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INTERSECTION PROPERTIES FOR CONES OF MONOTONE AND CONVEX FUNCTIONS IN SCALE OF LIPSCHITZ SPACES

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Abstract. The basic results of the real interpolation method is known to be valid for couples $(X_0 \cap Q, X_1 \cap Q)$ under the condition that the cone Q has the so-called *intersection property* with respect to the couple (X_0, X_1). In this paper we study this property for the cones of monotone and convex functions with respect to the couple of Lipschitz (Nikol'skii-Besov) spaces.

1. Introduction

Many important problems in harmonic analysis, PDE and approximation theory require interpolation of operators preserving a convex cone structure (positivity, monotonicity, convexity etc). The only robust interpolation method in this case is *the real interpolation method*, since all others are hardly rely on the linear structure of underlying data. It was noted by Y. Sagher ([12], [13]) that the basic results of the real method are valid under the condition named below *the cone intersection property*, see Definition 1.1. In this paper we study this property and its modification for cones of univariate monotone and convex functions with respect to the scale of Lipschitz (Nikol'skii-Besov) spaces. In all previous works on this problem (see references below) the underlying spaces were always taken to be lattices.

Let Y be a linear space over the field of reals. Suppose that $X \subset Y$ is a linear subspace and that $Q \subset Y$ is a cone. A *norm* on $X \cap Q$ is a map $\|\cdot\|$ of $X \cap Q$ to $[0, \infty)$ satisfying the properties usually required for a norm on a linear space, except that the formula $\|\lambda x\| = |\lambda|\|x\|$ is only required to hold for $\lambda \geqslant 0$.

Hereafter we shall use the basic notions of interpolation space theory, see [2] or [4].

DEFINITION 1.1. A cone Q has the **intersection property** (IP) with respect to a Banach couple $\bar{X} = (X_0, X_1)$ if for all $t > 0$

$$(X_0 + tX_1) \cap Q = (X_0 \cap Q) + t(X_1 \cap Q) \tag{1.1}$$

where the norms are equivalent up to constants independent of t.

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Here the norm of $(X_0+tX_1)\cap Q$ is taken to be simply the restriction to Q of the natural norm (K-functional) on X_0+tX_1 and the norm on $(X_0\cap Q)+t(X_1\cap Q)$ is taken to be

$$K(f,t;\bar{X}\cap Q)=\inf\{\|f_0\|_{X_0}+t\|f_1\|_{X_1}: f=f_0+f_1, f_i\in X_i\cap Q\},$$

i.e., it is the K-functional of the couple of cones $\bar{X}\cap Q:=(X_0\cap Q, X_1\cap Q)$.

Hence the intersection property (1.1) is equivalent to the two-sided inequality

$$K(f,t;\bar{X}\cap Q)\approx K(f,t;\bar{X}),\quad (f\in Q, t>0). \tag{1.2}$$

Here $F\approx G$ means that $C_1F\leqslant G\leqslant C_2F$ for some constants $0<C_1,C_2<\infty$ independent of the arguments of F,G. In particular, (1.2) holds uniformly with respect to $t>0$ and $f\in Q$. We also use the notation $F\prec G (G\succ F)$ if $F\leqslant C_1G$ for some constant $C_1>0$ independent of the arguments of F,G.

If Q satisfies the intersection property then it also satisfies:

$$(X_0\cap Q, X_1\cap Q)_{\theta,q}=Q\cap(X_0,X_1)_{\theta,q}\quad (0<\theta<1, 1\leqslant q\leqslant\infty) \tag{1.3}$$

with equivalence of the norms.

Let us recall that the norm of the cone on the left is determined by

$$\|f\|_{(X_0\cap Q,X_1\cap Q)_{\theta q}}:=\left(\int_0^\infty\left(\frac{K(t,f;\bar{X}\cap Q)}{t^\theta}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}.$$

DEFINITION 1.2. The cone Q has the **weak intersection property** (*WIP*) with respect to a Banach couple (or seminormed couple) $\bar{X}=(X_0,X_1)$ if (1.3) holds with equivalence of norms.

As it was mentioned, cones satisfying (1.3) were first introduced and studied by Y. Sagher ([12], [13]). He called them “Marcinkiewicz cones". He also gave some interesting applications of his results to harmonic analysis. The intersection property for the cone of concave nondecreasing nonnegative functions on $\mathbb{R}_+$ was studied by I. Asekritova [1]. She proved that it has the *IP* with respect to a couple of weighted L_∞ spaces where the weights are quasi-concave on $\mathbb{R}_+$. Recently J. Cerdà and J. Martin [5] have obtained a similar result for the cone of non-negative non-increasing functions on $\mathbb{R}_+$ with respect to (L_p,L_q) and also with respect to couples of Lorentz spaces.

To illustrate the role of the intersection property we introduce the following results, a version of Holmstedt's formula, which also be used below. Unfortunately, we cannot apply Holmstedt's proof ([8]) directly to the case of a couple of cones $\bar{X}^Q:=(X_0\cap Q, X_1\cap Q)$ because we have to avoid taking differences of two functions from a cone. Nevertheless, we can use the following version of the reiteration theorem in our setting. In order to formulate the results we set

$$\bar{K}^Q(f,s):=K(f,s,(\bar{X}^Q)_{\theta_0,q_0},(\bar{X}^Q)_{\theta_1,q_1}).$$

PROPOSITION 1. *Suppose that for fixed $\theta_j\in(0,1)$ and $q_j\in[1,\infty]$ the isomorphism*

$$(X_0\cap Q, X_1\cap Q)_{\theta_i,q_i}=(X_0,X_1)_{\theta_i,q_i}\cap Q \tag{1.4}$$

holds for $i=0,1$. For all $f\in\sum(\bar{X}^Q):=X_0\cap Q+X_1\cap Q$ and $j=0,1$ let

$$P_j^Q(t) = (\int_0^t [s^{-\theta_j} K(f, s, X_0 \cap Q, X_1 \cap Q)]^{q_j} \frac{ds}{s})^{1/q_j}$$

and

$$R_j^Q(t) = (\int_t^\infty [s^{-\theta_j} K(f, s, X_0 \cap Q, X_1 \cap Q)]^{q_j} \frac{ds}{s})^{1/q_j}.$$

Then

$$\bar{K}^Q(f, t^\lambda) \approx P_0^Q(t) + t^\lambda R_1^Q(t). \tag{1.5}$$

Here $\theta_0 < \theta_1$ *and* $\lambda := \theta_1 - \theta_0$.

Proof. We adapt Holmstedt's proof to to our case of a couple of cones. The inequality

$$P_0^Q(t) + t^\lambda R_1^Q(t) \leqslant C\bar{K}^Q(f, t^\lambda)$$

is proved as in [8], pp. 180–182.

To obtain the converse inequality we choose for $f \in \sum(\bar{X}^Q)$ and for $t > 0$ a decomposition $f = g_t + h_t \in X_0 \cap Q + X_1 \cap Q$, $g_t \in X_0 \cap Q$, $h_t \in X_1 \cap Q$ such that

$$\|g_t\|_{X_0} + t\|h_t\|_{X_1} \leqslant 2K(f, t, X_0 \cap Q, X_1 \cap Q). \tag{1.6}$$

With this choice we have

$$\begin{aligned} \bar{K}^Q(f, t^\lambda) &\leqslant \|g_t\|_{(\bar{X}^Q)_{\theta_0, q_0}} + t^\lambda \|h_t\|_{(\bar{X}^Q)_{\theta_1, q_1}} \\ &= (\int_0^\infty [s^{-\theta_0} K(g_t, s, X_0 \cap Q, X_1 \cap Q)]^{q_0} \tfrac{ds}{s})^{1/q_0} + \\ &+ t^\lambda (\int_0^\infty [s^{-\theta_1} K(h_t, s, X_0 \cap Q, X_1 \cap Q)]^{q_1} \tfrac{ds}{s})^{1/q_1}. \end{aligned} \tag{1.7}$$

To estimate the right hand side of (1.7) we have to apply (1.4):

$$\begin{aligned} \bar{K}^Q(f, t^\lambda) &\leqslant C\{(\int_0^\infty [s^{-\theta_0} K(g_t, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} + t^\lambda (\int_0^\infty [s^{-\theta_1} K(h_t, s, X_0, X_1)]^{q_1} \tfrac{ds}{s})^{1/q_1}\} \\ &\leqslant C\{(\int_0^t [s^{-\theta_0} K(g_t, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} + (\int_t^\infty [s^{-\theta_0} K(g_t, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} \\ &+ t^\lambda (\int_0^t [s^{-\theta_1} K(h_t, s, X_0, X_1)]^{q_1} \tfrac{ds}{s})^{1/q_1} + t^\lambda (\int_t^\infty [s^{-\theta_1} K(h_t, s, X_0, X_1)]^{q_1} \tfrac{ds}{s})^{1/q_1}\}. \end{aligned} \tag{1.8}$$

For the first term we obtain, by the triangle inequality,

$$\begin{aligned} (\int_0^t [s^{-\theta_0} K(g_t, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} &\leqslant (\int_0^t [s^{-\theta_0} K(f, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} + \\ + (\int_0^t [s^{-\theta_0} K(h_t, s, X_0, X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0}. \end{aligned} \tag{1.9}$$

Since for every cone Q

$$K(f, s, X_0, X_1) \leqslant K(f, s, X_0 \cap Q, X_1 \cap Q),$$

it follows that

$$(\int_0^t [s^{-\theta_0} K(f,s,X_0,X_1)]^{q_0} \frac{ds}{s})^{1/q_0} \leqslant P_0^Q(t).$$

According to the choice of h_t (see (1.6)) the latter integral in (1.9) is bounded by

$$\begin{aligned}&(\int_0^t [s^{-\theta_0} s\|h_t\|_{X_1}]^{q_0} \tfrac{ds}{s})^{1/q_0} = C_1(\|h_t\|_{X_1} t)t^{-\theta_0}\\ &\leqslant 2C_1 K(f,t,X_0\cap Q, X_1\cap Q)t^{-\theta_0}\\ &\leqslant C_2 \tfrac{K(f,t,X_0\cap Q,X_1\cap Q)}{t} \int_{\frac{t}{2}}^t s^{-\theta_0} ds.\end{aligned} \tag{1.10}$$

Since the function $t \to K(f,t,X_0\cap Q,X_1\cap Q)$ is concave, $\frac{K(f,t,X_0\cap Q,X_1\cap Q)}{t} \leqslant \frac{K(f,s,X_0\cap Q,X_1\cap Q)}{s}$, if $s \leqslant t$. Therefore the right hand side of the above inequality does not exceed

$$\begin{aligned}&C_2 \int_{\frac{t}{2}}^t [s^{-\theta_0} K(f,s,X_0\cap Q,X_1\cap Q) \tfrac{ds}{s} \leqslant C_2 (\int_{\frac{t}{2}}^t [s^{-\theta_0} K(f,s,X_0\cap Q,X_1\cap Q)]^{q_0} \tfrac{ds}{s})^{1/q_0}\\ &\leqslant C_3 (\int_0^t [s^{-\theta_0} K(f,s,X_0\cap Q,X_1\cap Q)]^{q_0} \tfrac{ds}{s})^{1/q_0} = C_3 P_0^Q(t).\end{aligned} \tag{1.11}$$

(The first inequality in (1.11) follows from Hölder's inequality).

The second term of (1.8) is estimated by similar arguments

$$\begin{aligned}&(\int_t^\infty [s^{-\theta_0} K(g_t,s,X_0,X_1)]^{q_0} \tfrac{ds}{s})^{1/q_0} \leqslant (\int_t^\infty [s^{-\theta_0} \|g_t\|_{X_0}]^{q_0} \tfrac{ds}{s})^{1/q_0}\\ &\leqslant 2(\int_t^\infty [s^{-\theta_0} K(f,t,X_0\cap Q,X_1\cap Q)]^{q_0} \tfrac{ds}{s})^{1/q_0} \leqslant CK(f,t,X_0\cap Q,X_1\cap Q)t^{-\theta_0}\\ &\leqslant C(\int_0^t [s^{-\theta_0} K(f,s,X_0\cap Q,X_1\cap Q)]^{q_0} \tfrac{ds}{s})^{1/q_0} = CP_0^Q(t)\end{aligned} \tag{1.12}$$

The remaining two terms of (1.8) are treated analogously. Summing the four estimates we obtain the required inequality

$$\bar{K}^Q(f,t^\lambda) \leqslant C(P_0^Q(t) + t^\lambda R_1^Q(t))$$

completing the proof of the theorem.

The following result is easily proved by an adaptation of the previous proof.

PROPOSITION 2. *Suppose that for fixed $\theta \in (0,1)$ and $q \in [1,\infty]$ the following isomorphism*

$$(X_0\cap Q, X_1\cap Q)_{\theta,q} = (X_0,X_1)_{\theta,q} \cap Q$$

holds. Then

$$K(f,t^\theta,X_0,(\bar{X}^Q)_{\theta,q}) \approx t^\theta (\int_t^\infty [s^{-\theta} K(f,s,X_0\cap Q,X_1\cap Q)]^q \frac{ds}{s})^{1/q}$$

Let us now define the main cones which are studied in this paper.

DEFINITION 1.3. $\mathbf{M}$ is the cone of nonnegative nondecreasing functions defined on $[0,1)$.

DEFINITION 1.4. $\mathbf{C}$ is the cone of nonnegative nondecreasing convex functions defined on $[0,1)$.

We shall let $\dot{H}_p^\alpha$ denote the (seminormed) Lipschitz(Nikol'skii-Besov) space which is defined as follows:

DEFINITION 1.5. For $\alpha > 0$ we write $r = [\alpha] + 1$. The space $\dot{H}_p^\alpha(0,1)$, $1 \leqslant p \leqslant \infty$ consists of all functions $f \in L_p(0,1)$ for which the seminorm

$$|f|_{\dot{H}_p^\alpha} := \sup_{t>0}(t^{-\alpha}\omega_r(f,t)_p)$$

is finite.

Here ω_r denotes the modulus of smoothness of order r, i.e.

$$\omega_r(f,t,[0,1])_p := \omega_r(f,t)_{L_p(0,1)} := \omega_r(f,t)_p := \sup_{0\leqslant h\leqslant t} \|\Delta_h^r f(\cdot)\|_{L_p(I_{rh})}$$

where $\Delta_h^r f(x) := \sum_{i=0}^r (-1)^{r-i}\binom{r}{i} f(x+ih)$ is the r^{th} order difference of f, and $I_{rh} = [0, 1-rh]$, $t \in [0, \frac{1}{r}]$; and $\omega_r(f,t)_p := \omega_r(f,\frac{1}{r})_p$, if $t > \frac{1}{r}$.

In turn, the norm of the (normed) space H_p^α is defined by

$$\|f\|_{H_p^\alpha} := |f|_{\dot{H}_p^\alpha} + \|f\|_{L_p}.$$

In section 2, we prove the following result.

THEOREM 1.

(i) If $0 < \alpha < \frac{1}{p}$ *and* $1 \leqslant p < \infty$ *then* $\mathbf{M}$ *has the IP with respect to the couple* (L_p, H_p^α).

(ii) If $\alpha \geqslant \frac{1}{p}$, $1 \leqslant p < \infty$ *or if* $\alpha > 0$, $p = \infty$ *then* $\mathbf{M}$ *does not have neither the WIP nor the IP with respect to the couple* $(L_p, \dot{H}_p^\alpha)$.

In section 3 we study the *IP* for the cone of convex functions $\mathbf{C}$ and prove the following theorem.

THEOREM 2.

(i) The cone $\mathbf{C} := \mathbf{C}$ *has the IP with respect to* (L_p, H_p^α), $p \in [1,\infty]$, *if* $0 < \alpha < 1$.

(ii) $\mathbf{C}$ *does not have the WIP with respect to* $(L_p, \dot{H}_p^\alpha)$, *if* $\alpha > 1$.

Note that the more restrictive assumptions on α in the first theorem related to the fact that smoothness of functions in $\mathbf{M}$ is, roughly speaking, 1 while that of functions in $\mathbf{C}$ equals 2.

Other results about the *IP* with respect to these couples and couples of L_p-spaces and the application relates to reverse embeddings for Lipschitz(Nikol'skii-Besov) spaces are obtained in [9].

2. Proof of Theorem 1

Recall first the following result (see [3]):

For each $f \in L_p[0,1]$ *and* $p \in [1,\infty]$

$$K(f,t,L_p,\dot{W}_p^r) \approx \omega_r(f,t^{1/r},[0,1])_p, \tag{2.13}$$

where the constants of equivalence depend only on r.

To prove (i) apply the reverse embedding theorem of Herz [7]

$$L_{q,\infty} \cap \mathbf{M} \subset H_p^\alpha$$

where $0 < \alpha < \frac{1}{p}$, $\alpha = 1/p - 1/q$, $p < q < \infty$. Then we have for each $f \in H_p^\alpha \cap \mathbf{M}$:

$$K(f,t,L_p \cap \mathbf{M}, H_p^\alpha \cap \mathbf{M}) \leqslant C_1 K(f,t,L_p \cap \mathbf{M}, L_{q,\infty} \cap \mathbf{M}).$$

Now we estimate the right hand side using the fact that the cone of monotone functions has the IP with respect to couples of Lorentz spaces ([5], [6]), that is,

$$K(f,t,L_p \cap \mathbf{M}, L_{q,\infty} \cap \mathbf{M}) \leqslant C_2 K(f,t,L_p,L_{q,\infty}).$$

Combining these two inequalities with the Peetre embedding theorem ([11]):

$$H_p^\alpha \subset L_{q,\infty}$$

with above mentioned α, p, q we, at least, obtain

$$K(f,t,L_p \cap \mathbf{M}, H_p^\alpha \cap \mathbf{M}) \leqslant C_3 K(f,t,L_p,H_p^\alpha)$$

for each $f \in H_p^\alpha \cap \mathbf{M}$. This proves the IP for $\alpha < 1/p$ and $1 \leqslant p < \infty$.

(ii) Let us,first, show that the cone $\mathbf{M}$ does not have any intersection property with respect to $(L_p, \dot{H}_p^\alpha)$, when $\alpha \geqslant 1/p$ and $1 \leqslant p < \infty$.

We begin with the case $\alpha = 1/p$. Assume, on the contrary, that

$$\begin{aligned} &[\int_0^\infty [t^{-\theta} K(f,t,L_p \cap \mathbf{M}, \dot{H}_p^{1/p} \cap \mathbf{M})]^q \tfrac{dt}{t}]^{\frac{1}{q}} \\ &\leqslant C[\int_0^\infty [t^{-\theta} K(f,t,L_p,\dot{H}_p^{1/p})]^q \tfrac{dt}{t}]^{\frac{1}{q}} \end{aligned} \tag{2.14}$$

for some positive C.

To obtain a contradiction we require the formula

$$\begin{aligned} K(f,t,L_p,\dot{H}_p^{n+\alpha}) &\approx t \sup_{s \geqslant t^{\frac{n+1}{n+\alpha}}} \frac{\omega_{n+1}(f,s^{1/(n+1)})_p}{s^{(n+\alpha)/(n+1)}} \\ &= t \sup_{s \geqslant t} \frac{\omega_{n+1}(f,s^{1/(n+\alpha)})_p}{s} \end{aligned} \tag{2.15}$$

for $0 \leqslant \alpha < 1$, $n \in \mathbb{N} \cup \{0\}$ and $n + \alpha > 0$. This result is a direct consequence of (2.13), Holmstedt's theorem ([2], p.53, Corollary 3.6.2) and following from them the isomorphism $\dot{H}_p^{n+\alpha} = (L_p, \dot{W}_p^{n+1})_{\frac{n+\alpha}{n+1},\infty}$.

Take the function

$$f(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \epsilon; \\ 1 & \text{if } \epsilon < x \leqslant 1. \end{cases} \tag{2.16}$$

Let us consider first the case $p > 1$. Since

$$\omega_1(f, s^p)_p = \begin{cases} s & \text{if } s < \epsilon^{1/p}; \\ \epsilon^{1/p} & \text{if } s \geqslant \epsilon^{1/p}, \end{cases} \tag{2.17}$$

using the formula (2.15) with $n = 0$ implies for $1 < p < \infty$:

$$K(f, t, L_p, \dot{H}_p^{1/p}) \prec \begin{cases} t & \text{if } t \leqslant \epsilon^{1/p}; \\ \epsilon^{1/p} & \text{if } t > \epsilon^{1/p} \end{cases}.$$

Hence

$$\begin{aligned} \int_0^\infty [t^{-\theta} K(f, t, L_p, \dot{H}_p^{1/p})]^q \tfrac{dt}{t} &\prec \int_0^{\epsilon^{1/p}} [t^{-\theta} t]^q \tfrac{dt}{t} + \int_{\epsilon^{1/p}}^\infty [t^{-\theta} \epsilon^{1/p}]^q \tfrac{dt}{t} \\ &= \frac{1}{\theta(1-\theta)q} \epsilon^{\frac{q}{p}(1-\theta)}. \end{aligned} \tag{2.18}$$

Now let us consider the K-functional with constraints:

$$\begin{aligned} &K(f, t, L_p \cap \mathbf{M}, \dot{H}_p^{1/p} \cap \mathbf{M}) \\ &= \inf\{\|f - g\|_{L_p} + t|g|_{H_p^{1/p}} \mid g \in \dot{H}_p^{1/p} \cap \mathbf{M}, f - g \in \mathbf{M}\}. \end{aligned} \tag{2.19}$$

It is clear, that g in (2.19) has to be 0 for $0 \leqslant x \leqslant \epsilon$ and a constant, say $0 \leqslant b \leqslant 1$ for $\epsilon < x \leqslant 1$:

$$g(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \epsilon; \\ b & \text{if } \epsilon < x \leqslant 1, \end{cases} \tag{2.20}$$

where $0 \leqslant b \leqslant 1$.

In this case

$$|g|_{H_p^{1/p}} = b$$

and

$$\|f - g\|_{L_p} = (1-b)(1-\epsilon)^{1/p}.$$

So we get

$$\begin{aligned} K(f, t, \ L_p \cap \mathbf{M}, \dot{H}_p^{1/p} \cap \mathbf{M}) &= \inf_{0 \leqslant b \leqslant 1} \{(1-b)(1-\epsilon)^{1/p} + tb\} \\ = \min\{(1-\epsilon)^{1/p}, t)\} &= \begin{cases} t & \text{if } t \leqslant (1-\epsilon)^{1/p}; \\ (1-\epsilon)^{1/p} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.21}$$

Hence

$$\begin{aligned} \int_0^\infty [t^{-\theta} K(f, t, \ L_p \cap \mathbf{M}, \dot{H}_p^{1/p} \cap \mathbf{M})]^q \tfrac{dt}{t} &= \int_0^{(1-\epsilon)^{1/p}} [t^{-\theta} t]^q \tfrac{dt}{t} + \\ +(1-\epsilon)^{q/p} \int_{(1-\epsilon)^{1/p}}^\infty t^{-\theta q} \tfrac{dt}{t} &= \tfrac{1}{\theta(1-\theta)q} (1-\epsilon)^{\frac{q}{p}(1-\theta)}. \end{aligned} \tag{2.22}$$

From (2.18) and (2.22) follows that if ϵ is sufficiently small we obtain a contradiction to the assumption (2.14).

The same conclusion can be drawn for $\alpha = \frac{1}{p}$ when $p = 1$. In fact, in this case the function f defined in (2.16) satisfies $\omega_2(f, h)_1 = \min\{2h, \epsilon\}$ and so $|f|_{\dot{H}_1^1} = 2$. Similarly the function g defined in (2.20) satisfies $|g|_{\dot{H}_1^1} = 2b$. Using the decomposition $f = 0 + f$ for $t \leqslant \epsilon$ and $f = -\chi_{[0,\epsilon]} + \chi_{[0,1)}$ for $t > \epsilon$, we obtain that

$$K(f, t, L_1, \dot{H}_1^1) \leqslant 2\min\{t, \epsilon\}.$$

By a similar argument to above we have

$$\begin{aligned} K(f, t, L_1 \cap \mathbf{M}, \quad \dot{H}_1^1 \cap \mathbf{M}) &= \inf_{0 \leqslant b \leqslant 1}\{(1-b)(1-\epsilon) + 2bt\} \\ &\geqslant (1+b)\min\{t, 1-\epsilon\} \geqslant \min\{t, 1-\epsilon\}. \end{aligned} \tag{2.23}$$

Thus the inequalities (2.18) and (2.22) also hold when $p = 1$ and again we obtain a contradiction to (2.14) for sufficiently small ϵ.

We now consider a counter-example for the WIP for $p < \infty$ and $\alpha > 1/p$. Let

$$f(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1/2; \\ 1 & \text{if } 1/2 < x \leqslant 1. \end{cases} \tag{2.24}$$

For $\alpha < 1$ we shall use the fact that

$$\omega_1(f, s)_p = \min(s^{1/p}, 2^{-1/p}), \quad 0 \leqslant s < \infty. \tag{2.25}$$

For $\alpha \geqslant 1$ we shall use the fact that

$$\omega_2(f, s)_1 = \min(2s, \frac{1}{2}), \quad 0 \leqslant s < \infty. \tag{2.26}$$

We have from these two estimates and the inequality $\omega_k(f, s)_p \leqslant 2^{k-j}\omega_j(f, s)_p$, $0 \leqslant j \leqslant k$, that for arbitrary $k \geqslant 2$ and $1 \leqslant p \leqslant \infty$

$$\omega_k(f, s)_p \prec \min(s^{1/p}, 2^{-1/p}), \quad 0 \leqslant s < \infty. \tag{2.27}$$

From here and (2.15) with appropriate n we estimate the K - functional for $1 \leqslant p < \infty$ and all $\alpha > 0$:

$$K(f, t, L_p, \dot{H}_p^\alpha) \prec \min(t^{\frac{1}{p\alpha}}, 2^{-1/p}).$$

Hence for $\theta < \frac{1}{p\alpha}$

$$\int_0^\infty [t^{-\theta} K(f, t, L_p, \dot{H}_p^\alpha)]^q \frac{dt}{t} < \infty. \tag{2.28}$$

From (2.25) and (2.26) it follows that $f \in \dot{H}_p^{\frac{1}{p}}$, and $f \notin \dot{H}_p^\alpha$, if $\frac{1}{p} < \alpha < 1$ for $p > 1$ and $1 < \alpha < 2$ for $p = 1$. But $\dot{H}_p^\alpha \subset \dot{H}_p^\beta$ if $\alpha > \beta$ and therefore $f \notin \dot{H}_p^\alpha$ for every $\alpha > \frac{1}{p}$.

Now estimate below the K-functional with constraints for $\alpha > \frac{1}{p}$

$$\begin{aligned} &K(f,t,L_p \cap \mathbf{M}, \dot{H}_p^\alpha \cap \mathbf{M}) \\ &= \inf\{\|f-g\|_{L_p} + t|g|_{H_p^\alpha} | g \in \dot{H}_p^\alpha \cap \mathbf{M}, f-g \in \mathbf{M}\} \\ &\geqslant \inf\{\|f-g\|_{L_p} \mid g \in \dot{H}_p^\alpha \cap \mathbf{M}, f-g \in \mathbf{M}\}. \end{aligned} \tag{2.29}$$

Every g such that $g \in \mathbf{M}$, and $f-g \in \mathbf{M}$ must have the form $g = \beta\chi_{[0,1/2]} + \gamma\chi_{(1/2,1)}$ with $\beta = 0$. If also $g \in \dot{H}_p^\alpha$, with $\alpha > 1/p$ then by the embedding theorem (see, for instance, [10], Section 6.3) $g \in C[0,1]$ and therefore $\gamma = \beta = 0$. Consequently

$$K(f,t,L_p \cap \mathbf{M}, \dot{H}_p^\alpha \cap \mathbf{M}) \geqslant (\frac{1}{2})^{1/p}.$$

Since for each $\theta \in (0,1)$ and $q \in [1,\infty]$

$$\int_0^\infty (K(f,t,L_p \cap \mathbf{M}, \dot{H}_p^\alpha \cap \mathbf{M})t^{-\theta})^q \frac{dt}{t} = \infty,$$

we deduce, using (2.28) that the WIP does not hold.

It remains to consider a counterexample for $p = \infty$. Let an integer $b > 2$ and C_b be a Cantor type set on $[0,1]$ constructed as follows. In the classical Cantor construction we will remove the middle interval of length $(1-\frac{2}{b})|I|$ from each interval appearing at each step of the construction. (If we choose $b = 3$ then C_b will be the usual Cantor set C_3). So, we have

$$C_b = [0,1] \setminus \bigcup_{k=1}^\infty (c_k, d_k),$$

where the right endpoints of the intervals (c_k, d_k) are numbers of the form

$$d_k = \sum_{n=1}^{N_k} \frac{a_n}{b^n}, \tag{2.30}$$

with a_n assumes only the values $b-1$ and 0 and $a_{N_k} \neq 0$. Now we define a Cantor-Lebesgue type function $u : [0,1] \to [0,1]$. Every point t of C_b can be uniquely represented as $t = \sum\limits_{n=1}^\infty \frac{a_n(t)}{b^n}$, where $a_n(t) \in \{0,1,\ldots,b-1\}$. Define now the one-to-one correspondence $\varphi_b : C_b \to C_3$ by

$$\varphi_b(t) := \sum_{n=1}^\infty \frac{\tilde{a}_n(t)}{3^n},$$

where $\tilde{a}_n(t) := 0$ if $a_n(t) = 0$ and $\tilde{a}_n(t) := 2$ if $a_n(t) = b-1$. Extend φ_b to each adjacent interval (c_k, d_k) continuously as a linear function. The extended function $\tilde{\varphi}_b$ is a monotone bijection of $[0,1]$ and therefore, it is continuous. Set now

$$u_b := u \circ \tilde{\varphi}_b,$$

where u is the classical Cantor function. Then u_b is continuous on $[0,1]$ and at every $t \in [c_k, d_k]$ with d_k determined by (2.30) we have

$$u_b(t) = \sum_{n=1}^{N_k} \frac{\tilde{a}_n(d_k)}{2^n}. \tag{2.31}$$

Show that $u_b \in \dot{H}_\infty^{\log_b 2}$. The case $b = 3$ is considered in [14]. For the sake of completeness we outline the proof for arbitrary b.

Choose $t_1 < t_2$ such that $t_i = d_{k_i}$, $i = 1,2$ with $k_1 \neq k_2$, where $d_{k_i} = \sum_{n=1}^{N_{k_i}} \frac{a_n^i}{b^n}$, $a_n^i \in \{0, b-1\}$ and suppose that $a_n^1 = a_n^2$ for every $1 \leqslant n \leqslant s$, and $a_{s+1}^1 \neq a_{s+1}^2$. Then

$$|u(t_2) - u(t_1)| \leqslant C|t_2 - t_1|^\beta \tag{2.32}$$

where $\beta = \log_b 2$. In fact,

$$0 \leqslant \frac{u(t_2)-u(t_1)}{(t_2-t_1)^\beta} = \frac{u(d_{k_2})-u(d_{k_1})}{(d_{k_2}-d_{k_1})^\beta} \leqslant C\frac{\frac{1}{2^{s+1}} + \sum\limits_{n\geqslant s+2} \frac{1}{2^n}}{(\frac{1}{b^{s+1}} - \sum\limits_{n\geqslant s+2} \frac{1}{b^n})^\beta} \leqslant C(\frac{b^\beta}{2})^{s+1} \frac{2}{[1-\frac{1}{b-1}]^\beta}$$

This proves (2.32) for such t_1 and t_2. But since the points of the form d_k are a dense subset of C_b it follows that (2.32) holds for all $t_1, t_2 \in C_b$. Next, suppose that t_1, t_2 are arbitrary points in $\bigcup_{k=1}^{\infty} [c_k, d_k]$. So, $t_1 \in [c_{k_1}, d_{k_1}]$ and $t_2 \in [c_{k_2}, d_{k_2}]$. If $k_1 = k_2$ obviously (2.32) holds. Otherwise suppose without loss of generality that $d_{k_1} < c_{k_2}$. Then, since $c_{k_j} \in C_b$,

$$|u(t_2) - u(t_1)| = |u(c_{k_2}) - u(d_{k_1})| \leqslant C|c_{k_2} - d_{k_1}|^\beta \leqslant C|t_2 - t_1|^\beta$$

Finally, since $\bigcup_{k=1}^{\infty} [c_k, d_k]$ is dense in $[0,1]$, we deduce (2.32) for all $t_1, t_2 \in [0,1]$. This precisely means that $u \in \dot{H}_\infty^\beta$, where $\beta = \log_b 2$.

It is easy also to see that u does not belong to $\dot{H}_\infty^\alpha$, for any $\alpha > \log_b 2$. Since $\dot{H}_\infty^\alpha \subset \dot{H}_\infty^\gamma$ whenever $0 < \gamma < \alpha$, it suffices to consider the case $\alpha < 1$. But, for such $\alpha < 1$ we have

$$0 \leqslant \frac{u(\frac{b-1}{b^n}) - u(0)}{(\frac{b-1}{b^n} - 0)^\alpha} = (b-1)^{1-\alpha}(\frac{b^\alpha}{2})^n \to \infty$$

as $n \to \infty$.

If we have the *WIP* for $(L_\infty, \dot{H}_\infty^\alpha)$ for some $0 < \alpha$ then there exists a constant C so that

$$\begin{aligned} &\int_0^\infty (K(f,t,L_\infty \cap \mathbf{M}, \dot{H}_\infty^\alpha \cap \mathbf{M})t^{-\theta})^q \frac{dt}{t} \\ &\leqslant C \int_0^\infty (K(f,t,L_\infty, \dot{H}_\infty^\alpha)t^{-\theta})^q \frac{dt}{t}. \end{aligned} \tag{2.33}$$

Let u be as above with b chosen so that $u \in \dot{H}^{\beta}_{\infty} \cap \mathbf{M}$ for some $\beta < \alpha$ ($0 < \beta < 1$) and $u \notin \dot{H}^{\alpha}_{\infty}$. Then for $f = u$ and arbitrary $k \geqslant 1$ we have

$$\omega_k(f,t,[0,1])_{L_\infty} \leqslant C(k)\omega_1(f,t,[0,1])_{L_\infty} \leqslant C(k)\min(t^{\beta},1).$$

Hence, similarly to the case $p < \infty$, we get

$$K(f,t,L_\infty,\dot{H}^{\alpha}_{\infty}) \prec \min(t^{\frac{\beta}{\alpha}},1),$$

and so the right hand side of (2.33) is finite for all $\theta \in (0,1)$, $q \in (0,\infty]$ such that $\theta < \frac{\beta}{\alpha}$.

On the other hand,

$$\begin{aligned} &K(f,t,L_\infty \cap \mathbf{M},\dot{H}^{\alpha}_{\infty} \cap \mathbf{M}) \\ &= \inf\{\|f-g\|_{L_\infty} + t|g|_{H^{\alpha}_{\infty}} \mid g \in G_f\}, \end{aligned} \tag{2.34}$$

where

$$G_f := \{g \in \dot{H}^{\alpha}_{\infty} \cap \mathbf{M}, f-g \in \mathbf{M}\}.$$

If $g \in \mathbf{M}$ and $f-g \in \mathbf{M}$, then for $0 < t_1 < t_2 \leqslant 1$

$$0 \leqslant g(t_2) - g(t_1) \leqslant f(t_2) - f(t_1) \leqslant C(t_2-t_1)^{\beta}$$

where C is independent of g. Besides, $0 \leqslant g \leqslant f \leqslant 1$. Thus, according to the Arzela-Ascoli theorem, G_f is precompact in $C[0,1]$. Since G_f is clearly closed in $C[0,1]$ it is compact in $C[0,1]$. Therefore there exists a function $g_0 \in G_f$ such that $\inf\limits_{g \in G_f}\{\|f-g\|_{L_\infty}\} = \|f-g_0\|_{L_\infty}$. Then we have

$$\begin{aligned} \inf_{g\in G_f}\{\|f-g\|_{L_\infty} + t|g|_{H^{\alpha}_{\infty}}\} &\geqslant \inf_{g\in G_f}\{\|f-g\|_{L_\infty}\} \\ &= \|f-g_0\|_{L_\infty}. \end{aligned} \tag{2.35}$$

Since $f-g_0 \in \mathbf{M}$, the function g_0 has to be constant on each interval of constancy of f. But g_0 cannot equal f a.e., since then $f \in \dot{H}^{\alpha}_{\infty}$ contradicting the condition $f \notin \dot{H}^{\alpha}_{\infty}, \alpha > \beta$ above.

Thus $\|f-g_0\|_{L_\infty} > 0$.

Then we get from (2.35) that

$$K(f,t,L_\infty \cap \mathbf{M},\dot{H}^{\alpha}_{\infty} \cap \mathbf{M}) \geqslant \|f-g_0\|_{L_\infty} > 0$$

so that

$$\int_0^\infty (K(f,t,L_\infty \cap \mathbf{M},\dot{H}^{\alpha}_{\infty} \cap \mathbf{M})t^{-\theta})^q \frac{dt}{t} = \infty$$

and we have a contradiction with (2.33).

REMARK. $\mathbf{M}$ does not also have the WIP with respect to $(L_\infty,\dot{W}^{r}_{\infty})$ for $r \geqslant 1$. As a counter-example for this couple we let f be the classical Cantor function. Then $f \in L_\infty \cap \mathbf{M}$. Let $g \in W^{r}_{\infty} \cap \mathbf{M}$ be a "good" approximation for the K-functional with constraints and $f-g\in\mathbf{M}$. This forces g to be constant on each interval of constancy of f. Thus $g'=0$ a.e. on $[0,1]$. Since $g\in W^1_\infty$ it is absolutely continuous and so it is constant. But this means that

$$\|f-g\|_{L_\infty}\geqslant\frac{1}{2}. \tag{2.36}$$

Since

$$K(f,t,L_\infty,\dot W^r_\infty)\approx\omega_r(f,t^{1/r})_\infty\leqslant 2^{r-1}\omega_1(f,t^{1/r})_\infty \tag{2.37}$$

and f is continuous, this K-functional tends to 0 as t tends to 0. This proves that the cone of monotone functions $\mathbf{M}$ does not have the IP with respect to $(L_\infty,\dot W^r_\infty)$. Applying the $L^\theta_\infty(\frac{dt}{t})$ norm to both sides of (2.37) and choosing $\theta:=(\log_3 2)/r$ we have

$$\sup_{t>0}t^{-\theta}K(f,t,L_\infty,\dot W^r_\infty)\prec 2^{r-1}\sup_{t>0}t^{-r\theta}\omega_1(f,t)_\infty=2^{r-1}|f|_{\dot H^{\log_3 2}_\infty}.$$

Since $f\in\dot H^{\log_3 2}_\infty$, $\sup_{t>0}t^{-\theta}K(f,t,L_\infty,\dot W^r_\infty)<\infty$.

On the other hand, by (2.36),

$$\sup_{t>0}t^{-\theta}K(f,t,L_\infty\cap\mathbf{M},\dot W^r_\infty\cap\mathbf{M})\geqslant\frac{1}{2}\sup_{t>0}t^{-\theta}=\infty.$$

Thus, the cone $\mathbf{M}$ also does not have the WIP with respect to $(L_\infty,\dot W^r_\infty)$.

3. Proof of Theorem 2

(i). If $f\in L_p\cap\mathbf{C}$, then the left derivative f'_- exists at every point of and is also a non decreasing function. Set for $t\leqslant 1$

$$g_t(x)=\begin{cases} f(x) & \text{if } 0\leqslant x\leqslant 1-t;\\ T(x) & \text{if } 1-t<x\leqslant 1,\end{cases} \tag{3.38}$$

where $T(x):=T_1(x,x_t):=f(1-t)+f'_-(1-t)(x-1+t)$.

We shall show that for $t\leqslant 1$

$$\|f-g_t\|_{L_p(0,1)}+t\|g'_t\|_{L_p(0,1)}\leqslant C\omega_1(f,t)_p. \tag{3.39}$$

It is readily seen that both g_t and $f-g_t$ belong to $\mathbf{C}$. Thus from (3.38) and (3.39) will follow that

$$\begin{aligned}K(f,t,L_p\cap\mathbf{C},W^1_p\cap\mathbf{C})&\leqslant\|f-g_t\|_{L_p(0,1)}+t(\|g'_t\|_{L_p(0,1)}+\|g_t\|_{L_p(0,1)})\\&\leqslant C(\omega_1(f,t)_p+\|f\|_{L_p(0,1)})\quad(0<t\leqslant 1).\end{aligned} \tag{3.40}$$

Since the right-hand side is equivalent to $K(f,t,L_p,W^1_p)$, we get for $t\leqslant 1$ that $K(f,t,L_p\cap\mathbf{C},W^1_p\cap\mathbf{C})\leqslant CK(f,t,L_p,W^1_p)$.

For $t>1$

$$K(f,t,L_p\cap\mathbf{C},W^1_p\cap\mathbf{C})\leqslant\|f\|_{L_p(0,1)}\leqslant K(f,t,L_p,W^1_p).$$

This means that $\mathbf{C}$ has the IP with respect to (L_p, W_p^1). From here we conclude that $\mathbf{C}$ has the IP with respect to $(L_p, (L_p, W_p^1)_{\alpha,\infty}) = (L_p, H_p^\alpha)$ with $0 < \alpha < 1$. In fact, applying the Proposition 2 and the Holmstedt's formula, see Corollary 3.6.2 (b) of [2] p. 53 we get

$$\begin{aligned} K(f,t, \; & L_p \cap \mathbf{C}, (L_p, W_p^1)_{\alpha,\infty} \cap \mathbf{C}) \\ & \leqslant Ct^{1-\alpha}\Big(\int_t^\infty [s^{-(1-\alpha)} K(f,s,L_p \cap \mathbf{C}, W_p^1 \cap \mathbf{C})]^q \tfrac{ds}{s}\Big)^{1/q} \\ & \leqslant Ct^{1-\alpha}\Big(\int_t^\infty [s^{-(1-\alpha)} K(f,s,L_p, W_p^1)]^q \tfrac{ds}{s}\Big)^{1/q} \leqslant CK(f,t,L_p,(L_p,W_p^1)_{\alpha,\infty}) \end{aligned} \tag{3.41}$$

Thus it remains to prove (3.39). According to (3.38) we have for $f \in L_p \cap \mathbf{C}$

$$\|f - g_t\|_{L_p(0,1)} = \|f - T\|_{L_p(1-t,1)} \leqslant \|f - f(1-t)\|_{L_p(1-t,1)}.$$

But

$$\|f - f(1-t)\|_{L_p(1-t,1)} \leqslant CE_0(f;[1-t,1])_{L_p}$$

and the right-hand side does not clearly exceed $C\omega_1(f,t)_p$. So we get

$$\|f - g_t\|_{L_p(0,1)} \leqslant C\omega_1(f,t)_p.$$

To estimate the second term in (3.39), write using definition (3.38)

$$t\|g_t'\|_{L_p(0,1)} \leqslant t\|f'\|_{L_p(0,1-t)} + t\|f_-'(1-t)\|_{L_p(1-t,1)}. \tag{3.42}$$

Since $f \in \mathbf{C}$, we get for every $x \in (1-t, 1-\frac{t}{2})$

$$f_-'(1-t) \leqslant f_-'(x) \leqslant \frac{\Delta_{\frac{t}{2}}(f,x)}{t/2}.$$

Thus

$$\begin{aligned} t\|f_-'(1-t)\|_{L_p(1-t,1)} & \leqslant 2^{1+1/p}\|\Delta_{\frac{t}{2}}(f)\|_{L_p(1-t,1-\frac{t}{2})} \\ & \leqslant 2^{1+1/p}\omega_1(f,\tfrac{t}{2})_{L_p(0,1)} \leqslant 2^{2+1/p}\omega_1(f,t)_{L_p(0,1)}. \end{aligned} \tag{3.43}$$

It remains to estimate the first term in (3.42). Since f is convex, for every $x \in [0, 1-t]$ we have

$$f_-'(x) \leqslant \frac{\Delta_{\frac{t}{2}}(f,x)}{t/2}.$$

Then we get, since $f' = f_-'$ a.e., that

$$t\|f'\|_{L_p(0,1-t)} \leqslant 2\|\Delta_{\frac{t}{2}}(f)\|_{L_p(0,1-t)} \leqslant 2\omega_1(f,\frac{t}{2})_{L_p(0,1)}.$$

Thus part (i) of the theorem is proved.

(ii) Prove now that $\mathbf{C}$ does not have the WIP (and therefore, also does not have the IP) with respect to $(L_p, \dot{H}_p^{1+\alpha})$ with $\alpha > 0$. First, assume that $\alpha \leqslant 1/p$ (this excludes the case $p = \infty$).

Suppose, on the contrary, that for every $f \in H_p^{1+\alpha} \cap \mathbf{C}$, $\theta \in (0,1)$ and $q \geqslant 1$

$$\|K(f,\cdot,L_p \cap \hat{M}_2, \dot{H}_p^{1+\alpha} \cap \mathbf{C})\|_{L_q^\theta(\frac{dt}{t})} \leqslant C\|K(f,\cdot,L_p,\dot{H}_p^{1+\alpha})\|_{L_q^\theta(\frac{dt}{t})} \tag{3.44}$$

for some absolute constant $C > 0$.

We shall show that the function

$$f_\epsilon(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \epsilon; \\ x-\epsilon & \text{if } \epsilon < x \leqslant 1. \end{cases} \tag{3.45}$$

for some suitable $\epsilon \in (0,1/2)$ gives the desired counterexample to (3.44).

Applying Holmstedt's formula (2.15) to the case $\dot{H}_p^{1+\alpha} = (L_p, \dot{W}_p^2)_{\theta,\infty}$ with $\theta := \frac{1+\alpha}{2} < 1$, we have

$$K(f_\epsilon,t,L_p,\dot{H}_p^{1+\alpha}) \approx t \sup_{s \geqslant t^{\frac{1}{1+\alpha}}} \frac{\omega_2(f_\epsilon,s)_p}{s^{1+\alpha}} \leqslant t \sup_{s \geqslant t^{\frac{1}{1+\alpha}}} \frac{\omega_1(f'_\epsilon,s)_p}{s^\alpha}.$$

But a direct calculation of modulus of continuity for the step-function f'_ϵ (cf. (2.17)) gives

$$\omega_1(f'_\epsilon,s)_p = \min\{s^{1/p}, \epsilon^{1/p}\}.$$

Hence

$$K(f_\epsilon,t,L_p,\dot{H}_p^{1+\alpha}) \leqslant C\begin{cases} t\epsilon^{1/p-\alpha}, & \text{if } t \leqslant \epsilon^{1+\alpha}, \\ t^{\frac{1}{1+\alpha}}\epsilon^{1/p}, & \text{if } t > \epsilon^{1+\alpha}, \end{cases}$$

and for $\frac{1}{1+\alpha} < \theta < 1$

$$\|K(f_\epsilon,\cdot,L_p,\dot{H}_p^{1+\alpha})\|_{L_q^\theta(\frac{dt}{t})} \leqslant C_1\epsilon^{1+1/p-\theta(1+\alpha)}. \tag{3.46}$$

Now we shall estimate the modified K-functional from below. To this end we denote by $G(f_\epsilon)$ the set

$$G(f_\epsilon) := \{g \in H_p^{1+\alpha} \cap \mathbf{C} | f_\epsilon - g \in \mathbf{C}\}.$$

From the definition of the function f_ϵ it follows that each $g \in G(f_\epsilon)$ has the form

$$g_b(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \epsilon; \\ b(x-\epsilon) & \text{if } \epsilon < x \leqslant 1, \end{cases} \tag{3.47}$$

for some $b \in [0,1]$. Thus, by definition of the modified K-functional and by equivalence of the norms of Lipschitz(Nikol'skii-Besov) spaces (see e.g. [10], 4.2.4, p.159), we have

$$K(f_\epsilon,t,L_p \cap \mathbf{C}, \dot{H}_p^{1+\alpha} \cap \mathbf{C}) \approx \inf_{b \in [0,1]} \{\|f_\epsilon - g_b\|_{L_p(0,1)} + t|g'_b|_{H_p^\alpha}\}. \tag{3.48}$$

We can easily calculate the norm and the seminorm on the right:

$$\|f_\epsilon - g_b\|_{L_p(0,1)} = \frac{(1-b)(1-\epsilon)^{1+1/p}}{(p+1)^{1/p}},$$

$$|g_b'|_{H_p^\alpha} = \sup_{s>0} \frac{\omega_1(g_b', s)_p}{s^\alpha} = b\epsilon^{1/p-\alpha}.$$

Thus the required K-functional equals

$$\inf_{b\in[0,1]}\{\frac{(1-b)(1-\epsilon)^{1+1/p}}{(p+1)^{1/p}} + tb\epsilon^{1/p-\alpha}\} = \min\{\frac{(1-\epsilon)^{1+1/p}}{(p+1)^{1/p}}, t\epsilon^{1/p-\alpha}\}.$$

Hence

$$\|K(f_\epsilon, t, L_p\cap \mathbf{C}, \dot{H}_p^{1+\alpha}\cap \mathbf{C})\|_{L_q^\theta(\frac{dt}{t})} \geqslant C_2\epsilon^{(1/p-\alpha)\theta} \tag{3.49}$$

for some absolute constant $C_2 > 0$. Applying (3.46) and (3.49) to estimate the terms of inequality (3.44) we then have

$$\epsilon^{(1/p-\alpha)\theta} \leqslant C_3\epsilon^{1+1/p-\theta(1+\alpha)} \quad (\frac{1}{1+\alpha} < \theta < 1)$$

and consequently

$$\epsilon^{(\theta-1)(1+1/p)} \leqslant C_3 < \infty$$

which is impossible if $\epsilon \to 0$.

It remains to consider the case $\alpha > 1/p$. Introduce the function f_0 defined by

$$f_0(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1/2; \\ x - 1/2 & \text{if } 1/2 < x \leqslant 1. \end{cases} \tag{3.50}$$

Since for $k \geqslant 2$

$$\omega_k(f_0, s)_p \leqslant 2^{k-2}s\omega_1(f_0', s)_p = 2^{k-2}s\min\{s^{1/p}, 2^{-1/p}\} \tag{3.51}$$

the function f_0 belongs to $\dot{H}_p^{1+1/p}$. On the other hand, f_0 does not belong to $\dot{H}_p^{1+\alpha}$ with $\alpha > 1/p$. Actually, by the embedding theorem ([10], Section 6.3) $\dot{H}_p^{1+\alpha} \subset C^1[0,1]$ if $\alpha > 1/p$ whereas f_0' is discontinuous.

Let us now estimate the K-functional of f_0 using Holmstedt's formula (2.15) and (3.51). We get

$$K(f_\epsilon, t, L_p, \dot{H}_p^{1+\alpha}) \approx t \sup_{s \geqslant t^{\frac{1}{1+\alpha}}} \frac{\omega_k(f_0, s)_p}{s^{1+\alpha}} \leqslant Ct \sup_{s\geqslant t^{\frac{1}{1+\alpha}}} \frac{\min\{s^{1/p}, 2^{-1/p}\}}{s^\alpha}.$$

Here k is the smallest integer more than $1+\alpha$. Then, similarly to the previous case we get for $\frac{1}{1+\alpha} < \theta < \frac{1+1/p}{1+\alpha}$ that

$$\|K(f_0, \cdot, L_p, \dot{H}_p^{1+\alpha})\|_{L_q^\theta(\frac{dt}{t})} < \infty. \tag{3.52}$$

On the other hand,

$$K(f_0, t, L_p\cap\mathbf{C}, \dot{H}_p^{1+\alpha}\cap\mathbf{C}) = \inf_{g\in G(f_0)}\{\|f_0 - g\|_{L_p(0,1)} + t|g|_{H_p^{1+\alpha}}\},$$

where

$$G(f_0) := \{g \in H_p^{1+\alpha} \cap \mathbf{C} | f_0 - g \in \mathbf{C}\}.$$

Because of these conditions on g, it should have the form

$$g(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1/2; \\ b(x - 1/2) & \text{if } 1/2 < x \leqslant 1, \end{cases} \tag{3.53}$$

for some $b \in [0,1]$. Since $g \in \dot{H}_p^{1+\alpha} \subset C^1[0,1]$, the number b has to be 0. So $G(f_0) = \{0\}$ and therefore we have

$$K(f_0, t, L_p \cap \mathbf{C}, \dot{H}_p^{1+\alpha} \cap \mathbf{C}) = \|f_0\|_{L_p(0,1)} = (\frac{1}{2})^{1/p}.$$

Thus, this K-functional does not belong to $L_q^\theta(\frac{dt}{t})$. Comparing this statement with inequality (3.52), we conclude that inequality (3.44) is also impossible in this case.

The proof of the theorem is complete.

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