quad k^{r+1} A_k = o(1)\,, \quad k \to \infty\,. \tag{1}$$

It is obvious that $S_{r+1} \subset S_r$ but the converse of that inclusion is false.

EXAMPLE. For $n=1,2,3,\ldots$ define $\Delta a_n = \dfrac{1}{n^{r+2}}$, $r=0,1,2,3,\ldots$. Firstly, we shall show that $\{a_n\} \notin S_{r+1}$.

Really, $a_n = \sum_{k=n}^{\infty} \Delta a_k = \sum_{k=n}^{\infty} \dfrac{1}{k^{r+2}} \to 0$ as $n \to \infty$.

Let we denote $A_n^* = \max_{i \geqslant n} |\Delta a_i|$. Then $A_n^* = |\Delta a_n|$ and $A_n^* \downarrow 0$ as $n \to \infty$. Let A_n is an arbitrary positive sequence such that $A_n \downarrow 0$ and $A_n^* \leqslant A_n$.

Then $\sum_{n=1}^{\infty} n^{r+1} A_n \geqslant \sum_{n=1}^{\infty} n^{r+1} \dfrac{1}{n^{r+2}} = \sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent, i.e. $\{a_n\} \notin S_{r+1}$.

Now, for all n, let $A_n = \dfrac{1}{n^{r+2}}$, $r = 0,1,2,\dots$.

Then $A_n \downarrow 0$, $|\Delta a_n| \leqslant A_n$ and $\sum_{n=1}^{\infty} n^r A_n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, i.e. $\{a_n\} \in S_r$.

Garrett and Stanojević [3] introduced the following class C. A null sequence $\{a_k\}$ belongs to the class C if for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n and such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon, \quad \text{for all } n,$$

where D_n is the Dirichlet's kernel.

Applying the Lemma 1, Garrett, Rees and Stanojević in [2] proved that $S \subseteq BV \cap C$, where BV denote the class of null-sequences of bounded variation.

In [7] we defined an extension of the Garrett-Stanojević class as follows, a null sequence $\{a_k\}$ belongs to the class C_r, $r = 0,1,2,3,\dots$ i.e. $\{a_k\} \in C_r$, if for every $\varepsilon > 0$ there exists $\delta > 0$ independent of n such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \varepsilon, \quad \text{for all } n$$

where $D_n^{(r)}$ is the r-th derivate of the Dirichlet kernel.

In this paper we shall prove the following main results:

THEOREM 2. *Let $\{\alpha_k\}_{k=0}^{n}$ be a sequence of real numbers such that $|\alpha_k| \leqslant 1$ for all k. Then there exists a constant $M > 0$ such that for any $n \geqslant 0$ and $r = 0,1,2,\dots$*

$$\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\|_1 \leqslant M(n+1)^{r+1},$$

where $||\cdot||$ is the L^1-norm.

THEOREM 3. *Let the coeficients of the series* (C) *satisfy the condition S_r, $r = 0,1,2,\dots$. Then the r-th derivate of the series* (C) *is a Fourier series of some $f^{(r)} \in L^1(0,\pi)$ and the following relation holds:*

$$\int_0^{\pi} \left| f^{(r)}(x) \right| dx \leqslant M \sum_{n=1}^{\infty} n^r A_n, \quad \text{where} \quad M > 0.$$

THEOREM 4. *For each $r = 0,1,2,\dots$ the following embedding relation holds: $S_r \subseteq BV \cap C_r$.*

2. Lemmas

For the proofs of the our new results, we need the following lemmas.

LEMMA 2. [4] *Let* r *be a nonnegative integer and* $x \in (0, \pi]$. *Then*

$$D_n^{(r)}(x) = \sum_{k=0}^{r-1} \frac{\left(n+\frac{1}{2}\right)^k \sin\left[\left(n+\frac{1}{2}\right)x + \frac{k\pi}{2}\right]}{\left(\sin(x/2)\right)^{r+1-k}} \varphi_k(x) + \frac{\left(n+\frac{1}{2}\right)^r \sin\left[\left(n+\frac{1}{2}\right)x + \frac{r\pi}{2}\right]}{2\sin(x/2)},$$

where the same φ_k *denotes various analytical function of* x *independent of* n.

LEMMA 3. [8] *If* $T_n(x)$ *is a trigonometrical polynomial of order* n, *then* $||T_n^{(r)}|| \leqslant n^r ||T_n||$. This is S. Bernstein's inequality in the $L^1(0,\pi)$-metric (see [8], Vol. 2, p. 11).

LEMMA 4. *Let* $\{\alpha_j\}_{j=0}^k$ *be a sequence of real numbers. Then the following relation holds for* $\nu = 0, 1, 2, \ldots, r$ *and* $r = 0, 1, 2, \ldots$

$$\begin{aligned} U_k &= \int_{\pi/(k+1)}^{\pi} \left| \sum_{j=0}^{k} \alpha_j \frac{\left(j+\frac{1}{2}\right)^\nu \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu\pi}{2}\right]}{\left(\sin(x/2)\right)^{r+1-\nu}} \right| dx \\ &= O\left((k+1)^{r+\frac{1}{2}-\nu} \left(\sum_{j=0}^{k} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right). \end{aligned}$$

Proof. Applying first Cauchy-Bunjakovskii inequality, yields,

$$U_k \leqslant \Big[\int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{\left(\sin(x/2)\right)^{2(r+1-\nu)}}\Big]^{1/2} \Big\{\int_{\frac{\pi}{k+1}}^{\pi} \Big[\sum_{j=0}^{k} \alpha_j \Big(j+\frac{1}{2}\Big)^\nu \sin\Big[\Big(j+\frac{1}{2}\Big)x + \frac{\nu\pi}{2}\Big]\Big]^2 dx\Big\}^{1/2}.$$

Since

$$\begin{aligned} \int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{\left(\sin(x/2)\right)^{2(r+1-\nu)}} &\leqslant \pi^{2(r+1-\nu)} \int_{\frac{\pi}{k+1}}^{\pi} \frac{dx}{x^{2(r+1-\nu)}} \leqslant \frac{\pi(k+1)^{2(r+1-\nu)-1}}{2(r+1-\nu)-1} \\ &\leqslant \pi(k+1)^{2(r+1-\nu)-1}, \end{aligned}$$

we have:

$$\begin{aligned} U_k &\leqslant \sqrt{\pi}\big[(k+1)^{2(r+1-\nu)-1}\big]^{1/2} \Big\{\int_0^{\pi} \Big[\sum_{j=0}^{k} \alpha_j \Big(j+\frac{1}{2}\Big)^\nu \sin\Big[\Big(j+\frac{1}{2}\Big)x + \frac{\nu\pi}{2}\Big]\Big]^2 dx\Big\}^{1/2} \\ &\leqslant \sqrt{2\pi}\Big[(k+1)^{2(r+1-\nu)-1}\Big]^{1/2} \Big\{\int_0^{2\pi} \Big[\sum_{j=0}^{k} \alpha_j \Big(j+\frac{1}{2}\Big)^\nu \sin\Big[(2j+1)t + \frac{\nu\pi}{2}\Big]\Big]^2 dt\Big\}^{1/2}. \end{aligned}$$

Then, applying the Parseval's equality, we obtain:

$$U_k \leqslant \sqrt{2\pi}\Big[(k+1)^{2(r+1-\nu)-1}\Big]^{1/2}\Big[\sum_{j=0}^{k}\alpha_j^2(j+1)^{2\nu}\Big]^{1/2},$$

i.e.

$$U_k = O\Big((k+1)^{r+\frac{1}{2}-\nu}\Big(\sum_{j=0}^{k}\alpha_j^2(j+1)^{2\nu}\Big)^{1/2}\Big).$$

3. Proofs of the main results

3.1. First proof of the Theorem 2.

We have:

$$\int_0^{\pi}\left|\sum_{i=0}^{n}\alpha_i D_i^{(r)}(x)\right| dx = \int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi} = I_n + J_n\,.$$

Applying the inequality $D_n^{(r)}(x) = O(n^{r+1})$, we have:

$$I_n \leqslant \sum_{i=0}^{n}|\alpha_i| i^{r+1}\frac{\pi}{n+1} \leqslant \beta(n+1)^{r+1}\,, \qquad \text{where} \qquad \beta > 0\,.$$

Applying the Lemma 2, let us estimate the second integral:

$$J_n \leqslant \int_{\pi/(n+1)}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\left(\sum_{\nu=0}^{r-1}\frac{\big(j+\frac{1}{2}\big)^{\nu}\,\sin\big[\big(j+\frac{1}{2}\big)x+\frac{\nu\pi}{2}\big]}{\big(\sin(x/2)\big)^{r+1-\nu}}\,\varphi_\nu(x)\right)\right| dx$$
$$+\int_{\pi/(n+1)}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\frac{\big(j+\frac{1}{2}\big)^{r}\,\sin\big[\big(j+\frac{1}{2}\big)x+\frac{r\pi}{2}\big]}{2\sin(x/2)}\right| dx = \lambda_n + \mu_n\,.$$

Since φ_ν are bounded, we have:

$$\int_{\pi/(n+1)}^{\pi}\left|\sum_{j=0}^{n}\alpha_j\frac{\big(j+\frac{1}{2}\big)^{\nu}\,\sin\big[\big(j+\frac{1}{2}\big)x+\frac{\nu\pi}{2}\big]}{\big(\sin(x/2)\big)^{r+1-\nu}}\,\varphi_\nu\right| dx \leqslant K\,U_n\,,$$

where U_n is the integral as in the Lemma 4, and K is a positive constant.

Applying the Lemma 4, to the last integral, we obtain:

$$\int_{\pi/(n+1)}^{\pi} \left| \sum_{j=0}^{n} \alpha_j \frac{\left(j+\frac{1}{2}\right)^{\nu} \sin\left[\left(j+\frac{1}{2}\right)x + \frac{\nu\pi}{2}\right]}{\left(\sin(x/2)\right)^{r+1-\nu}} \varphi_\nu(x) \right| dx$$
$$= O\left((n+1)^{r+\frac{1}{2}-\nu} \left(\sum_{j=0}^{n} \alpha_j^2 (j+1)^{2\nu} \right)^{1/2} \right)$$
$$= O\left((n+1)^{r+\frac{1}{2}-\nu} (n+1)^{\nu+\frac{1}{2}} \right) = O\left((n+1)^{r+1} \right) .$$

Since r is finite value, we have $\lambda_n = O\left((n+1)^{r+1}\right)$. Similarly, we can get $\mu_n = O\left((n+1)^{r+1}\right)$. Hence $J_n = O\left((n+1)^{r+1}\right)$. Finally the our inequality is satisfied, where M is a positive constant.

3.2. Second proof of the Theorem 2.

We note that $\sum_{k=0}^{n} \alpha_k D_k(x)$ is a cosine trigonometric polynomial of order n.

Applying first Bernstein's inequality, then Sidon-Fomin's inequality yields:
$\left\| \sum_{k=0}^{n} \alpha_k D_k^{(r)}(x) \right\| \leqslant (n+1)^r \left\| \sum_{k=0}^{n} \alpha_k D_k(x) \right\| \leqslant M(n+1)^{r+1}$, where $M > 0$.

3.3. Proof of the Theorem 3.

We have:

$$\sum_{k=1}^{n} |\Delta(k^r a_k)| \leqslant \sum_{k=1}^{n} |(k+1)^{r+1} a_k - k^r a_{k+1}| + \sum_{k=1}^{n} |k^r a_{k+1} - k^r a_k|$$
$$= \sum_{k=1}^{n} |\Delta(k^r) a_{k+1}| + \sum_{k=1}^{n} k^r |\Delta a_k|$$
$$= O\left(\sum_{k=1}^{n} k^{r-1} |a_{k+1}| \right) + O\left(\sum_{k=1}^{n} k^r A_k \right) .$$

Applying Abel's transformation, we have:

$$\sum_{k=1}^{n} k^{r-1} |a_{k+1}| = \sum_{k=1}^{n-1} \Delta |a_{k+1}| \sum_{j=1}^{k} j^{r-1} + |a_{n+1}| \sum_{j=1}^{n} j^{r-1}$$
$$\leqslant \sum_{k=1}^{n-1} |\Delta a_{k+1}| k^r + |a_{n+1}| n^r$$

$$\leqslant \sum_{k=1}^{n-1} k^r A_{k+1} + \sum_{k=n+1}^{\infty} k^r |\Delta a_k|$$
$$\leqslant \sum_{k=1}^{n-1} k^r A_k + \sum_{k=n+1}^{\infty} k^r A_k .$$

Letting $n \to \infty$, we get $\sum_{k=1}^{\infty} |\Delta(k^r a_k)| < \infty$, i.e. $\lim_{n\to\infty} S_n^{(r)}(x) = f^{(r)}(x)$. It is well-known that

$$f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x) . \tag{2}$$

From inequality $|D_n^{(r)}(x)| = O\left(\frac{n^r}{x}\right)$ (see [4]), we have that series $\sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$ is uniformly convergent on any compact subset of $(0, \pi)$. Thus the representation (2) implies that $f^{(r)}(x) = \sum_{k=0}^{\infty} \Delta a_k D_k^{(r)}(x)$.

From Theorem 2 and from the formulae (1), we obtain:

$$A_N \int_0^{\pi} \left| \sum_{j=0}^{N} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx = O\left((N+1)^{r+1} A_N\right) = o(1) \quad N \to \infty . \tag{3}$$

Again applying the Abel's transformation, (3) and Theorem 2, we get:

$$\int_0^{\pi} |f^{(r)}(x)|\, dx \leqslant \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k) \int_0^{\pi} \left| \sum_{j=0}^{k} \frac{\Delta a_j}{A_j} D_j^{(r)}(x) \right| dx$$
$$= O(1) \lim_{N\to\infty} \sum_{k=0}^{N-1} (\Delta A_k)(k+1)^{r+1}$$
$$= O(1) \lim_{N\to\infty} \left\{ \sum_{k=0}^{N} \left[(k+1)^{r+1} - k^{r+1}\right] A_k - (N+1)^{r+1} A_n \right\}$$
$$= O(1) \left(\sum_{k=0}^{\infty} k^r A_k \right) ,$$

since $(N+1)^{r+1} A_N = o(1)$, $N \to \infty$.

3.4. Proof of the Theorem 4.

It is obvious that $S_r \subset S \subset BV$.

Thus it suffices to show that

$$\int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx = o(1), \quad n \to \infty .$$

Applying the Abel's trandformation, we have:

$$\int_0^\pi \left|\sum_{k=n}^\infty \Delta a_k D_k^{(r)}(x)\right| dx \leqslant \sum_{k=n}^\infty (\Delta A_k) \int_0^\pi \left|\sum_{j=0}^k \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx + A_n \int_0^\pi \left|\sum_{j=0}^{n-1} \frac{\Delta a_j}{A_j} D_j^{(r)}(x)\right| dx .$$

Since $\left|\dfrac{\Delta a_j}{A_j}\right| \leqslant 1$, applying the Theorem 2 and (1), we get:

$$\int_0^\pi \left|\sum_{k=n}^\infty \Delta a_n D_k^{(r)}(x)\right| dx \leqslant O(1)\left[\lim_{N\to\infty} \sum_{k=n}^{N-1} (\Delta A_k)(k+1)^{r+1} + A_n n^{r+1}\right]$$

$$= O(1) \lim_{N\to\infty} \left\{\sum_{k=n}^{N} \left[(k+1)^{r+1} - k^{r+1}\right] A_k - (N+1)^{r+1} A_N\right\} + O\left(n^{r+1} A_n\right)$$

$$= O\left(\sum_{k=n}^\infty k^r A_k\right) + o(1) = o(1), \quad n \to \infty .$$

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