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A NOTE ON GENERALIZED HERONIAN MEANS

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Abstract. In this note several inqualities for generalized Heronian Means of two numbers are proved. These means are defined for the first time in this paper and they are compared with some well-known means. All inequalities are best possible.

0. Introduction

In [3] the special case $w = 4$ of the generalized Heronian mean $H_w(a,b)$ now defined as

$$H_w(a,b) := \begin{cases} \dfrac{a + w\sqrt{ab} + b}{w+2}, & 0 \leqslant w < \infty \\ \sqrt{ab}, & w = \infty \end{cases}$$

is dealt with. (a and b are non-negative real numbers.)

$H_4(a,b)$ is denoted by $\widetilde{H}(a,b)$ in [3]. Among other things the double-inequality $M_{1/3}(a,b) \leqslant H_4(a,b) \leqslant M_{1/2}(a,b)$ is shown in [3], where as usual

$$M_p(a,b) := \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0 \\ \sqrt{ab}, & p = 0 \end{cases}$$

denotes the p-th power-mean of a and b. (It is also proven there that "$1/3$" cannot be increased without letting the inequality be true no longer in general.)

Furthermore, for the "classical" Heronian mean $He(a,b) = H_1(a,b)$ is known that the sharpest double-inequality ot type

$$M_\alpha(a,b) \leqslant He(a,b) \leqslant M_\beta(a,b) \tag{1}$$

is given by $\alpha = \ln 2/\ln 3$ and $\beta = 2/3$. (See [1] and [2], p. 350.)

It is one of the aims of this note to establish the best inequality of type (1) for $He(a,b)$ replaced by $H_w(a,b)$, $w > 0$. We thereby answer one of the open questions posed in [1] by the present author.

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We also determine the optimum values α and β for two comparison double-inequalities of type

$$H_\alpha(a,b) \leqslant F(a,b) \leqslant H_\beta(a,b) \tag{2}$$

where $F(a,b)$ stands for the logarithmic mean $L(a,b)$ and the identric mean $I(a,b)$ defined as $L(a,b) := \dfrac{a-b}{\ln a - \ln b}$ and $I(a,b) := \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{1/(b-a)}$, resp.

1. Elementary properties of $H_w(a,b)$

• $H_0(a,b)$, $H_2(a,b)$ and $H_\infty(a,b)$ equal the arithmetic mean $A(a,b)$, the power-mean $M_{1/2}(a,b)$ and the geometric mean $G(a,b)$, resp., of a and b.

• $H_w(a,b)$ is a non-increasing function of the variable w, i.e. we have $H_\alpha(a,b) \geqslant H_\beta(a,b)$ whenever $0 \leqslant \alpha \leqslant \beta \leqslant \infty$.

• $G(a,b) \leqslant H_w(a,b) \leqslant A(a,b)$ whenever $0 \leqslant w \leqslant \infty$.

• $H_w(a,b) = \dfrac{w}{w+2}G(a,b) + \dfrac{2}{w+2}A(a,b)$, i.e. $H_w(a,b)$ is a convex combination of $G(a,b)$ and $A(a,b)$.

• For all $w \geqslant 0$ the mean $H_w(a,b)$ is concave.

(This follows from H_w being the sum of a linear function and the square-root, both of which are concave.)

2. A preliminary lemma

In one of the subsequent proofs we shall use the following auxiliary result stated as

LEMMA.
(i) *For* $w \in (0,2)$ *there holds* $(w+2)^4 > 4^{w+2}$.
(ii) *For* $w > 2$ *the reversed inequality holds true.*

Proof. Upon taking logarithm of $f(w) := (w+2)^4/4^{w+2}$, $w > 0$, and differentiating twice the log-concavity of $f(w)$ follows. This yields both inequalities. □

3. Generalized Heronian means and power means

In this section we shall prove the following

THEOREM 1. *Let* $w > 0$ *be given. Then the optimum values* α *and* β *such that*

$$M_\alpha(a,b) \leqslant H_w(a,b) \leqslant M_\beta(a,b) \tag{3}$$

holds true in general, are

1.) *in case of* $w \in (0,2]$: $\alpha_{max} = \dfrac{\ln 2}{\ln(w+2)}$ *and* $\beta_{min} = \dfrac{2}{w+2}$

2.) *in case of* $w \in [2,\infty)$: $\alpha_{max} = \dfrac{2}{w+2}$ *and* $\beta_{min} = \dfrac{\ln 2}{\ln(w+2)}$

Proof.

• For $w = 2$ we are done due to $H_2(a,b) = M_{1/2}(a,b)$. Thus let further on be $w \neq 2$.

• Letting $a = 1$ and $b = 0$ in (3) we obtain $2^{-1/\alpha} \leqslant \dfrac{1}{w+2} \leqslant 2^{-1/\beta}$ i.e. $\alpha \leqslant \ln 2/\ln(w+2) \leqslant \beta$. This shows that neither α_{max} for $w \in (0,2)$ nor β_{min} for $w \in (2,\infty)$ can be improved.

• Let us compare now $H_w(a,b)$ and $M_{\ln 2/\ln(w+2)}(a,b)$ by considering the difference $\ln H_w(a,b) - \ln M_{\ln 2/\ln(w+2)}(a,b)$, i.e. upon putting $b = 1$ (due to homogeneity) and $a = s^{2\ln(w+2)}$, $s \geqslant 1$ (due to symmetry), and using the abbreviation $p := \ln(w+2)$ we have to look at the function

$$A(s) := \ln 2 \cdot \ln(1 + w \cdot s^p + s^{2p}) - p \cdot \ln(1 + s^{2\ln 2}), \quad s \geqslant 1.$$

But

$$A'(s) = p \cdot \ln 2 \cdot s^{-1} \cdot \left[\frac{w \cdot s^p + 2s^{2p}}{1 + w \cdot s^p + s^{2p}} - \frac{2s^{2\ln 2}}{1 + s^{2\ln 2}}\right]$$

i.e. (as a short simplification shows)

$$A'(s) = \frac{p \cdot \ln 2 \cdot s^{2\ln 2 - 1} \cdot K(s)}{(1 + ws^p + s^{2p})(1 + s^{2\ln 2})},$$

where

$$K(s) := 2s^{2p-2\ln 2} - w \cdot s^p + w \cdot s^{p-2\ln 2} - 2.$$

But $K'(s) = s^{p-2\ln 2-1} \cdot L(s)$, with

$$L(s) := 2(2p - 2\ln 2)s^p - p \cdot w \cdot s^{2\ln 2} + w(p - 2\ln 2),$$

whence finally

$$L'(s) = p \cdot s^{2\ln 2 - 1} \cdot [(4p - \ln 16)s^{p-2\ln 2} - 2w\ln 2].$$

We now distinguish two cases.

(i) $w \in (0,2)$. Then (via Lemma) $L'(1) > 0$. Because of $p - 2\ln 2 < 0$ it follows $L'(s) > 0$ for $s \in (1,s_0)$ and $L'(s) < 0$ as $s \in (s_0,\infty)$ where $s_0 = \left(\dfrac{2w\ln 2}{4p-16}\right)^{1/(p-2\ln 2)}$. Thus, $L(1) > 0$ (once more via Lemma) and $L(s) \to -\infty$ as $s \to \infty$ (note $p < 2\ln 2$) yield the existence of precisely one $s_1 \in (s_0,\infty)$ such that $K(s)$ increases as $s \in (1,s_1)$ and $K(s)$ decreases as $s \in (s_1,\infty)$. Because of $A'(1) = 0$ and $\lim\limits_{s\to\infty} A'(s) = 0$ we finally arrive at $A(s) \geqslant 0$, $s \geqslant 1$.

But this means $\alpha_{max} = \dfrac{\ln 2}{\ln(w+2)}$, as claimed.

(ii) $w \in (2,\infty)$. Reasoning in a similar fashion as before we have $L'(s) < 0$ as $s \in (1,s_0)$ and $L'(s) > 0$ for $s \in (s_0,\infty)$. As in $i)$ there follows the existence of two intervals $(1, s_2)$ and (s_2, ∞) on which $K(s)$ decreases and increases, resp., finally leading to $A(s) \leqslant 0$, $s \geqslant 1$. This in turn means $\beta_{min} = \dfrac{\ln 2}{\ln(w+2)}$.

• Next we consider $M_{2/(w+2)}(a,b)$ and $H_w(a,b)$. Proceding as before (i.e. taking logarithm and putting now $a = s^{2(w+2)}$) for $\ln M_{2/(w+2)}(a,b) - \ln H_w(a,b)$ we have to discuss the function

$$B(s) := (w+2)\ln(1+s^4) - 2\ln(1+w\cdot s^{w+2}+s^{2(w+2)}) - (w+2)\ln 2 + 2\ln(w+2), \quad s \geqslant 1.$$

Now

$$B'(s) = \frac{(-2)(w+2)s^3 \cdot F(s)}{(1+w\cdot s^{w+2}+s^{2(w+2)})(1+s^4)},$$

where $F(s) := 2s^{2w} - w\cdot s^{w+2} + w\cdot s^{w-2} - 2$. But $F'(s) = ws^{w-3}\cdot G(s)$ with $G(s) := 4s^{w+2} - (w+2)s^4 + w - 2$. Finally $G'(s) = 4(w+2)s^3\cdot(s^{w-2}-1)$, $s \geqslant 1$, showing $G'(s) \leqslant 0$ as $w \in (0,2)$ and $G'(s) \geqslant 0$ for $w \in (2,\infty)$. Therefore, due to $G(1) = F'(1) = 0$ there follows for $s \geqslant 1$: $F'(s) \leqslant 0$ for $w \in (0,2)$ and $F'(s) \geqslant 0$ as $w \in (2,\infty)$. This and $F(1) = 0$ imply for $s \geqslant 1$: $F(s) \leqslant 0$ if $w \in (0,2)$ and $F(s) \geqslant 0$ if $w \in (2,\infty)$, whence finally $B(s) \geqslant 0$, $w \in (0,2)$ and $B(s) \leqslant 0$, $w \in (2,\infty)$, as $s \geqslant 1$.

These inequalities mean $\beta_{min} \leqslant \dfrac{2}{w+2}$ as $w \in (0,2)$ and $\alpha_{max} \geqslant \dfrac{2}{w+2}$ as $w \in (2,\infty)$.

• We finally have to show that neither one of these values can be improved. Indeed, the Taylor series expansion of the difference

$$d_p(s) := \ln M_p(s^2,1) - \ln H_w(s^2,1)$$

at $s = 1$ starts

$$d_p(s) = \frac{p(w+2)-2}{2(w+2)}\cdot[(s-1)^2 - (s-1)^3] + \ldots$$

Therefore in case of $w \in (0,2)$ we have to have (via $d_p(s) \geqslant 0$) $p(w+2) - 2 \geqslant 0$, i.e. $p \geqslant 2/(w+2)$ and $\beta_{min} = 2/(w+2)$ follows. In a similar way the claim $\alpha_{max} = 2/(w+2)$ for $w \in (2,\infty)$ is settled. □

Remark. Theorem 1 in different notation $\left(w = \dfrac{2\lambda}{1-\lambda}, 0 < \lambda < 1\right)$ answers one of the open questions in [1].

4. Generalized Heronian means and the logarithmic mean

In this section we deal with inequality (2) where $F(a,b) \equiv L(a,b)$ and we shall prove

THEOREM 2. *The optimum numbers* α *and* β *such that*

$$H_\alpha(a,b) \leqslant L(a,b) \leqslant H_\beta(a,b) \tag{4}$$

is true in general, are $\alpha_{min} = \infty$ *and* $\beta_{max} = 4$.

Proof.

- For the left-hand-inequality let $b = 1$ and $a \to 0$. Because of $L(a,1) = \dfrac{a-1}{\ln a} \to 0$ as $a \to 0$ we infer $\dfrac{1}{\alpha+2} \leqslant 0$, i.e. $\alpha = \infty$. Due to [2], p.347 the inequality $G(a,b) \leqslant L(a,b)$ is valid and we are done.
- For the right-hand-inequality we put $a = s^2$ $(s > 1)$ and $b = 1$ and get $\dfrac{s^2-1}{2\ln s} \leqslant \dfrac{1+\beta s+s^2}{\beta+2}$ i.e. $f(s) \geqslant 0$ where

$$f(s) := \ln s - \frac{\beta+2}{2} \cdot \frac{s^2-1}{1+\beta s+s^2}, \quad s > 1.$$

Now the Taylor – series expansion of $f(s)$ at $s = 1$ starts $f(s) = \dfrac{(4-\beta)(s-1)^3}{6(\beta+2)} + \dots$ immediately showing $\beta \leqslant 4$. But in [3] the inequality $L(a,b) \leqslant H_4(a,b)$ is proven and we are done for this case, too. □

5. Generalized Heronian means and the identric mean

Now we deal with inequality (2) for $F(a,b) \equiv I(a,b)$ and we shall show

THEOREM 3. *The optimum numbers* α *and* β *such that*

$$H_\alpha(a,b) \leqslant I(a,b) \leqslant H_\beta(a,b) \tag{5}$$

is valid in general, are $\alpha_{min} = 1$ *and* $\beta_{max} = e-2$.

Proof.

- For the right hand inequality we firstly note upon letting $b = 1$ and $a = t$:

$$\frac{1}{e} \cdot t^{t/(t-1)} \leqslant \frac{1+\beta\sqrt{t}+t}{\beta+2}.$$

Therefore $t \to 0$ leads to $\dfrac{1}{e} \leqslant \dfrac{1}{\beta+2}$, i.e. $\beta \leqslant e-2$. We now show the claimed validity for $\beta = e-2$. Taking logarithm and collecting terms we get $g(t) \geqslant 0$, where

$$g(t) := (t-1)\ln(1+(e-2)\sqrt{t}+t) - t\ln t, \qquad t \geqslant 1$$

(due to symmetry). Now

$$g'(t) = \ln(1+(e-2)\sqrt{t}+t) - \ln t - \frac{(e-2)t+4\sqrt{t}+e-2}{2\sqrt{t}(1+(e-2)\sqrt{t}+t)}$$

and

$$g''(t) = \frac{(1-t)((e-2)t + 2(e^2-4e+2)\sqrt{t} + e - 2)}{4t\sqrt{t}(1+(e-2)\sqrt{t}+t)^2}.$$

From these expressions we deduce $g''(t) > 0$ for $t \in (1, t_0)$ and $g''(t) < 0$ for $t \in (t_0, \infty)$. Here t_0 denotes the only solution of $g''(t) = 0$ lying in $(1, \infty)$. Because of $g'(1) = 0$ and $\lim_{t\to\infty} g'(t) = 0$ there follows $g'(t) \geqslant 0$ as $t \geqslant 1$. This and $g(1) = 0$ finally yield the claimed inequality.

• Proceeding similarly for the left-hand-inequality we firstly get upon putting $a = t$ and $b = 1$ $(t \geqslant 1)$ and taking logarithm: $k_\alpha(t) \geqslant 0$, where

$$k_\alpha(t) := t\ln t + (t-1)(\ln(\alpha+2) - 1) - (t-1)\ln(1+\alpha\sqrt{t}+t).$$

Developing this function about $t = 1$ gives $k_\alpha(t) = \dfrac{\alpha-1}{12(\alpha+2)}(t-1)^3 + \ldots$ whence $\alpha \geqslant 1$. In order to prove the validity of $k_1(t) \geqslant 0$, $t \geqslant 1$, we note

$$k_1'(t) = \ln t - \ln(1+\sqrt{t}+t) + \frac{(1-t)(2\sqrt{t}+1)}{2\sqrt{t}(1+\sqrt{t}+t)}$$

and

$$k_1''(t) = \frac{(\sqrt{t}+1)(\sqrt{t}-1)^3}{4t\sqrt{t}(1+\sqrt{t}+t)^2}.$$

Therefore $k_1(t)$ is convex on $(1, \infty)$ and $k_1'(1) = k_1(1) = 0$ immediately yields $k_1(t) \geqslant 0$, as claimed. □

In [4] the following identities are stated

$$A(a,b) = G(a,b) \cdot \exp\Big(\sum_{k=1}^{\infty} \frac{1}{2k}\Big(\frac{a-b}{a+b}\Big)^{2k}\Big)$$

and

$$I(a,b) = G(a,b) \cdot \exp\Big(\sum_{k=1}^{\infty} \frac{1}{2k+1}\Big(\frac{a-b}{a+b}\Big)^{2k}\Big).$$

Therefore theorem 3 allows the curious.

COROLLARY 1. *Let* $x \in (-1, 1)$.

(i) Then $\alpha + 2\exp\Big(\sum_{k=1}^{\infty} \frac{1}{2k}x^{2k}\Big) \leqslant (\alpha+2)\exp\Big(\sum_{k=1}^{\infty} \frac{1}{2k+1}x^{2k}\Big)$ *whenever* $\alpha \in [1, \infty)$.

(ii) The reversed inequality holds true whenever $\alpha \in [0, e-2]$. □

Using familiar series – expansions this can also be summarized as

COROLLARY 2. *Let* $x \in (-1, 1)$.

(i) Then $\alpha + \dfrac{2}{\sqrt{1-x^2}} \leqslant \dfrac{\alpha+2}{e}\Big(\dfrac{1+x}{1-x}\Big)^{\frac{1}{2x}}$ *whenever* $\alpha \geqslant 1$.

(ii) The reversed inequality is valid whenever $\alpha \in [0, e-2]$. □

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