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ON THE STABILITY OF FUNCTIONAL EQUATIONS WITH SQUARE–SYMMETRIC OPERATION

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Abstract. In this paper, we obtain the modified Hyers-Ulam-Rassias stability for the family of functional equations $f(x \circ y) = H(f(x), f(y))$ $(x, y \in S)$, where H is a homogeneous function and $\circ$ is a square-symmetric operation on the set S. As a consequence we obtain the Hyers-Ulam stability of its functional equation.

1. Introduction

In 1940, S. M. Ulam [12] raised the following problem: Under what condition does there exist an additive mapping near an approximately additive mapping?

In 1941, this problem was solved by D. H. Hyers [3]. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ has the Hyers- Ulam stability if for an approximate solution f of this equation, i.e. for a function f with $|E_1(f) - E_2(f)| \leqslant \delta$ there exists a function g such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \leqslant \epsilon$. In 1978, the Hyers-Ulam stability for approximate linear mapping was generalized by Th. M. Rassias [8]. Thereafter P. Găvruta [2] generalized the stability of Rassias for the case of the bounded function as follows: If for an approximate solution f of the equation $E_1(h) = E_2(h)$, i.e. for a function f such that $|E_1(f) - E_2(f)| \leqslant \phi$ holds with a given function ϕ there exists a function g such that $E_1(g) = E_2(g)$ and $|g(x) - f(x)| \leqslant \Phi(x)$ for some fixed function Φ. It is called the modified Hyers-Ulam-Rassias stability(or stability in the spirit of P. Găvruta). Namely the result of Rassias is the case of a special type of ϕ in this stability. One is referred to [1], [4], [5], [6], [7], [8], [9], [10], [11] for further generalizations and new open problems.

The aim of the present paper is to investigate the modified Hyers-Ulam-Rassias stability for the following family of functional equations:

$$f(x \circ y) = H(f(x), f(y)) \qquad (x, y \in S), \tag{1.1}$$

where S is a nonempty set, $\circ : S \times S \to S$ is a binary operation and $H : G \times G \to G$ is a G-homogeneous function of two variables, that is, H satisfies

$$H(uv, uw) = uH(v, w) \qquad (u, v, w \in G), \tag{1.2}$$

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and G is a multiplicative subsemigroup of the real or complex field. A particular case of (1.1) is the Cauchy functional equation

$$f(x+y)=f(x)+f(y), \qquad (x,y\in S)$$

where S is a semigroup with the operation $+$ and $f: S\to\mathbb{C}$.

If the operation $\circ$ satisfies the following identity:

$$(x\circ y)\circ(x\circ y)=(x\circ x)\circ(y\circ y) \qquad (x,y\in S),$$

the operation $\circ$ will be called *square symmetric*.

In the proof of this paper we use the ideas and methods that are analogous to the ones used in [7], and the following lemmata are the results of Zsolt Páles, Peter Volkmann and Duncan Luce [7]. Let S be a nonempty set and $\circ: S\times S\to S$ be an arbitrary operation.

LEMMA 1. [7, corollary 1]. *Let G be a multiplicative subsemigroup of $\mathbb{C}$, let $H: G\times G\to G$ satisfy (1.2), and let $\phi: S\to G$ be an arbitrary bijective function. Then the binary operation $\circ: S\times S\to S$ defined by*

$$x\circ y:=\phi^{-1}\Big(H(\phi(x),\phi(y)\Big) \qquad (x,y\in S)$$

is square symmetric.

LEMMA 2. [7, lemma 1]. *Let $\circ$ be a square-symmetric operation on S. Define, for $x\in S$, the sequence $x[2^n]$ $(n=0,1,2,\cdots)$ by*

$$x[1]=x[2^0]:=x, \qquad x[2^{n+1}]:=x[2^n]\circ x[2^n], \qquad n\in\mathbb{N}.$$

Then, for each $n\in\mathbb{N}$, the mapping $x\mapsto x[2^n]$ is an endomorphism of $(S,\circ)$, that is

$$(x\circ y)[2^n]=x[2^n]\circ y[2^n] \qquad \textit{for } x,y\in S.$$

2. The modified Hyers-Ulam-Rassias stability of (1.1)

In this section we shall investigate the modified Hyers-Ulam-Rassias stability for the functional equation (1.1).

In each theorem of this paper, φ and $\Phi_i (i=1,2,3,4)$ are mappings from $S\times S$ into G, and each mapping Φ_i will be used in each theorem, respectively.

By using an idea in P. Găvruta [2] we can obtain the following results:

THEOREM 1. *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1\in G$ and $H: G\times G\to G$ be a continuous G-homogeneous function such that $|H(1,1)|\neq 0$ and $\dfrac{1}{H(1,1)}\in G$. Assume that a function $g: S\to G$ satisfies the inequality*

$$|g(x\circ y)-H(g(x),g(y))|\leqslant\varphi(x,y) \qquad (x,y\in S). \tag{2.1}$$

Then there exists a unique function $f : S \to G$ such that f is a solution of (1.1) *and*

$$|f(x) - g(x)| \leqslant \Phi_1(x,x) \qquad (x \in S), \tag{2.2}$$

where $\Phi_1(x,y) = \sum_{n=1}^{\infty} \frac{\varphi(x[2^{n-1}],y[2^{n-1}])}{|H(1,1)|^n} < \infty$ *for all* $x,y \in S$.

Proof. Substituting $x = y$ into (2.1) and using the G-homogeneity of H, we get

$$|g(x \circ x) - g(x)H(1,1)| \leqslant \varphi(x,x) \qquad (x \in S). \tag{2.3}$$

Let $x \in S$ be fixed, and replace x by $x[2^{n-1}]$ in (2.3).

Then we obtain

$$\left|\frac{g(x[2^n])}{H(1,1)^n} - \frac{g(x[2^{n-1}])}{H(1,1)^{n-1}}\right| \leqslant \frac{\varphi(x[2^{n-1}],x[2^{n-1}])}{|H(1,1)|^n} \tag{2.4}$$

for all $x \in S$ and $n \in \mathbb{N}$. Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := \frac{g(x[2^n])}{H(1,1)^n} \qquad (x \in S).$$

Then $g_n : S \to G$ (since $\frac{1}{H(1,1)} \in G$) and, due to (2.4), we have

$$\begin{aligned} |g_n(x) - g_m(x)| &\leqslant \sum_{j=m+1}^{n} |g_j(x) - g_{j-1}(x)| \\ &\leqslant \sum_{j=m+1}^{n} \frac{\varphi(x[2^{j-1}],x[2^{j-1}])}{|H(1,1)|^j} \end{aligned}$$

for $n > m > 0$. Taking the limit $m \to \infty$ in the last inequality, the sequence $g_n(x)$ is a Cauchy sequence for all fixed $x \in S$ from the definition of Φ_1. Since the set G is closed, we can define a mapping $f : S \to G$ by

$$f(x) := \lim_{n \to \infty} g_n(x) \qquad (x \in S).$$

It follows from (2.4) that

$$\begin{aligned} |g_n(x) - g_0(x)| &\leqslant \sum_{j=1}^{n} \frac{\varphi(x[2^{j-1}],x[2^{j-1}])}{|H(1,1)|^j} \\ &\leqslant \sum_{j=1}^{\infty} \frac{\varphi(x[2^{j-1}],x[2^{j-1}])}{|H(1,1)|^j}. \end{aligned}$$

Taking the limit $n \to \infty$, we obtain (2.2).

To see that f satisfies (1.1), let $x,y \in S$ and replace x,y by $x[2^n]$, $y[2^n]$ in (2.1). Using Lemma 2, we get

$$|g((x \circ y)[2^n]) - H(g(x[2^n]),g(y[2^n]))| \leqslant \varphi(x[2^n],y[2^n]).$$

Hence, by the G-homogeneity of H,

$$|g_n(x \circ y) - H(g_n(x), g_n(y))| \leqslant \frac{\varphi(x[2^n], y[2^n])}{|H(1,1)|^n}$$

for all $x, y \in S$ and $n \in \mathbb{N}$. Taking the limit $n \to \infty$, by the continuity of H and the definition of Φ_1 it follows that

$$|f(x \circ y) - H(f(x), f(y))| = 0 \qquad (x, y \in S).$$

Thus (1.1) holds.

Assume that $f^* : S \to G$ is another function which satisfies (1.1) and (2.2). It follows from (2.2) that

$$\begin{aligned} |f^*(x) - f(x)| &= \frac{1}{|H(1,1)|^n} |f^*(x[2^n]) - f(x[2^n])| \\ &\leqslant \frac{1}{|H(1,1)|^n} \left(|f^*(x[2^n]) - g(x[2^n])| + |g(x[2^n]) - f(x[2^n])|\right) \\ &\leqslant \frac{2}{|H(1,1)|^n} \Phi_1(x[2^n], x[2^n]) \end{aligned}$$

for all $x \in S$ and $n \in \mathbb{N}$. By letting $n \to \infty$ in the preceeding inequality, we immediately see the uniqueness of f from the definition of Φ_1, and the theorem is proved. □

COROLLARY 1. [7, theorem 2]. *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| > 1$ and $\frac{1}{H(1,1)} \in G$. Assume that, for some $\varepsilon \geqslant 0$, a function $g : S \to G$ satisfies the inequality*

$$|g(x \circ y) - H(g(x), g(y))| \leqslant \varepsilon \qquad (x, y \in S). \tag{2.5}$$

Then there exists a unique function $f : S \to G$ such that f is a solution of (1.1) *and*

$$|g(x) - f(x)| \leqslant \frac{\varepsilon}{|H(1,1)| - 1} \qquad (x \in S).$$

Proof. Apply Theorem 1 with $\varphi(x, y) = \varepsilon$. □

We say that the operation $\circ$ has the *divisibility property* if, for each $x \in S$, there exists a unique element $y \in S$ such that $y \circ y = x$. In this case the equation $y[2^n] = x$ has a unique solution y for each fixed $x \in S$ and $n \in \mathbb{N}$. Denote this unique element y by $x[2^{-n}]$. Clearly, the mapping $x \to x[2^{-n}]$ is also an endomorphism of $(S, \circ)$.

THEOREM 2. *Let S be a nonempty set and $\circ$ be a square-symmetric operation with the divisibility property on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function. Assume* *that a function* $g : S \to G$ *satisfies the inequality* (2.1). *Then there exists a function* $f : S \to G$ *such that* f *is a solution of* (1.1) *and*

$$|f(x) - g(x)| \leqslant \Phi_2(x,x) \qquad (x \in S), \tag{2.6}$$

where $\Phi_2(x,y) = \sum_{n=1}^{\infty} \varphi(x[2^{-n}], y[2^{-n}])|H(1,1)|^{n-1} < \infty$ *for all* $x, y \in S$.

Proof. The proof of this theorem is analogous to that of Theorem 1.

Replacing x and y by $x[2^{-n}]$ in (2.1) and using the G-homogeneity of H, we get

$$|g(x[2^{1-n}]) - g(x[2^{-n}])H(1,1)| \leqslant \varphi(x[2^{-n}], x[2^{-n}]) \qquad (x \in S,\ n \in \mathbb{N}).$$

Thus

$$\begin{aligned} |g(x[2^{1-n}])H(1,1)^{n-1} &- g(x[2^{-n}])H(1,1)^n| \\ &\leqslant \varphi(x[2^{-n}], x[2^{-n}])|H(1,1)|^{n-1} \end{aligned} \tag{2.7}$$

for $x \in S$, $n \in \mathbb{N}$. Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := g(x[2^{-n}])H(1,1)^n \qquad (x \in S).$$

Then $g_n : S \to G$ and, by (2.7), as in the proof of Theorem 1, we can deduce that the sequence $g_n(x)$ is a Cauchy sequence for all fixed $x \in S$ from the definition of Φ_2. Define f as the pointwise limit function of the sequence g_n. It follows from (2.7) that

$$\begin{aligned} |g_n(x) - g_0(x)| &\leqslant \sum_{j=1}^{n} \varphi(x[2^{-j}], x[2^{-j}])|H(1,1)|^{j-1} \\ &\leqslant \sum_{j=1}^{\infty} \varphi(x[2^{-j}], x[2^{-j}])|H(1,1)|^{j-1}. \end{aligned}$$

Taking the limit $n \to \infty$, we obtain (2.6).

To see that f satisfies (1.1), let $x, y \in S$, replace x, y by $x[2^{-n}]$, $y[2^{-n}]$ in (2.1). Then we get

$$|g((x \circ y)[2^{-n}]) - H(g(x[2^{-n}]), g(y[2^{-n}]))| \leqslant \varphi(x[2^{-n}], y[2^{-n}]).$$

Hence, by the G-homogeneity of H and an endomorphism of the above mapping $x \to x[2^{-n}]$,

$$|g_n(x \circ y) - H(g_n(x), g_n(y))| \leqslant \varphi(x[2^{-n}], x[2^{-n}])|H(1,1)|^n$$

for all $x, y \in S$, $n \in \mathbb{N}$. Taking the limit $n \to \infty$, by using the continuity of H and the definition of Φ_2, it follows that

$$|f(x \circ y) - H(f(x), f(y))| = 0 \qquad (x, y \in S).$$

Therefore (1.1) holds and the uniquness can be proved in a similar manner of Theorem 1, Hence the theorem is proved. □

COROLLARY 2. [7, theorem 3]. *Let S be a nonempty set and $\circ$ be a square-symmetric operation with the divisibility property on S. In addition, let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| < 1$. Assume that, for some $\varepsilon \geqslant 0$, a function $g : S \to G$ satisfies the inequality* (2.5). *Then there exists a unique function $f : S \to G$ such that f is a solution of* (1.1) *and*

$$|f(x) - g(x)| \leqslant \frac{\varepsilon}{1 - |H(1,1)|} \qquad (x \in S).$$

Proof. Apply Theorem 2 with $\varphi(x,y) = \varepsilon$. □

COROLLARY 3. *Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| \neq 0$ and $\dfrac{1}{H(1,1)} \in G$. Assume that a function $g : G \to G$ satisfies the inequality*

$$|g(H(x,y)) - H(g(x), g(y))| \leqslant \varphi(x,y) \qquad (x, y \in G).$$

Then there exists a unique function $f : G \to G$ such that f is a solution of

$$f(H(x,y)) = H(f(x), f(y)) \qquad (x, y \in G). \tag{2.8}$$

and

$$|g(x) - f(x)| \leqslant \begin{cases} \Phi_1(x,x) \\ \Phi_2(x,x) \end{cases} \qquad (x \in G).$$

Proof. In order to apply Theorem 1, let $S := G$ and $x \circ y := H(x,y)$. By Lemma 1, $\circ$ is a square-symmetric operation. In the case $|H(1,1)| \neq 0$, the statement of Theorem 1 is also equivalent to that of this corollary.

For application of the Theorem 2, it suffices to show that $\circ$ satisfies the divisibility assumption. Let $x \in G$. Then the equation $y \circ y = x$ is equivalent to $yH(1,1) = x$. Since the element $\frac{1}{H(1,1)}$ is in G, we have

$$y = \frac{x}{H(1,1)} \in G.$$

Thus, in this case, Theorem 2 can be applied. □

COROLLARY 4. [7, Corollary 2]. *Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| \neq 0, 1$ and $\dfrac{1}{H(1,1)} \in G$. Assume that, for some $\varepsilon \geqslant 0$, a function $g : G \to G$ satisfies the inequality*

$$|g(H(x,y)) - H(g(x), g(y))| \leqslant \varepsilon \qquad (x, y \in G).$$

Then there exists a function $f : G \to G$ such that f is a solution of the equation (2.8) *and*

$$|f(x) - g(x)| \leqslant \frac{\varepsilon}{|1 - |H(1,1)||} \qquad (x \in G).$$

Proof. Apply Corollary 3 with $\varphi(x,y) = \varepsilon$. By separating $|H(1,1)| > 1$ and $|H(1,1)| < 1$ for $\Phi_1(x,y)$ and $\Phi_2(x,y)$, the proof is completed. □

In the next results we obtain that the below equation (2.9) is stable in Găvruta's sense if $|a+b| \neq 0$. In the case when $X = \mathbb{C}$ or $X = \mathbb{R}$ these results are also corollaries of Theorems 1 and 2 if one takes the homogeneous function $H(x,y) = ax + by$.

THEOREM 3. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let $a,b \in \mathbb{K}$, such that $|a+b| \neq 0$. Assume that a function $g : S \to X$ satisfies the inequality*

$$|g(x \circ y) - ag(x) - bg(y)| \leqslant \varphi(x,y) \qquad (x,y \in S). \tag{2.9}$$

Then there exists a unique function $f : S \to X$ such that f is a solution of

$$f(x \circ y) = af(x) + bf(y) \qquad (x,y \in S) \tag{2.10}$$

and

$$|f(x) - g(x)| \leqslant \Phi_3(x,x) \qquad (x \in S), \tag{2.11}$$

where $\Phi_3(x,y) = \sum_{n=1}^{\infty} \frac{\varphi(x[2^{n-1}],y[2^{n-1}])}{|a+b|^n} < \infty$ for all $x,y \in S$.

Proof. The proof of this theorem is analogous to that of Theorem 1. Substituting $x = y$ into (2.9),

$$|g(x \circ x) - (a+b)g(x)| \leqslant \varphi(x,x) \qquad (x \in S). \tag{2.12}$$

Let $x \in S$ be fixed, and replace x by $x[2^{n-1}]$ in (2.12). Then, for $x \in S$ and $n \in \mathbb{N}$,

$$\left| \frac{g(x[2^n])}{(a+b)^n} - \frac{g(x[2^{n-1}])}{(a+b)^{n-1}} \right| \leqslant \frac{\varphi(x[2^{n-1}],x[2^{n-1}])}{|a+b|^n}. \tag{2.13}$$

Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := \frac{g(x[2^n])}{(a+b)^n} \qquad (x \in S).$$

Then, for the inequality (2.13), by arguing in the same way as in the proof of Theorem 1, we can see that the sequence $g_n(x)$ is a Cauchy sequence for all fixed $x \in S$ from the definition of Φ_3. Defining $f : S \to X$ by

$$f(x) := \lim_{n\to\infty} g_n(x) \qquad (x \in S),$$

and making analogous steps, we can see that f satisfies (2.10) and the estimate (2.11). □

COROLLARY 5. [7, theorem 4]. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let $a,b \in \mathbb{K}$, such that $|a+b| > 1$. Assume that, for some $\varepsilon \geqslant 0$, a function $g : S \to X$ satisfies the inequality*

$$|g(x \circ y) - ag(x) - bg(y)| \leqslant \varepsilon \qquad (x,y \in S). \tag{2.14}$$

Then there exists a unique function $f : S \to X$ such that f is a solution of (2.10) *and*

$$|f(x) - g(x)| \leqslant \frac{\varepsilon}{|a+b|-1} \qquad (x \in S).$$

THEOREM 4. *Let X be a Banach space over $\mathbb{K}$, S be a nonempty set and $\circ$ be a square-symmetric operation on S with the divisibility property. Assume that a function $g : S \to X$ satisfies the inequality* (2.9).

Then there exists a function $f : S \to X$ such that f is a solution of (2.10) *and*

$$|g(x) - f(x)| \leqslant \Phi_4(x,x) \qquad (x \in S),$$

where $\Phi_4(x,y) = \sum_{n=1}^{\infty} \varphi(x[2^{-n}], y[2^{-n}])|a+b|^{n-1} < \infty$ for all $x,y \in S$.

Proof. The proof of this theorem is analogous to that of Theorem 2. Define $x[2^{-n}]$ exactly as it was defined therein. Replacing x and y by $x[2^{-n}]$ in (2.9), we get

$$|g(x[2^{1-n}]) - (a+b)g(x[2^{-n}])| \leqslant \varphi(x[2^{-n}], x[2^{-n}]) \qquad (x \in S,\ n \in \mathbb{N}).$$

Thus, for $x \in S$ and $n \in \mathbb{N}$,

$$\begin{aligned} &|(a+b)^{n-1} g(x[2^{1-n}]) - (a+b)^n g(x[2^{-n}])| \\ &\qquad \leqslant \varphi(x[2^{-n}], x[2^{-n}])|a+b|^{n-1}. \end{aligned} \tag{2.15}$$

Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := (a+b)^n g(x[2^{-n}]) \qquad (x \in S).$$

Now, for the inequality (2.15), using the definition of Φ_4, an analogous argument such as Theorem 2 shows that f satisfies the desired conditions of this theorem. □

COROLLARY 6. [7, theorem 5]. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S with the divisibility property. Let $a,b \in \mathbb{K}$, such that $|a+b| < 1$. Assume that, for some $\varepsilon \geqslant 0$, a function $g : S \to X$ satisfies the inequality* (2.14).

Then there exists a function $f : S \to X$ such that f is a solution of (2.10) *and*

$$|f(x) - g(x)| \leqslant \frac{\varepsilon}{1-|a+b|} \qquad (x \in S).$$

Proof. Apply Theorem 4 with $\varphi(x,y) = \varepsilon$. □

REMARK. The assumption (stability in the spirit of Găvruta) that each infinite series Φ_i $(i = 1, 2, 3, 4)$ converges can be weakly changed. In Theorem 1, the Găvruta's assumption can replace $\limsup \sqrt[n]{\varphi(x[2^n], x[2^n])} \leqslant \gamma < \infty$ and $|H(1,1)| > \gamma$. This is because the term of series Φ_1 can be represented by the forms

$$P_n^n = \left(\frac{\limsup \sqrt[n]{\varphi(x[2^{n-1}], x[2^{n-1}])}}{|H(1,1)|} \right)^n,$$

where P_n is less than 1. Thus the series Φ_1 converges by using the Cauchy-criterion (or Test of infity series). Of course this idea could be extended to all the well-known results of this form. The results of Theorem 2,3 and 4 also can be applied to similiar corollaries without the assumption that each infinite series Φ_i converges. Corollary 1 could be considerd as an example of the particular case: $\limsup \sqrt[n]{\varphi(x[2^n], x[2^n])} = \varepsilon$.

In view of the above remark, we can obtain the similar results of the following:

COROLLARY 7. *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H : G \times G \to G$ be a continuous G-homogeneous function such that $\frac{1}{H(1,1)} \in G$. Assume that a function $g : S \to G$ satisfies the stability inequality* (2.1) *and* $\limsup \sqrt[n]{\varphi(x[2^n], x[2^n])} \leqslant \gamma < \infty \qquad (x, y \in S)$.

Then the series Φ_1 converges if $|H(1,1)| > \gamma$. There exists a unique function $f : S \to G$ such that f is a solution of (1.1) *and*

$$|f(x) - g(x)| \leqslant \Phi_1(x, x) \qquad (x \in S).$$

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