Mathematical
Inequalities
& Applications
Volume 4, Number 3 (2001), 327–330

SOME GENERALIZATIONS FOR A THEOREM BY LANDAU

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(*communicated by J. Sándor*)

Abstract. Let $\pi(x)$ be the number of primes not exceeding x. E. Landau made the following conjecture: $\pi(2x) \leqslant 2\pi(x)$ for integer $x \geqslant 2$. In 1966 Rosser and Schoenfeld proved this conjecture. In the present paper we establish upper bounds for $\pi(x+y)$. Taking the particular case $x = y$, we find again Landau's inequality.

1. Introduction

In [2] E. Landau proved the existence of a real x such that whenever $x > x_0$ one has $\pi(2x) \leqslant 2\pi(x)$ and at the same time he asserted that the inequality is satisfied for every integer $x \geqslant 2$.

This inequality was later proved by J. B. Rosser and L. Schoenfeld [6]. For informations on an elementary proof see [3], p. 236. We will begin with the presentation of the same classical inequalities.

In [5] Rosser and Schoenfeld show that for $x \geqslant 67$ one has

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x) < \frac{x}{\log x - \frac{3}{2}}. \tag{1}$$

Hardy and Littlewood [3] made the following conjecture:

For all integers $x, y \geqslant 2$,

$$\pi(x+y) \leqslant \pi(x) + \pi(y). \tag{2}$$

This problem is not yet solved, but for $x = y$ it is exactly the Landau-Rosser-Schoenfeld's inequality.

H.L. Montgomery and R.C. Vaughan (see [3] p. 237) have proved that

$$\pi(x+y) < \pi(x) + 2\pi(y)$$

for all integers $x \geqslant 1$, $y \geqslant 2$.

Ishikawa [1] proved that if integers $x, y \geqslant 1$ then

$$\pi(xy) \geqslant \pi(x) + \pi(y). \tag{3}$$

Mathematics subject classification (2000): 11A25, 11N05.

Key words and phrases: prime numbers, inequalities, convex function.

The author gratefully acknowledges partial support from Grant #7D/2000 awarded by the Consiliul Naţional al Cercetării Ştiinţifice din Învăţământul Superior, România..

Paper MIA-04-31

Recently I was able to prove in [4] that

$$\pi(x) < \frac{x}{\log x - 1.12} \text{ for any } x \geqslant 4, \tag{4}$$

and

$$\pi(x) < \frac{x}{\log x - \frac{28}{29}} \text{ for any } x \geqslant 3299. \tag{5}$$

2. Upper bounds for $\pi(x+y)$

The following inequalities are connected with (2), but weaker than (2). In order to obtain them it is necessary to use (1), (4), (5) and Schinzel's inequality [7]:

$$\pi(x+y) \leqslant \pi(x) + \pi(y) \text{ for } \min(x,y) \leqslant 146 \tag{6}$$

and also Segal's inequality [8]:

$$\pi(x+y) \leqslant \pi(x) + \pi(y) \text{ for } x+y \leqslant 101,081. \tag{7}$$

THEOREM 1. *Let* $x \geqslant y \geqslant 2$ *be integer numbers. The following inequality holds:*

$$\pi(x+y) \leqslant \pi(x) + \pi(y) + \pi(x-y). \tag{8}$$

Proof. Let $x \geqslant y \geqslant 2$.

For $y \leqslant 146$, according to (6), the inequality (8) is valid.

Let $x \geqslant y > 146$.

a) If $x - y \geqslant 67$, using (1) we have

$$\begin{aligned} E(x,y) = \pi(x) + \pi(y) + \pi(x-y) &> \frac{x}{\log x - \frac{1}{2}} + \frac{y}{\log y - \frac{1}{2}} + \frac{x-y}{\log(x-y) - \frac{1}{2}} \\ &> \frac{2x-y}{\log x - \frac{1}{2}} + \frac{y}{\log y - \frac{1}{2}} = \frac{2x}{\log x - \frac{1}{2}} + \frac{y \log \frac{x}{y}}{\left(\log x - \frac{1}{2}\right)\left(\log y - \frac{1}{2}\right)} \\ &> \frac{2x}{\log x - \frac{1}{2}} > \frac{2x}{\log 2x - 1.12}. \end{aligned}$$

Using (5) we will obtain $E(x,y) > \pi(2x) \geqslant \pi(x+y)$. It remains to study the case

b) $x - y < 67$ and $x \geqslant y \geqslant 146$.

If $x \leqslant 55,00$ according to (7), (8) is true.

Let now $x > 55,000$. It follows immediately that $y > y - 67 > 50,000$ and we have $\pi(x+y) \leqslant \pi(2y+67)$.

It is enough to prove that $\pi(2y+67) \leqslant 2\pi(y)$ for $y > 50,000$.

Using the inequalities (4) and (5) we will obtain

$$\pi(2y+67) < \frac{2y+67}{\log(2y+67) - 1.12} < \frac{2y+67}{\log 2y - 1.12} < \frac{2y+67}{\log y - 0.43}$$

and $2\pi(y) > 2y/\left(\log y - \frac{28}{29}\right)$.

It is enough to prove that $2y/\left(\log y - \frac{28}{29}\right) > (2y+67)/(\log y - 0.43)$, which is true for $y > 50,000$.

THEOREM 2. *For integers $x \geqslant y \geqslant 2$ we have*

$$2\frac{\pi(x+y)}{x+y} \leqslant \frac{\pi(x)}{x} + \frac{\pi(y)}{y} \tag{9}$$

excepting the following cases: $x=3, y=2$ and $x=5, y=2$.

Proof. Let $x, y \geqslant 67$ integer numbers. From (1) and the inequality $\frac{1}{a}+\frac{1}{b} \geqslant \frac{4}{a+b}$ $(a,b>0)$ we obtain

$$\frac{\pi(x)}{x} + \frac{\pi(y)}{y} > \frac{1}{\log x - \frac{1}{2}} + \frac{1}{\log x - \frac{1}{2}} > \frac{4}{\log xy - 1}.$$

Taking into account formula (4) we have successively

$$\begin{aligned} 2\frac{\pi(x+y)}{x+y} &< \frac{2}{\log(x+y) - 1.12} \leqslant \frac{2}{\log 2\sqrt{xy} - 1.12} \\ &= \frac{4}{\log xy - 2(1.12 - \log 2)} < \frac{2}{\log xy - 1} < \frac{\pi(x)}{x} + \frac{\pi(y)}{y}. \end{aligned}$$

In case $\min(x,y) < 67$ we choose $x \geqslant y$ and therefore $y < 67$. From (6) we have

$$2\frac{\pi(x+y)}{x+y} \leqslant 2\frac{\pi(x)+\pi(y)}{x+y}.$$

On the other hand, we have

$$2\frac{\pi(x)+\pi(y)}{x+y} \leqslant \frac{\pi(x)}{x} + \frac{\pi(y)}{y} \text{ if an only if } \frac{\pi(x)}{x} \leqslant \frac{\pi(y)}{y}.$$

It remains to study the case $\dfrac{\pi(x)}{x} > \dfrac{\pi(y)}{y}$. It is enough to prove that we have $\dfrac{\pi(x+y)}{x+y} < \dfrac{\pi(y)}{y}$ for $2 \leqslant y \leqslant 66$.

Because $\min\limits_{2\leqslant y\leqslant 66} \dfrac{\pi(y)}{y} = \dfrac{3}{11}$ we will show that $\dfrac{\pi(x+y)}{x+y} \leqslant \dfrac{3}{11}$, which is true for $x+y \geqslant 81$.

It remains to study the case $x+y \leqslant 80$. The author checked this finite number of inequalities using a P.C. After this check the proof is complete.

THEOREM 3. *For integers $x, y \geqslant 2$ one has*

$$\pi^2(x+y) \leqslant 2\left(\pi^2(x) + \pi^2(y)\right). \tag{10}$$

Proof. We will use (1) and (4). For $x \geqslant 67$ define $f(x) = x^2/\left(\log x - \frac{1}{2}\right)^2$. It follows:

$$f''(x) = \frac{2}{\left(\log x - \frac{1}{2}\right)^2}\left(\log^2 x - 4\log x + \frac{19}{4}\right) > 0.$$

So f is convex. Consequently for $x, y \geqslant 67$ we get

$$2\left(\pi^2(x) + \pi^2(y)\right) \geqslant 2(f(x) + f(y)) \geqslant 4f\left(\frac{x+y}{2}\right) = \frac{(x+y)^2}{\left(\log\frac{x+y}{2} - \frac{1}{2}\right)^2}$$

$$> \left(\frac{x+y}{\log(x+y) - 1.12}\right)^2 \geqslant \pi^2(x+y)$$

In case $x \geqslant y$ and $y \leqslant 66$ from (6) it follows

$$\pi(x+y) \leqslant \pi(x) + \pi(y) < \pi(x) + 18.$$

Because $\pi(y) \geqslant 1$ it is enough to show that $(\pi(x) + 18)^2 \leqslant 2(\pi^2(x) + 1)$ that is $\pi(x) \geqslant 44$ so $x \geqslant 193$.

The cases $2 \leqslant y \leqslant x \leqslant 193$ are consequences of (7).

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