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SOME INEQUALITIES AND PROPERTIES CONCERNING CHORDAL SEMI–POLYGONS

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(*communicated by V. Volenec*)

Abstract. The paper deals with some inequalities and properties concerning semi-chordal polygons.

1. Preliminaries

Semi-polygon. Let $A_1, \dots, A_n$ be any given different points in a plane. Then the union

$$A_1A_2 \cup A_2A_3 \cup \cdots \cup A_{n-1}A_n \cup S \tag{1}$$

of the line segments $A_1A_2, \dots, A_{n-1}A_n$ and the set S wich is either empty set or the segment A_nA_1, will be called a semi-polygon and denoted by $A_1 \dots A_n$ or briefly by $\underline{A}$.

So, each polygon may be called semi-polygon, but not conversely if S is empty set.

If $A_1 \dots A_n$ is a semi-polygon which is not a polygon, then its vertices A_1 and A_n will be called *end-vertices*.

Chordal semi-polygon. A semi-polygon $A_1 \dots A_n$ will be called a chordal semi-polygon if there is a circle $\mathcal{C}$ such that each of the vertices $A_1, \dots, A_n$ lie on $\mathcal{C}$.

Now in short about the angles wich play an important role in the following considerations.

Let $A_1 \dots A_n$ be a chordal semi-polygon and let C be the centre of its circumcircle. Then

$$\beta_i = \text{ measure of } \sphericalangle CA_iA_{i+1}, \quad i = 1, \dots, n \tag{2}$$

$$\varphi_i = \text{ measure of } \sphericalangle A_iCA_{i+1}, \quad i = 1, \dots, n. \tag{3}$$

In this paper we shall use oriented angles. As it is known, an angle $\sphericalangle PQR$ is positively or negatively oriented if it is going from QP to QR counter-clockwise or clockwise.

It is very important to remark that the angles β_i and φ_i given by (2) and (3) have opposit orientation, e. g. see Fig. 1

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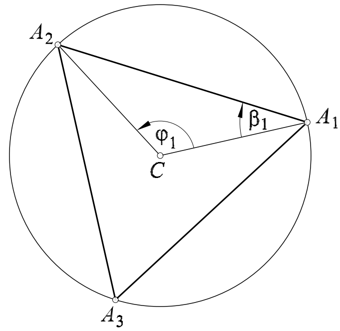

Fig. 1

The measure of an oriented angle will be taken in radians and will be with sign $+$ or $-$ depending on if the orientation of the angle is positive or negative.

Obviously, for β_i and φ_i is valid

$$0 \leqslant |\beta_i| < \frac{\pi}{2}, \quad 0 < |\varphi_i| \leqslant \pi,$$

since no two of the consecutive vertices in $A_1 \dots A_n$ are the same (by the given definition).

For simplicity, the measures of the oriented angles given by (2) and (3) we shall also write as β_i and φ_i.

Notice 1. In the following, when we speak about chordal semi-polygon, it will be meant that this polygon is a such one that no two of its consecutive vertices are the same and that no one of its sides is a diameter of its circumcircle, that is, no one of $\beta_1, \dots, \beta_n$ is zero. Also we suppose that its vertices are not collinear.

Of course, in the case when some of β_i is zero, then we have radius r readily. Also, in the case when all vertices are collinear, the polygon is very simple one.

Notice 2. Whenever $\underline{A} = A_1 \dots A_n$ is a chordal semi-polygon under consideration, then by C, r and $\mathcal{C}$ will be denoted center, radius and circumcircle of $\underline{A}$ respectively. Also β_i and φ_i are given by (2) and (3), $\alpha_1, \dots, \alpha_n$ are given by

$$\alpha_i = \text{ measure of } \sphericalangle A_{n-1+i}A_iA_{i+1}, \quad i = 1, \dots, n.$$

Of course, $\sphericalangle A_{n-1+i}A_iA_{i+1}$ is oriented and indices are calculated modulo n.

Bay $a_1, \dots, a_n$ will be denoted the lengths of the sides of $\underline{A}$.

DEFINITION 1. Let $\underline{A} = A_1 \dots A_n$ be a chordal semi-polygon and let

$$\phi = |\varphi_1 + \dots + \varphi_n|, \tag{4}$$

$$u = \text{ the number of the elements in the set } \{i : \beta_i > 0\}, \tag{5}$$

$$v = \text{ the number of the elements in the set} \{i : \beta_i < 0\}, \tag{6}$$

where φ_n in (4) is omited if $\underline{A}$ is not a polygon.

Then we say that (ϕ, u, v) is a characteristic of $\underline{A}$.

For example, the semi-polygon in Fig. 2 has $(\phi, 2, 1)$ as a characteristic, where $\phi =$ measure of $\sphericalangle A_1CA_4$.

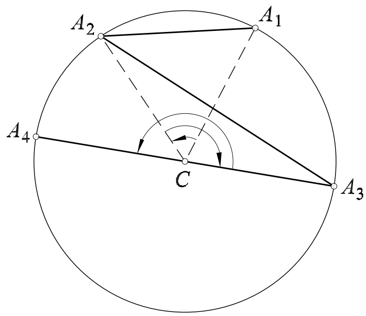

Fig. 2

DEFINITION 2. Let $\underline{A} = A_1 \dots A_n$ be a chordal semi-polygon. We say that $\underline{A}$ is of the *first kind* if inside $\mathcal{C}$ there is a point 0 such that all of the angles $\sphericalangle A_iOA_{i+1}$, $i = 1, \dots, n$, have the same orientation. If such a point 0 does not exists, that is, not all of the angles $\sphericalangle A_iOA_{i+1}$ have the same sign, we say that $\underline{A}$ is of the *second kind*.

DEFINITION 3. Let $\underline{A} = A_1 \dots A_n$ be a chordal polygon and let 0 be a point inside $\mathcal{C}$ such that

$$|\psi_1 + \cdots + \psi_n| = 2k\pi, \tag{7}$$

where k is a positive integer ant $\psi_i =$ measure of $\sphericalangle A_iOA_{i+1}$. If the point 0 is a such one that k is maximal, then we say that $\underline{A}$ is a k-*inscribed polygon* of the first kind or the second kind depending on if $\underline{A}$ is of the first or of the second kind.

If

$$|\varphi_1 + \cdots + \varphi_n| = 2j\pi, \tag{8}$$

where $\varphi_i =$ measure of $\sphericalangle A_iCA_{i+1}$, then we say that j is an *index* of $\underline{A}$. Of course, $j \in \{0, 1, \dots, k\}$.

A k-inscribed chordal polygon of the first kind whose index is k will be briefly called k-*chordal polygon*. Using Definition 1 it can be said: A polygon is k-chordal if its characteristic (ϕ, u, v) has property that $\phi = 2k\pi$, $u = n$ or $v = n$.

It is easy to see that $A_1 \dots A_n$ is a k-chordal polygon iff

$$|\beta_1 + \cdots + \beta_n| = (n - 2k)\frac{\pi}{2}, \tag{9}$$

where $\beta_1 > 0$, $i = 1, \dots, n$ or $\beta_1 < 0$, $i = 1, \dots, n$.

For example, if $\beta_i > 0$, then $\varphi_i < 0$, and $\varphi_i = -\pi + 2\beta_i$, so that the equation

$$|\varphi_1 + \cdots + \varphi_n| = 2k\pi$$

when $\beta_i > 0$, $i = 1, \dots, n$ can be written as

$$(-\pi + 2\beta_1) + \cdots + (-\pi + 2\beta_n) = -2k\pi,$$

from which follows (9).

Thus, if $\beta_i > 0$, $i = 1, \dots, n$ then

$$\beta_1 + \cdots + \beta_n = (n - 2k)\frac{\pi}{2}.$$

But if $\beta_i < 0$, $i = 1, \dots, n$, then $\beta_1 + \cdots + \beta_n = (-n + 2k)\frac{\pi}{2}$.

The sign of the sum $\beta_1 + \cdots + \beta_n$ depends on the orientation of the polygon. In brief about this.

Let $\underline{A} = A_1 \dots A_n$ be a chordal n-gon and let $\underline{B} = B_1 \dots B_n$ be such that $B_i = A_{n+1-i}$, $i = 1, \dots, n$. Then $\underline{A}$ and $\underline{B}$ are in fact the same since their vertices lie just in opposite order. But if it is a question of an oriented polygon, then $\underline{A}$ and $\underline{B}$ have opposite orientations.

An orientation of $A_1 \dots A_n$ is positive or negative depending on if starting from A_1 and going to A_n the polygon is circumscribed counter-clockwise or clockwise.

If $\underline{A} = A_1 \dots A_n$ is a k-chordal polygon, then β_i, $i = 1, \dots, n$ is negative if $\underline{A}$ is positively oriented and vice versa. But in the case when $\underline{A}$ is a chordal polygon of the second kind, then some β_i are negative and some positive.

Notice 3. In the following, for the sake of simplicity, we shall suppose that a polygon and a semi-polygon as well are negatively oriented.

Then $\varphi_1 + \cdots + \varphi_n \leqslant 0$ but $\beta_1 + \cdots + \beta_n \geqslant 0$. So, for example, instead of $|\beta_1 + \cdots + \beta_n| = (n - 2k)\frac{\pi}{2}$ we can write $\beta_1 + \cdots + \beta_n = (n - 2k)\frac{\pi}{2}$.

Therefore, in the following when we say chordal semi-polygon or chordal polygon it will always be meant that the assumptions in Notice 1 and Notice 3 are fulfiled.

Now about notation which will be used.

Symbol $\lfloor\frac{n-1}{2}\rfloor$. If n is a positive integer, then $\lfloor\frac{n-1}{2}\rfloor = \frac{n-1}{2}$ if n is odd, $\lfloor\frac{n-1}{2}\rfloor = \frac{n-2}{2}$ if n is even.

Symbol P_j^n. If j and n are positive integers and $j \leqslant n$, then P_j^n is the sum of $\binom{n}{j}$ products of the form

$$\cos\beta_{i_1} \dots \cos\beta_{i_j} \sin\beta_{i_{j+1}} \dots \sin\beta_{i_n},$$

where $(i_1, i_2, \dots, i_n)$ is a permutation of $\{1, 2, \dots, n\}$.

For example:

$$P_1^3 = \cos\beta_1 \sin\beta_2 \sin\beta_3 + \sin\beta_1 \cos\beta_2 \sin\beta_3 + \sin\beta_1 \sin\beta_2 \cos\beta_3.$$

2. Some inequalities and properties concerning chordal semi-polygons

We commence with the following theorem.

THEOREM 1. *Let $\underline{A} = A_1 \dots A_n$ be a chordal polygon. Then it is valid*

$$\alpha_1 + \cdots + \alpha_n = 2(\beta_1 + \cdots + \beta_n). \tag{10}$$

Proof. Obviously, if $\underline{A}$ is convex and C is inside of $\underline{A}$, then

$$\alpha_1 = \beta_n + \beta_1$$
$$\alpha_2 = \beta_1 + \beta_2$$
$$\alpha_3 = \beta_2 + \beta_3$$
$$\dots\dots\dots$$
$$\alpha_n = \beta_{n-1} + \beta_n$$

that is

$$\alpha_i = \beta_{n-1+i} + \beta_i, \quad i = 1, \dots, n. \tag{11}$$

It is not difficult to see that (11) is valid in all other cases too. So, if $A_1 \dots A_5$ is a pentagon as in Fig. 3, then

$$\alpha_i = \beta_{4+i} + \beta_i, \quad i = 1, \dots, 5.$$

Let us remark that all $\alpha_1, \dots, \alpha_5$ are positive and that β_1 and β_4 are negative.

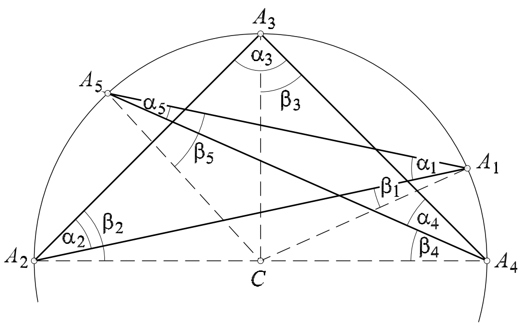

Fig. 3

Generally, no matter which is a chordal polygon, beginning from its first vertex, it is easy to see that $\alpha_1 = \beta_n + \beta_1$, then that is $\alpha_2 = \beta_{n-1} + \beta_2$, and so on. The situation is like in a proof by induction.

So, Theorem 1 is proved. $\square$

Now we state some of its corollaries.

COROLLARY 1.1. *Let $A_1 \dots A_n$ be a chordal polygon. Then there is an integer z such that*

$$\beta_1 + \dots + \beta_n = z\frac{\pi}{2}, \tag{12}$$

$$\alpha_1 + \dots + \alpha_n = z\pi. \tag{13}$$

Proof. It follows from the equation $\varphi_1 + \cdots + \varphi_n = 2j\pi$ since

$$\varphi_i = \pi + 2\beta_i \text{ or } \varphi_i = -\pi + 2\beta_i$$

depending on if $\beta_i < 0$ or $\beta_i > 0$.

COROLLARY 1.2. *Let $A_1 \ldots A_n$ be a chordal n-gon whose index is j and let v of the angles $\beta_1, \ldots, \beta_n$ be negative. Then*

$$\alpha_1 + \cdots + \alpha_n = [n - 2(j+v)]\pi. \tag{14}$$

Proof. Let, for simplicity, be $\beta_i < 0$, $i = 1, \ldots, v$. Then $\beta_i > 0$ for $i = v+1, \ldots, n$. Thus, the equality

$$\varphi_1 + \cdots + \varphi_n = -2j\pi$$

can be written as

$$\sum_{i=1}^{v}(\pi + 2\beta_i) + \sum_{i=v+1}^{n}(-\pi + 2\beta_i) = -2j\pi$$

or

$$2\beta_i + \cdots + 2\beta_n = [n - 2(j+v)]\pi, \tag{15}$$

and by (10) it can be written as (14).

COROLLARY 1.3. *Let $A_1 \ldots A_n$ be as in Corollary 1.2. Then*

$$|\beta_1| + \cdots + |\beta_n| = [n - 2(j+v)]\frac{\pi}{2} + 2\tau, \tag{16}$$

where

$$\tau = - \text{ (sum of all } \beta_i \text{ which are negative).}$$

Proof. From (15) we have

$$\begin{aligned} \beta_1 + \cdots + \beta_n &= [n - 2(j+v)]\frac{\pi}{2}, \\ 2\tau + \beta_1 + \cdots + \beta_n &= [n - 2(j+v)]\frac{\pi}{2} + 2\tau. \end{aligned} \tag{17}$$

COROLLARY 1.4. *Let $A_1 \ldots A_n$ be a chordal n-gon. If n is odd, then*

$$\beta_1 + \cdots + \beta_n > 0,$$

but if n is even then may be $\beta_1 + \cdots + \beta_n = 0$. (Instead of $|\beta_1 + \cdots + \beta_n|$ we may write $\beta_1 + \cdots + \beta_n$ by Notice 3.)

Proof. If n is even then may be $[n - 2(j+v)]\frac{\pi}{2} = 0$ or $n = 2(j+v)$. So, for example, it $A_1A_2A_3A_4$ is a chordal quadrangle which is not convex, then $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$.

COROLLARY 1.5. *If* $A_1 \dots A_n$ *is* k*-inscribed chordal* n*-gon, then*

$$\beta_1 + \cdots + \beta_n \leqslant (n-2k)\frac{\pi}{2}. \tag{18}$$

Proof. If $A_1 \dots A_n$ is k-chordal n-gon, that is, a k-inscribed chordal n-gon whose index is maximal, then $\beta_1 + \cdots + \beta_n = (n-2k)\dfrac{\pi}{2}$.

So, fox example, the pentagon in Fig. 4a is 1-chordal pentagon, and that in Fig. 4b is 1-inscribed chordal pentagon (where A_2 is A_3 in Fig. 4a and vice versa). For the first one it is valid.

$$\alpha_1 + \cdots + \alpha_5 = 3\pi, \quad \beta_1 + \cdots + \beta_5 = 3\frac{\pi}{2}, \tag{19}$$

but for that in Fig. 4b it is obviously $|\alpha + \cdots + \alpha_5| < 3\pi$.

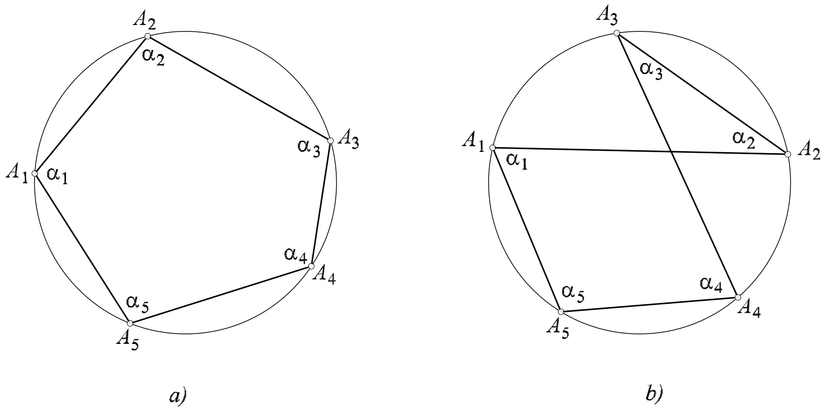

Fig. 4

COROLLARY 1.6. *If* $A_1 \dots A_n$ *is a* k*-inscribed chordal* n*-gon, then*

$$j + v \geqslant k. \tag{20}$$

Proof. From $[n-2(j+v)]\dfrac{\pi}{2} \leqslant (n-2k)\dfrac{\pi}{2}$ it follows $j + v \geqslant k$.

COROLLARY 1.7. *If* $A_1 \dots A_n$ *is a* k*-inscribed chordal* n*-gon, then*

$$2k \leqslant n-2 \ \text{ if } n \text{ is even,}$$
$$2k \leqslant n-1 \ \text{ if } n \text{ is odd.}$$

In the following corollary we shall use the following definition.

DEFINITION 4. Let $A_1 \dots A_n$ be a k-inscribed chordal n-gon. Then the chordal semi-polygon whose end vertices are A_1 and A_n will be called *k-inscribed chordal semi-polygon*.

COROLLARY 1.8. *If $A_1 \dots A_n$ is a k-chordal semi-polygon, then*

$$(n-1-2k)\frac{\pi}{2} < \beta_1 + \cdots + \beta_{n-1} < (n-2k)\frac{\pi}{2}. \tag{21}$$

THEOREM 2. *If $\underline{A} = A_1 \dots A_n$ is a k-chordal polygon and p is an positive integer, then*

$$\cos^p \beta_1 + \cdots + \cos^p \beta_n > n\Big(\frac{2k}{n}\Big)^p. \tag{22}$$

Proof. In [1], Theorem 1, we have the following result:

Let k and n be any given positive integers such that $n-2k > 0$, and let β_1, $\dots, \beta_n$ satisfy

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2},\ \ 0 < \beta_i < \frac{\pi}{2}. \tag{23}$$

Then

$$\cos\beta_1 + \cdots + \beta_n > 2k\cos\beta_j,\ \ j = 1, \dots, n. \tag{24}$$

The proof is as follows.

Since $\cos\pi x > 1-2x$ if $0 < x < \frac{1}{2}$, putting $\alpha = \pi x$ we obtain

$$\cos\alpha > 1 - \frac{2}{\pi}\alpha,\ \ 0 < \alpha < \frac{\pi}{2}. \tag{25}$$

Consequently

$$\sum_{i=1}^{n}\cos\beta_i > n - \frac{2}{\pi}\sum_{i=1}^{n}\beta_i = n - \frac{2}{\pi}(n-2k)\frac{\pi}{2} = 2k > 2k\cos\beta_j.$$

Here we need to prove that (22) holds for $p = 2, 3, \dots$ too.

From (25), since $0 < \beta_i < \frac{\pi}{2}$, we have

$$\cos^p \beta_i > (1 - \frac{2}{\pi}\beta_i)^p. \tag{26}$$

So for $p = 2$ we can write (using the property of arithemetical mean)

$$\begin{aligned}\sum_{i=1}^{n}\cos^2\beta_i &> n - \frac{4}{\pi}(\beta_1 + \cdots + \beta_n) + \Big(\frac{2}{\pi}\Big)^2(\beta_1^2 + \cdots + \beta_n^2)\\ &\geqslant n - 2(n-2k) + n\Big(\frac{\beta_1 + \cdots + \beta_n}{n}\Big)^2\\ &= n - 2(n-2k) + \frac{(n-2k)^2}{n} = n\Big[\frac{n-(n-2k)}{n}\Big]^2 = n\Big(\frac{2k}{n}\Big)^2\end{aligned}$$

Similarly for $p = 3$ we have

$$\begin{aligned}\sum_{i=1}^{n}\cos^3\beta_i &> n - 3(n-2k) + 3n\Big(\frac{n-2k}{n}\Big)^2 - n\Big(\frac{n-2k}{n}\Big)^3\\ &= n\Big[\frac{n-(n-2k)}{n}\Big]^3 = n\Big(\frac{2k}{n}\Big)^3.\end{aligned}$$

It is easy to see that it generally holds

$$\sum_{i=1}^{n} \cos^p \beta_i > n\left[\frac{n-(n-2k)}{n}\right]^p = n\left(\frac{2k}{n}\right)^p.$$

Let us remark that $\sum_{i=1}^{n} \cos^p \beta_i > n\left(\frac{2k}{n}\right)^p$ for $\beta_1 = \ldots = \beta_n = (n-2k)\frac{\pi}{2n}$, since

$$n\cos^p(n-2k)\frac{\pi}{2n} = n\sin^p \frac{k\pi}{n},$$

$$\sin\frac{k\pi}{n} > \frac{2k}{n} \quad \text{for each} \quad n \geqslant 3 \quad \text{and} \quad k = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor.$$

Also let us remark that $\sum_{i=1}^{n} \cos^p \beta_i = 2k$ for $\beta_1 = \ldots = \beta_{n-2k} = \frac{\pi}{2}$, $\beta_{n-2k+1} = \ldots = \beta_n = 0$ and that $2k > n\left(\frac{2k}{n}\right)^p$ if $p > 1$.

So Theorem 2 is proved. □

COROLLARY 2.1. *Let $\underline{A} = A_1 \ldots A_n$ be a k-chordal polygon and let $a_1, \ldots, a_n$ be the lengths of its sides. Then*

$$\sum_{i=1}^{n} a_i^p > n\left(\frac{2k}{n}\right)^p a_j^p, \quad j = 1, \ldots, n. \tag{27}$$

Proof. It is valid $a_i = 2r\cos\beta_i$, $i = 1, \ldots, n$.

COROLLARY 2.2. *If $2k$ is maximal, that is*

$$2k = n-1 \quad \textit{if} \quad n \quad \textit{is odd}, \quad 2k = n-2 \quad \textit{if} \quad n \quad \textit{is even},$$

then

$$\sum_{i=1}^{n} a_i^p > n\left(1-\frac{1}{n}\right)^p a_j^p, \quad j = 1, \ldots, n \quad \textit{for} \quad n \quad \textit{odd},$$

$$\sum_{i=1}^{n} a_i^p > n\left(1-\frac{2}{n}\right)^p a_j^p, \quad j = 1, \ldots, n \quad \textit{for} \quad n \quad \textit{even}.$$

COROLLARY 2.3. *Let $\underline{A} = A_1 \ldots A_n$ be a k-inscribed chordal n-gon.Then*

$$\sum_{i=1}^{n} \cos^p \beta_i > n\left[\frac{2(j+v-\sigma)}{n}\right]^p, \tag{28}$$

where σ is the number such that $\sigma \cdot \frac{\pi}{2} = \tau$, and j, v, τ are as in Corollary 1.3.

Proof. If $|\beta_1| + \cdots + |\beta_n| = s \cdot \frac{\pi}{2}$, $0 < |\beta_i| < \frac{\pi}{2}$, $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} \cos^p \beta_i > n\left(2-\frac{s}{n}\right)^p. \tag{29}$$

Now, the proof is similar to the proof of Theorem 2. Let us remark that from

$$|\beta_1| + \cdots + |\beta_n| = [n - 2(j + v - \sigma)]\frac{\pi}{2} \tag{30}$$

it follows that $s = n - 2(j + v - \sigma)$ and

$$1 - \frac{s}{n} = \frac{2(j + v - \sigma)}{n}.$$

Also let us remark that in (28) we must write $|\beta_1| + \cdots + |\beta_n|$ instead of $\beta_1 + \cdots + \beta_n$. It because od (25). For example, if $p = 1$, we have

$$\sum_{i=1}^{n} \cos\beta_i > n - \frac{2}{\pi}(|\beta_1| + \cdots + |\beta_n|) = 2(j + v - \sigma).$$

COROLLARY 2.4. *Let $\underline{A} = A_1 \ldots A_n$ be as in Corollary 2.3. Then*

$$\sum_{i=1}^{n} a_i^p > n\left[\frac{2(j + v - \sigma)}{n}\right]^p \cdot a_j^p, \ \ j = 1, \ldots, n. \tag{31}$$

where $a_1, \ldots, a_n$ are the lengths of the sides of $A_1 \ldots A_n$.

COROLLARY 2.5. *Let $A_1 \ldots A_n$ be as in Corollary 2.4 and let $w = j + v - \sigma$. Then*

$$\frac{a_1 + \cdots + a_n}{n \sin \frac{w\pi}{n}} \leqslant 2r < \frac{a_1 + \cdots + a_n}{2w}, \tag{32}$$

$$\frac{2wa_i}{a_i + \cdots + a_n} < \cos\beta_i \leqslant \frac{na_i \sin \frac{w\pi}{n}}{a_1 + \cdots + \alpha_n}. \tag{33}$$

Proof. From

$$2r\cos\beta_i = a_i, \ \ |\beta_i| = \arccos\frac{a_i}{2r}$$

$$\arcsin x + \arccos x = \frac{\pi}{2}, \ \ 0 \leqslant x \leqslant 1$$

$$\sum_{n=1}^{n} \arcsin\frac{a_i}{2r} + \sum_{i=1}^{n} \arccos\frac{a_i}{2r} = n \cdot \frac{\pi}{2},$$

$$\sum_{i=1}^{n} \arccos\frac{a_i}{2r} = (n - 2w)\frac{\pi}{2},$$

it follows

$$\sum_{i=1}^{n} \arcsin\frac{a_i}{2r} = w\pi$$

or

$$\frac{a_1 + \cdots + a_n}{2r} + \frac{1}{2} \cdot \frac{1}{3}\left[\left(\frac{a_1}{2r}\right)^3 + \cdots + \left(\frac{a_n}{2r}\right)^3\right] + \cdots = w\pi. \tag{34}$$

Thence, using the property of arithmetical mean, we get

$$\arcsin \frac{a_1 + \cdots + a_n}{2rn} \leqslant \frac{w\pi}{n}$$

or

$$2r \geqslant \frac{a_1 + \cdots + a_n}{n \sin \frac{w\pi}{n}}. \tag{35}$$

Now, from

$$\cos \beta_i = \frac{a_i}{a_1} \cos \beta_1$$

$$\sum_{i=1}^{n} \cos \beta_i = \frac{a_1 + \cdots + a_n}{a_1} \cos \beta_1$$

$$2w < \frac{a_1 + \cdots + a_n}{a_1} \cos \beta_1$$

it follows

$$\cos \beta_1 > \frac{2wa_1}{a_1 + \cdots + a_n} \tag{36}$$

As $\cos \beta_1 = \dfrac{a_1}{2r}$ we have the following inequality

$$\frac{a_1}{2r} > \frac{2wa_1}{a_1 + \cdots + a_n}$$

or

$$2r < \frac{a_1 + \cdots + a_n}{2w}. \tag{37}$$

Finally, since $a_i = 2r \cos \beta_i$, the equality (34) can be rewritten as

$$(\cos \beta_1 + \cdots + \cos \beta_n) + \frac{1}{2} \cdot \frac{1}{3} (\cos^3 \beta_1 + \cdots + \cos^3 \beta_n) + \cdots = w\pi$$

or, using (35) and supposing that $a_1 = \max\{a_1, \ldots, a_n\}$,

$$\left(\frac{a_1 + \cdots + a_n}{a_1} \right) \cos \beta_1 + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{a_1^3 + \cdots + a_n^3}{a_1^3} \right) \cos^3 \beta_1 + \cdots = w\pi,$$

from which, using the property of arithmetical mean, it follows that

$$\frac{a_1 + \cdots + a_n}{na_1} \cos \beta_1 \leqslant \sin \frac{w\pi}{n}.$$

As $\cos \beta_1 = \dfrac{a_1}{a_i} \cos \beta_i$, we have the inequality

$$\cos \beta_i \leqslant \frac{na_i \sin \frac{w\pi}{n}}{a_1 + \cdots + a_n}. \tag{38}$$

COROLLARY 2.6. *We have that*

$$\frac{2w}{n} \leqslant \sin\frac{w\pi}{n}. \tag{39}$$

Proof. The above inequality follows from $\sum_{i=1}^{n}\cos\beta_i > 2w$ and (38).

COROLLARY 2.7. *Let $A_1 \dots A_n$ be a k-chordal polygon such that*

$$\sum_{i=1}^{n} a_i^p > 2ka_j^p,\ \ j = 1,\dots,n, \tag{40}$$

where $k = \lfloor\frac{n-1}{2}\rfloor$. Then for enough large n it is valid $a_1 \approx a_2 \approx \dots \approx a_n$.

Proof. In n is odd then from (40) it follows that

$$\frac{a_1^p + \ldots + a_n^p}{n} > \Big(1 - \frac{1}{n}\Big)a_j^p, \qquad j = 1,\dots,n.$$

Similar is for even n.

COROLLARY 2.8. *If $A_1 \dots A_n$ is a k-inscribed chordal semi-polygon and $2\varepsilon\cdot\frac{\pi}{2} = |\beta_n|$, $s = n - 2(j + v - \sigma + \varepsilon)$, then*

$$\sum_{i=1}^{n-1}\cos^p\beta_i > (n-1)\Big(1 - \frac{s}{n-1}\Big)^p, \tag{41}$$

$$\sum_{i=1}^{n-1} a_i^p > (n-1)\Big(1 - \frac{s}{n-1}\Big)^p a_j^p,\ \ j = 1,\dots,n-1. \tag{42}$$

For example, if $|\beta_1| + \dots + |\beta_{n-1}| = \frac{\pi}{3}$, then

$$\sum_{i=1}^{n-1} a_i^p > (n-1)\Big[1 - \frac{2}{3(n-1)}\Big]^p a_j^p,\ \ j = 1,\dots,n-1.$$

Let us remark that from $s\cdot\frac{\pi}{2} = \frac{\pi}{3}$ it follows that $s = \frac{2}{3}$.

The following theorem and its corollaries are concerning the condition which the lengths $a_1,\dots,a_n$ satisfy in the case when a polygon is k-chordal polygon.

THEOREM 3. *Let $A_1 \dots A_n$ be a k-chordal polygon and let $a_1 = \max\{a_1,\dots,\alpha_n\}$. Then*

$$\frac{a_1 + \dots + a_n}{a_1} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{a_1^3 + \dots + a_n^3}{a_1^3} + \dots > k\pi. \tag{43}$$

Proof. Let $\cos\gamma_i = \frac{a_i}{a_1}$, $i = 1,\dots,n$. Then

$$\gamma_1 + \dots + \gamma_n < (n-2k)\frac{\pi}{2}$$

since

$$\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2},$$
$$\cos\beta_i = \frac{a_i}{a_1}\cos\beta_1, \quad i = 1, \ldots, n. \tag{44}$$

Let us remark that $\beta_1 > 0$ if $\underline{A}$ is a k-chordal polygon.

Now from

$$\sum_{i=1}^{n} \arcsin\frac{a_i}{a_1} + \sum_{i=1}^{n} \arccos\frac{a_i}{a_1} = n\cdot\frac{\pi}{2},$$
$$\sum_{i=1}^{n} \arccos\frac{a_i}{a_1} < (n-2k)\frac{\pi}{2}$$

it follows

$$\sum_{i=1}^{n} \arcsin\frac{a_i}{a_1} > k\pi,$$

which can be written as (43). And Theorem 3 is proved. □

COROLLARY 3.1. *The condition (43) is not only necessary for $a_1, \ldots, a_n$ to be the lengths of the sides of a k-chordal polygon, but also sufficient.*

Proof. Let (43) be fulfiled and let

$$\cos\gamma_i = \frac{a_i}{a_1}\cos\gamma_1, \quad i = 1, \ldots, n.$$

Since $\gamma_i \to \frac{\pi}{2}$ when $\cos\gamma_1 \to 0$, it is clear that there is $0 < \gamma_1 < \frac{\pi}{2}$ such that

$$\gamma_1 + \cdots + \gamma_n = (n-2k)\frac{\pi}{2},$$
$$\frac{a_1}{\cos\gamma_1} = \cdots = \frac{a_n}{\cos\gamma_n} = c,$$

where $c = 2r$.

COROLLARY 3.2. *There is a k-chordal polygon whose sides have the lengths a_1, $\ldots, a_n$ iff there is $0 < \beta_1 < \frac{\pi}{2}$ such that*

$$\frac{a_1 + \cdots + a_n}{a_1}\cos\beta_1 + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{a_1^3 + \cdots + a_n^3}{a_1^3}\cos^3\beta_1 + \cdots = k\pi. \tag{45}$$

Proof. Iff $\sum_{i=1}^{n}\arccos\left(\frac{a_i}{a_1}\cos\beta_1\right) = (n-2k)\frac{\pi}{2}$ then (45) is valid.

COROLLARY 3.3. *If* $\beta_1+\cdots+\beta_n=(n-2k)\frac{\pi}{2}$, $0<\beta_i<\frac{\pi}{2}$, $i=1,\ldots,n$, *then*

$$\cos\beta_1+\cdots+\cos\beta_n\leqslant n\sin\frac{k\pi}{n}. \tag{46}$$

Proof. It follows from (44) and (45) (using the property of arithmetical mean).

Before making a statement of the following corollary here are some examples which may bi interesting.

1. If $a_i=b+i$, $i=1,\ldots,5$ and b is a positive integer less than 27, then there is no 2-chordal pentagon whose sides have the lengths $a_i=b+i$, $i=1$, $\ldots,5$. So if $a_i=10+i$, $i=1,\ldots,5$, then

$$\sum_{i=1}^{4}\arccos\frac{10+i}{15}\approx 130.066^\circ>90^\circ.$$

2. If $a_i=27+i$, $i=1,\ldots,5$, then

$$\sum_{i=1}^{4}\arccos\frac{27+i}{32}\approx 88.689^\circ<90^\circ.$$

3. If $n=6$, $k=2$, then $\beta_1+\cdots+\beta_6=\pi$ and b may be 9, but not less then 9. For $b=9$ we have

$$\sum_{i=1}^{5}\arccos\frac{9+i}{15}\approx 178.855^\circ<180^\circ.$$

4. If $n=7$, $k=3$, then b can not be less than 89. If $b=89$, then

$$\sum_{i=1}^{6}\arccos\frac{89+i}{96}\approx 89.889^\circ<90^\circ.$$

Of course, if m is a positive integer such that $m\leqslant\lfloor\frac{n-1}{2}\rfloor$ and if there is m-chordal polygon whose sides have the lengths $a_1,\ldots,a_n$, then for each $k=l,\ldots,m$ there is a k-chordal polygon whose sides have the lengths a_1, $\ldots,a_n$. Namely, if there is $\beta_1^{(m)}$ such that

$$\sum_{i=1}^{n}\arccos\Big(\frac{a_i}{a_1}\cos\beta_1^{(m)}\Big)=(n-2m)\frac{\pi}{2},$$

then there is $\beta_i^{(k)}$ such that $\sum_{i=1}^{n}\arccos\Big(\frac{a_i}{a_1}\cos\beta_1^{(k)}\Big)=(n-2k)\frac{\pi}{2}$.

So if $a_1=89+i$, $i=1,\ldots,7$ there is k-chordal heptagon for each $k=1,2,3$.

COROLLARY 3.4. *If the lengths* $a_1, \ldots, a_n$ *are such that*

$$a_1 + \cdots + a_n > n\sqrt[q]{\frac{2m}{n}}a_j, \ \ j = 1, \ldots, n \tag{47}$$

where $m = \lfloor \frac{n-1}{2} \rfloor$, $q = 2m-1$, *then at least for* $n \geqslant 12$ *there is* m*-chordal polygon whose sides have the lengths* $a_1, \ldots, a_n$.

Proof. Supposing that (47) holds, we need to prove that

$$\sum_{i=1}^{n} \arccos \frac{a_i}{a_1} < (n-2m)\frac{\pi}{2}. \tag{48}$$

For this purpose we shall prove the following lemma.

LEMMA. *If* $\cos\gamma_1 + \cdots + \cos\gamma_n \geqslant c$, $0 < \gamma_i < \frac{\pi}{2}$, $i = 1, \ldots, n$, *then*

$$\gamma_1 + \cdots + \gamma_n \leqslant n \arccos \frac{c}{n}.$$

In the proof we shall use the following fact: If

$$\gamma_1 + \ldots \gamma_n = a, \ \ 0 < \gamma_i < \frac{\pi}{2}, \ \ i = 1, \ldots, n$$

then $\max(\cos\gamma_1 + \cdots + \cos\gamma_n) = n \cdot \cos \frac{a}{n}$, that is $\cos\gamma_1 + \cdots + \cos\gamma_n =$ maximal for $\gamma_1 = \cdots = \gamma_n = \frac{a}{n}$.

Thus, if $\gamma_1 + \cdots + \gamma_n = n \arccos \frac{c}{n}$, then $\cos\gamma_1 + \cdots + \cos\gamma_n$ will be maximal if $\gamma_1 = \cdots = \gamma_n = \arccos \frac{c}{n}$.

It is not possibly to be $\gamma_1 + \cdots + \gamma_n = s$, where $s > n \arccos \frac{c}{n}$ since then

$$\cos\gamma_1 + \cdots + \cos\gamma_n \leqslant n \cos \frac{s}{n} < n \cos(\arccos \frac{c}{n}) = c.$$

So our lemma is proved.

Now from (47) we have

$$\cos\gamma_1 + \cdots + \cos\gamma_n > n\sqrt[q]{\frac{2m}{n}},$$

where $\cos\gamma_i = \frac{a_i}{a_1}$, $i = 1, \ldots, n$, $a_1 = \max\{a_1, \ldots, a_n\}$. Since

$$\sum_{i=1}^{n} \arccos \frac{a_i}{a_1} < n \cdot \arccos \sqrt[q]{\frac{2m}{n}}$$

and

$$n \cdot \arccos \sqrt[q]{\frac{2m}{n}} < (n-2m)\frac{\pi}{2},$$

it follows that (48) is valid.

Here are some examples.

1. If $n = 12$, then $m = 5$, $12 \arccos\left(\frac{10}{12}\right)^{\frac{1}{9}} = 126.4330954^\circ < 180^\circ$.
2. If $n = 13$, then $m = 6$, $13 \arccos\left(\frac{12}{13}\right)^{\frac{1}{11}} = 89.74674144^\circ < 90^\circ$.
3. If $n = 20$, then $m = 9$, $20 \arccos\left(\frac{18}{20}\right)^{\frac{1}{17}} = 127.44811^\circ < 180^\circ$.
4. If $n = 51$, then $m = 25$, $50 \arccos\left(\frac{50}{51}\right)^{\frac{1}{49}} = 83.06960318^\circ < 90^\circ$.
5. If $n = 100$, then $m = 49$, $100 \arccos\left(\frac{98}{100}\right)^{\frac{1}{97}} = 116.9342793^\circ < 180^\circ$.
6. If $n = 201$, then $m = 100$, $200 \arccos\left(\frac{200}{100}\right)^{\frac{1}{999}} = 81.53632288^\circ < 90^\circ$.
7. If $n = 500$, then $m = 249$, $500 \arccos\left(\frac{498}{500}\right)^{\frac{1}{497}} = 115.0490895^\circ < 180^\circ$.
8. If $n = 501$, then $m = 250$, $501 \arccos\left(\frac{500}{501}\right)^{\frac{1}{499}} = 81.23113761^\circ < 90^\circ$.
9. If $n = 1\,001$, then $m = 500$, $1\,001 \arccos\left(\frac{1\,000}{1\,001}\right)^{\frac{1}{999}} = 81.10950368^\circ < 90^\circ$.
10. If $n = 10\,001$, then $m = 5\,000$, $81.03657137^\circ < 90^\circ$.
11. If $n = 20\,000$, then $m = 9\,999$, $114.591559^\circ < 180^\circ$.

From (47) it follows that each of the sequences

$$u_n = n \arccos\left(\frac{n-1}{n}\right)^{\frac{1}{n-2}}, \ n \text{ is od and } n \geqslant 3, \tag{49}$$

$$v_n = n \arccos\left(\frac{n-2}{n}\right)^{\frac{1}{n-3}}, \ n \text{ is even and } n \geqslant 4, \tag{50}$$

must be convergent. Namely, it is clear that $|a_i - a_j| \to 0$ when $n \to \infty$. Thus, $A_1 \dots A_n$ converge to an equilateral polygon, and equilateral polygon has the property: If $a_1, \dots, a_n$ are equal, then for each $k = l, \dots, \lfloor \frac{n-1}{2} \rfloor$ there is a k-chordal polygon whose sides have the lengths $a_1, \dots, a_n$.

From the above examples it is clear that the sequences (49) and (50) are very slowly convergent which may be interesting in itself.

Also may be interesing that

$$0 < n \arccos\left(\frac{n-1}{n}\right)^{\frac{1}{n-2}} < \frac{\pi}{2}, \quad n \text{ is odd}$$
$$\frac{\pi}{2} < n \arccos\left(\frac{n-2}{n}\right)^{\frac{1}{n-3}} < \pi, \quad n \text{ is even}$$

or

$$1 > \left(\frac{n-1}{n}\right)^{\frac{1}{n-2}} > \cos\frac{\pi}{2n},$$
$$\cos\frac{\pi}{2n} > \left(\frac{n-2}{n}\right)^{\frac{1}{n-3}} > \cos\frac{\pi}{n}.$$

COROLLARY 3.5. *Let the lengths* $a_1, \ldots, a_n$ *be so that*

$$a_1^{2m-1} + \cdots + a_n^{2m-1} > 2m \; a_j^{2m-1}, \; j = 1, \ldots, n \tag{51}$$

where $m = \lfloor \frac{n-1}{2} \rfloor$. *Then at least for* $n \geqslant 12$ *there is* m*-chordal polygon whose sides have the lengths* $a_1, \ldots, a_n$.

Proof. The proof is similar to the proof of Corollary 3.4. Namely, from (51) we have

$$\sum_{i=1}^{n} \left(\frac{a_i}{a_1}\right)^{2m-1} > 2m$$

or

$$\sum_{i=1}^{n} \cos^{2m-1} \gamma_i > 2m,$$

from wich it follows that there are angles $\gamma_1, \ldots, \gamma_n$ such that

$$\sum_{i=1}^{n} \gamma_i < n \arccos\left(\frac{2m}{n}\right)^{\frac{1}{2m-1}}.$$

For example, if

$$\cos^{11}\gamma_1 + \cdots + \cos^{11}\gamma_{13} > 12$$

then $\gamma_1 + \cdots + \gamma_{13} < 13 \arccos\left(\frac{12}{13}\right)^{\frac{1}{11}} = 89.74674144^\circ < 90^\circ$.

Let us remark that from (51) it follows that for each $i, j \in \{1, \ldots, n\}$ it is valid $|a_i - a_j| \to 0$ when $n \to \infty$.

Notice 4. The assertion in Corollary 3.5 is a hypothesis in the paper [1]. So this hypothesis is now proved.

Notice 5. At this point we restrict ouerselves to the k-chordal polygon. Of course, similarly holds for k-inscribed chordal polygon.

Previous to stating the following theorem we give one definition.

DEFINITION 5. Let $a_1, \ldots, a_n$ be given lengths. If there exists a k-inscribed chordal polygon whose sides have the lengths $a_1, \ldots, a_n$, then it will be denoted by $\underline{A}^{(k)}(a_1, \ldots, a_n)$ and instead of notation $\beta_1, \ldots, \beta_n$ will be used notation $\beta_1^{(k)}$, $\ldots, \beta_n^{(k)}$.

If $\beta_{u_1}^{(k)}, \ldots, \beta_{u_j}^{(k)}$ are negative, then it will be denoted by

$$A^{(k)}(a_1, \ldots, a_n;\ u_1, \ldots, u_j).$$

In the case when $a_1 = \cdots = a_n = a$, then will be written

$$\underline{A}_n^{(k)}(a;\ u_1, \ldots, u_j).$$

Each two polygons $\underline{A}^{(k)}(a_1, \ldots, a_n)$ and $\underline{A}^{(l)}(a_1, \ldots, a_n)$, if both of them exist, will be called related chordal polygons and it will be written

$$\underline{A}^{(k)}(a_1, \ldots, a_n)\mathcal{R}\underline{A}^{(l)}(a_1, \ldots, a_n)$$

if $\operatorname{sign}\beta_i^{(k)} = \operatorname{sign}\beta_i^{(l)}$, $i = 1, \ldots, n$.

Thus $\underline{A}^{(k)}(a_1, \ldots, a_n)\mathcal{R}\underline{A}^{(l)}(a_1, \ldots, a_n)$ if and only if

$$\underline{A}^{(k)}(a_1, \ldots, a_n;\ u_1, \ldots, u_j)\mathcal{R}A^{(l)}(a_1, \ldots, a_n;\ u_1, \ldots, u_j).$$

Here are some examples.

1. $\underline{A}_8^{(1)}(1;2)$ and $\underline{A}_8^{(2)}(1;2)$ in Fig. 5 are related chordal octagons.

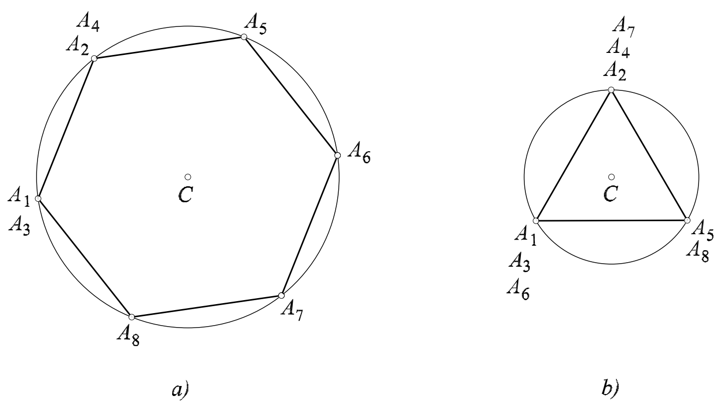

Fig. 5

2. Each of the polygons $\underline{A}_{11}^{(1)}(1;2,6)$, $\underline{A}_{11}^{(2)}(1;2,6,)$ and $A_{11}^{(3)}(1;2,6)$ (see Fig. 6) are related chordal 11-gons.

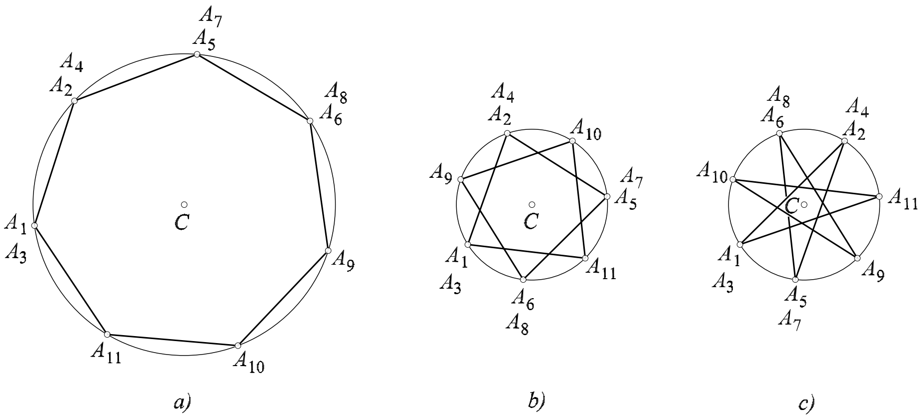

Fig. 6

3. The polygons $\underline{A}_{11}^{(1)}(1;1)$, $\underline{A}_{11}^{(2)}(1;1)$, $\underline{A}_{11}^{(3)}(1;1)$, $\underline{A}_{11}^{(4)}(1;1)$, are related chordal 11-gons, but, of course, they are not related to the polygons in the previous example.
4. The polygons $\underline{A}_{11}^{(1)}(1;2,4,6,8)$ Fig. 7 is only one 11-gon whose angles β_2, β_4, β_6, β_8 are negative and $a_1 = \cdots = a_{11} = 1$.

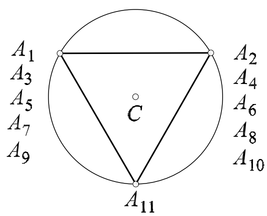

Fig. 7

It is easy to see that the number of related chordal n-gons can be at most $\lfloor \frac{n-1}{2} \rfloor$. This number depends not only of the lengths $a_1, \ldots, a_n$ but also of the number v (on the number of the negative angles β_i). For example, polygon shown in Fig. 7 is only one if β_2, β_4, β_6, β_8 are negative and $a_1 = \cdots = a_{11} = 1$.

Now we can state the following theorem in which the symbol P_j^n (given in Preliminaries) will be used.

THEOREM 4. *If $A_1 \ldots A_n$ is a k-inscribed chordal polygon, then $\beta_1, \ldots, \beta_n$ satisfy*

$$\sum_{i=1}^{m} (-1)^{i+1} P_{2i-1}^{n-1} = (-1)^{w+1} \cos \beta_n, \tag{52}$$

where $w = j + v$, $m = \dfrac{n-1}{2}$ if n is odd and $m = \dfrac{n}{2}$ if n is even.

Proof. If $A_1 \ldots A_n$ is k-inscribed chordal polygon then

$$\beta_1 + \cdots + \beta_n = (n - 2w)\frac{\pi}{2}, \tag{53}$$

where $w = j + v$. In the paper [1], Corollary 3.1, is proved that (52) holds if $A_1 \dots A_n$ is a k-chordal polygon, that is, when $v = 0$. Obviously, (52), is valid when $v \neq 0$ as well. Putting w instead k all essentially remain the same. So Theorem 4 is proved. □

Previous to we state some of its corollaries here are some examples.

1. Let the polygon be $\underline{A}_5^{(1)}(1;2)$. Then $\beta_1 = \beta_3 = \beta_4 = \beta_5 = \frac{\pi}{6}$, $\beta_2 = -\frac{\pi}{6}$ and we have

$$\begin{aligned} P_1^4 - P_3^4 =& \cos\beta_1 \sin\beta_2 \sin\beta_3 \sin\beta_4 + \sin\beta_1 \cos\beta_2 \sin\beta_3 \sin\beta_4 \\ &+ \sin\beta_1 \sin\beta_2 \cos\beta_3 \sin\beta_4 + \sin\beta_1 \sin\beta_2 \sin\beta_3 \cos\beta_4 \\ &- \cos\beta_1 \cos\beta_2 \cos\beta_3 \sin\beta_4 - \cos\beta_1 \cos\beta_2 \sin\beta_3 \cos\beta_4 \\ &- \cos\beta_1 \sin\beta_2 \cos\beta_3 \cos\beta_4 - \sin\beta_1 \cos\beta_2 \cos\beta_3 \cos\beta_4 \\ &- \frac{\sqrt{3}}{2} = (-1)^{2+1} \cos\frac{\pi}{6}. \end{aligned}$$

   Let us remark that $w = 2$ since $j = 1$, $v = 1$.

2. Let the polygon be $\underline{A}_6^{(1)}(1;4)$. Then $\beta_1 = \beta_2 = \beta_3 = \beta_5 = \beta_6 = \frac{\pi}{4}$, $\beta_4 = -\frac{\pi}{4}$ and it is easy to find that

$$P_1^5 - P_3^5 + P_5^5 = (-1)^{2+1} \cos\frac{\pi}{4}.$$

3. Let the polygon be $\underline{A}_7^{(1)}(1;2,5)$. Then $w = 1 + 2$ and we have

$$P_1^6 - P_3^6 + P_5^6 = (-1)^{3+1} \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

COROLLARY 4.1. *Let $\underline{A}^{(k)}(a_1, \dots, a_n)$ be a k-inscribed chordal polygon where j_k is its index and v is the number of β_i which are negative. The radius of the circumcircle of $\underline{A}^{(k)}(a_i, \dots, A_n)$ is a root of the equation*

$$\sum_{i=1}^{m} (-1)^{i+1} P_{2i-i}^{n-1}(a_1, \dots, a_n; r) = (-1)^{w_k+1} \frac{a_n}{2r}, \tag{54}$$

where $P_{2i-i}^{n-1}(a_1, \dots, a_n; r)$ is obtained by P_{2i-i}^{n-1} putting $\frac{a_i}{2r}$ instead of $\cos\beta_i$ and $\operatorname{sign}\beta_i \sqrt{1 - \left(\frac{a_i}{2r}\right)^2}$ instead of $\sin\beta_i$. Of course, $w_k = j_k + v$.

Here are some examples.

1. The radius of $\underline{A}_7^{(1)}(1;2)$ is the root of the equation

$$\sum_{i=1}^{4} (-1)^{i+1} P_{2i-1}^6(1,1,1,1,1,1,1;r) = (-1)^{2+1} \frac{1}{2r}.$$

   It can be found that $r = 0.85065$.

2. The radius of $\underline{A}_9^{(2)}(1;2,6)$ is the root of the equation

$$\sum_{i=1}^{4}(-1)^{i+1}P_{2i-1}^{8}(1,1,1,1,1,1,1,1,1;r)=(-1)^{4+1}\frac{1}{2r}.$$

It can be found that $r=0.52573$.

COROLLARY 4.2. *Let $\underline{A}^{(1)}(a_1,\dots,a_n)$ be a 1-inscribed chordal polygon. If there are l chordal polygons related to $A^{(1)}(a_1,\dots,a_n)$ then their radii $r_1,\dots,r_l$ are the roots of the equations*

$$\sum_{i=1}^{m}(-1)^{i+1}P_{2i-i}^{n-1}(a_1,\dots,a_n)=(-1)^{w_k}\frac{a_n}{2r_k},\quad k=1,\dots,l \tag{55}$$

where w_k as in Corollary 4.1.

Of course, the number v in the sum j_k+v is the same (by Definition 5) for all chordal polygon related to the polygon $\underline{A}^{(1)}(a_1,\dots,a_n)$.

Here are some examples.

1. If $n=11$, $a_1=\dots=a_{11}=1$ (see Fig. 6), then (55) can be written as

$$\sum_{i=1}^{5}(-1)^{i+1}P_{2i-1}^{10}(1,\dots,1;r)=(-1)^{j_k+2}\frac{1}{2r_k},\ k=1,2,3$$

from which it follows $r_1=1.15238$, $r_2=0.63952$, $r_3=0.51286$.

2. If $n=12$, $a_1=\dots=a_{12}=1$, $v=0$ then (55) can be written as

$$\sum_{i=1}^{6}(-1)^{i+1}P_{2i-1}^{11}(1,\dots,1;r)=(-1)^{k+1}\frac{1}{2r_k},\ k=1,\dots,5$$

from which it follows $r_1=1.93185$, $r_2=1$, $r_3=0.70711$, $r_4=0.57735$, $r_5=0.51176$.

Since each chordal polygon determines a chordal semi-polygon, each of the polygon in the above examples determines a chordal semi-polygon.

The radius of a chordal semi-polygon which is not a polygon, can be obtained in the following way.

Let $A_1\dots A_n$ be a chordal semi-polygon whose sides have the lengths $a_1,\dots,a_{n-1}$ and let the sum $\beta_1+\dots+\beta_{n-1}$ be given with sign of each β_i. If $\beta_1+\dots+\beta_{n-1}=\tau$, then the equation

$$\cos(\beta_1+\dots+\beta_{n-1})=\cos\tau$$

can be used. For example, if $n=5$, $a_1=a_2=a_3=a_4=1$, $\beta_1=\beta_2=\beta_3=\beta_4=\frac{\pi}{6}$, $\beta_2=-\frac{\pi}{6}$ then we have the equation

$$\begin{aligned}\cos(\beta_1+\beta_2+\beta_3+\beta_4)=&c_1c_2c_3c_4-s_1s_2c_3c_4-s_1c_2s_3c_4-c_1s_2s_3c_4\\&-s_1c_2c_3s_4-c_1s_2c_3s_4-c_1c_2s_3s_4+s_1s_2s_3s_4,\end{aligned}$$

where, for brevity, instead od $\sin\beta_i$ is written s_i and c_i instead of $\cos\beta_i$. Hence, using the expressions $\cos\beta_i = \frac{1}{2r}$, $\sin\beta_i = \operatorname{sign}\beta_i\sqrt{1-\left(\frac{1}{2r}\right)^2}$ we find that $r = \frac{\sqrt{3}}{3}$.

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